Scenario-based Risk Evaluation

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Abstract

Risk measures such as Expected Shortfall (ES) and Value-at-Risk (VaR) have been prominent in banking regulation and financial risk management. Motivated by practical considerations in the assessment and management of risks, including tractability, scenario relevance and robustness, we consider theoretical properties of scenario-based risk evaluation. We propose several novel scenario-based risk measures, including various versions of Max-ES and Max-VaR, and study their properties. We establish axiomatic characterizations of scenario-based risk measures that are comonotonic-additive or coherent and an ES-based representation result is obtained. These results provide a theoretical foundation for the recent Basel III & IV market risk calculation formulas. We illustrate the theory with financial data examples.

Keywords: Scenarios, risk measures, Expected Shortfall, model uncertainty, Basel Accords, stress adjustment, dependence adjustment

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1 Introduction

1.1 Background

Risk measures are used in various contexts in banking and insurance, such as regulatory capital calculation, optimization, decision making, performance analysis, and risk pricing; see e.g. McNeil et al. (2015) for a general review of quantitative risk management. In practice, risk measures have to be estimated from data. Therefore, it is often argued that one has to use a *law-based risk measure* (or a *statistical functional*), such as a Value-at-Risk (VaR) or an Expected Shortfall (ES), both standard risk measures used in banking and insurance.

However, even assuming that the distribution of a risk is accurately obtained, it may not be able to comprehensively describe the nature of the risk. From the regulatory perspective, a regulator is more concerned about the behavior of a risk in an adverse environment, e.g. during a catastrophic financial event; see e.g. Acharya et. al. (2012) for related discussions. Only the distribution of the risk may not be enough to distinguish a potentially huge loss in a financial crisis from a potentially huge loss in a common economy but no loss in a financial crisis.[1] Therefore, it may be useful to evaluate a risk under *different stress scenarios*. Summing up these evaluations in a single number would necessarily lead to a *non-law-based risk measure*.

Finally, it is usually unrealistic to assume that the distribution of a risk may be accurately obtained. Model uncertainty is a central component of the current challenges in risk measurement and regulation, and its importance in practice has been pivotal after the 2007 financial crisis (see e.g. OCC (2011)) in both the banking (e.g. BCBS (2016)) and the insurance sectors (e.g. IAIS (2014)). Model uncertainty may be due to statistical/parameter uncertainty or more generally, structural uncertainty of the model or of the economic system. A *robust approach* should take into account the distribution of the underlying risk under several plausible model assumptions.

In the framework of Basel III & IV (BCBS (2016)), the standard risk measure for market risk is an Expected Shortfall (ES_p) at level $p = 0.975$. Thus, the Basel Committee on Banking Supervision has opted for a law-based risk measures. However, while ES is the basic building block for marked risk assessment, the initial ES estimates are subsequently modified, in particular, two important adjustments are a *stress adjustment* and a *dependence adjustment* (p.52 - p.69 of BCBS (2016)), which then leads to the *capital charge for modellable risk factors* (abbreviated as IMCC in BCBS (2016)).

The aim of this paper is to present a theoretical approach to the construction of risk

[1] As another simple example, the profit/loss from a lottery and that from an insurance contract may have the same distribution, but they represent very different types of risks and can have very different effects on the decision maker or the society.

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measures that incorporates modifications such as a stress and dependence adjustment of an initial law-based risk measure into the risk measure itself. We call such risk measures *scenario-based risk measures*; see Definition 1. Our approach has the advantage that the final result of the risk estimation can be understood theoretically and properties such as coherence and comonotonic additivity can be studied not only for the initial law-invariant risk measure but for the final risk measure that is the relevant output for further actions and decisions, such as the IMCC in the Basel III & IV framework.

Before presenting our theoretical framework, let us give some details on the latest regulatory framework of the Basel Committee on Banking Supervision to illustrate how they deal with the issues mentioned above.

1.2 The Basel formulas for market risk and other motivating examples

In the framework of Basel III & IV (BCBS (2016)) for market risk, the time horizon is 10 days (two trading weeks), and each risk position (random loss) is modelled as a function of risk factors, such as equity prices, interest rates, credit spreads, and volatilities. Each risk factor is adjusted according to their category of liquidity. Let $X = \sum_{i=1}^{n} X_i$ be the aggregate portfolio loss at a given day, where $X_1, \ldots, X_n$ are the corresponding risk factors in the aggregation (with weights included).

(i) Stress adjustment

(a) Specify a set R of reduced risk factors which has a sufficiently long history of observation (at least span back to and including 2007), such that the ratio

$$\theta = \max\left\{\frac{\mathrm{ES}_F(X)}{\mathrm{ES}_R(X)}, 1\right\}$$

is less than 4/3, where $\mathrm{ES}_F(X) = \mathrm{ES}_p(\sum_{i=1}^{n} X_i)$ is the current ES value calculated using all risk factors, and $\mathrm{ES}_R(X) = \mathrm{ES}_p(\sum_{i\in R} X_i)$ is the current ES value calculated using the reduced risk factors. The ratio θ is treated like a constant and only needs to be updated weekly.

(b) Compute ES for a model with the reduced risk factors, *"calibrated to the most severe 12-month period of stress"*, and this is denoted by $\mathrm{ES}_{R,S}(X)$. The period of "most severe stress", also called the *stress scenario* corresponds to the rolling window of data of length one year that leads to the maximum possible value of ES using the reduced risk factor model (p.6 of BCBS (2017)). Mathematically, $\mathrm{ES}_{R,S}(X)$ involves taking a maximum over a set $\mathcal{Q}$ of distributions estimated from sequences of data of length one year (many of them

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overlapping), namely

$$\mathrm{ES}_{R,S}(X) = \max_{Q \in \mathcal{Q}} \mathrm{ES}_p^Q \left(\sum_{i \in R} X_i \right).$$

(c) Use the formula

$$\widetilde{\mathrm{ES}}(X) = \mathrm{ES}_{R,S}(X) \times \theta$$

to get the stress-adjusted ES value.

In particular, if the portfolio loss is modelled by only risk factors of sufficiently long history (spanning back to 2007), then $R = \{1, \ldots, n\}$ and the adjusted ES value is

$$\widetilde{\mathrm{ES}}(X) = \max_{Q \in \mathcal{Q}} \mathrm{ES}_p^Q \left(\sum_{i=1}^{n} X_i \right) = \max_{Q \in \mathcal{Q}} \mathrm{ES}_p^Q(X).$$

(ii) Dependence adjustment

(a) Risk factors in the portfolio are grouped into a range of broad regulatory risk classes (interest rate risk, equity risk, foreign exchange risk, commodity risk and credit spread risk). For the stress scenario (see (i)(b)), compute the ES of each risk class (according to (i)), and denote their sum by $\widetilde{\mathrm{ES}}_C(X)$. By comonotonic-additivity and subadditivity of ES (see Section 2 for details), this calculation is equivalent to using a model where all classes of risk factors are comonotonic ("non-diversified"), and it represents the worst-case value of ES among all possible dependence structures (e.g. Embrechts et al. (2014)).

(b) Use the formula

$$\mathrm{ES}(X) = \lambda \widetilde{\mathrm{ES}}(X) + (1 - \lambda) \widetilde{\mathrm{ES}}_C(X),$$

where λ is a constant (right now, λ is chosen as 0.5). The quantity $\mathrm{ES}(X)$ is called the IMCC of the portfolio.

Intuitively, the logic behind adjustment (i) is that risk assessment should be made based on stressed financial periods, and that behind adjustment (ii) is that the dependence structure between risk factors is difficult to specify and a worst-case value is combined with the original model to protect from overly optimistic diversification effects in the model specification. See Embrechts et al. (2014, 2015) for discussions on the aggregation of risk measures under dependence uncertainty[2].

In summary, in the framework of Basel III & IV (BCBS (2016)) for market risk, ES of the same portfolio is estimated under different scenarios and models: stress (stressed, non-stressed),

[2] In addition to (i) and (ii), the IMCC value will finally be adjusted by using the maximum of its present calculation and a moving average calculation of 60 days times a constant (currently 1.5).

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and dependence (diversified, non-diversified), and these values are aggregated with mainly two operations (iteratively): maximum and linear combination. In Theorem 4, we show that these two operations indeed are the two most crucial operations which lead to a coherent risk measure in the sense of Artzner et al. (1999) for scenario-based risk measures. Section 5.2 contains a detailed data analysis for the stress adjustment (i) outlined above.

We briefly mention two other prominent examples of risk evaluation using scenarios. First, the margin requirements calculation developed by the Chicago Mercantile Exchange (CME (2010)) relies on the maximum of the portfolio loss over several specified hypothetical scenarios; see p.63 of McNeil et al. (2015). Our data example in Section 5.1 is similar to this approach. The second example comes from the practice of credit rating, where a structured finance security (e.g. a defaultable bond) is rated according to its behavior (conditional distributions) under each economic stress scenario. This approach, in different specific forms, appear in both the Standard and Poor's and Moody's rating methodologies; see Standard and Poor's (2009) and Moody's (2010).

In this paper, we propose an axiomatic framework of scenario-based risk evaluation, which has the three merits mentioned above, and is consistent with many existing risk measurement procedures including the above examples. We shall keep the Basel formulas as our primary example in mind.

1.3 Our contribution and the structure of the paper

The contributions of our paper are summarized below. In Section 2, we introduce scenario-based risk measures. They includes classic law-based risk measures, non-law-based risk measures such as the systemic risk measures CoVaR and CoES (Adrian and Brunnermeier (2016)), and many practically used risk calculation principles such as the Basel formulas for market risk, the margin requirements by the Chicago Mercantile Exchange, and the common rating measures used in credit rating, as mentioned above. We introduce several novel scenario-based measures of risk in Section 3. In particular, we study the properties of Max-ES and Max-VaR, and related families of risk measures. Axiomatic characterizations of scenario-based risk measures are studied in Section 4. In particular, we characterize scenario-based comonotonic-additive as well as coherent risk measures, in the sense of Artzner et al. (1999) and Kusuoka (2001). Many surprising mathematical challenges emerge. Data analyses are given in Section 5, highlighting the broad range of possible interpretations of scenarios. In particular, scenario-based risk measures can be easily implemented for stress analysis and capital calculation.

Our framework builds upon the axiomatic theory of coherent risk measures as pioneered by Artzner et al. (1999). A comprehensive review on risk measures can be found in Delbaen (2012)

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and Föllmer and Schied (2016). The class of scenario-based risk measures is quite general. In addition to classic law-invariant risk measures, it also includes various forms of risk evaluation procedures such as the ones studied in Delbaen (2002), Cherny and Madan (2009), Adrian and Brunnermeier (2016), Kou and Peng (2016) and Righi (2018); see Sections 2 and 3 for details. For recent developments of risk measures, including various practical issues of statistical analysis, robustness, model uncertainty, and optimization, we refer to Fissler and Ziegel (2016), Cambou and Filipovic (2017), Krätschmer et al. (2017), Du and Escanciano (2017), Embrechts et al. (2018) and the references therein.

2 Theory of scenario-based risk measures

2.1 Definitions

Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{P}$ be the set of all probability measures on $(\Omega, \mathcal{F})$. For any probability measure Q on $(\Omega, \mathcal{F})$, write $F_{X,Q}$ for the cumulative distribution function (cdf) of X under Q, and denote by $X \sim_Q F$ if $F = F_{X,Q}$. For two random variables X and Y and a probability measure Q, we write $X \overset{\mathrm{d}}{=}_Q Y$ if $F_{X,Q} = F_{Y,Q}$. For any cdf F, its generalized inverse is defined as

$$F^{-1}(t) = \inf\{x \in \mathbb{R} : F(x) \geqslant t\},\ t \in (0, 1].$$

Let $\mathcal{X}$ be the set of bounded random variables in $(\Omega, \mathcal{F})$, and $\mathcal{Y}$ be a convex cone of random variables containing $\mathcal{X}$, representing the set of random variables of interest. We fix $\mathcal{X}$ throughout, whereas $\mathcal{Y}$ is specific to the functional considered[3]. A probability measure $\mathbb{P} \in \mathcal{P}$ shall be chosen as a reference probability measure in this paper, which may be interpreted as the real-world probability measure in some applications.

In this paper we use the term *scenario* for a probability measure $Q \in \mathcal{P}$. The reason behind this choice of terminology is from the perspective of scenario analysis, as in the following example. This example will be referred to a few times throughout the paper.

Example 1. Let Θ be a random economic factor taking values in a set K and $Q_\theta(\cdot) = \mathbb{P}(\cdot|\Theta = \theta)$, $\theta \in K$, are regular conditional probabilities with reference to Θ. The set $\{\Theta = \theta\} \in \mathcal{F}$ represents a possible economic event for each $\theta \in K$. To analyze the behavior of a risk X under each scenario $\Theta = \theta$, $\theta \in K$, the respective distributions of X under the probability measures Q_θ are of interest.

Suppose that there is a collection $\mathcal{Q}$ of scenarios of interest. As mentioned in the introduction, there may be different interpretations for the set $\mathcal{Q}$. In what follows, we take a collection of

[3]For instance, when considering the expectation $\mathbb{E}^Q$ for some $Q \in \mathcal{P}$, its domain $\mathcal{Y}$ is often chosen as the Q-integrable random variables, which depends on the choice of Q. However, it does not hurt to think of $\mathcal{Y} = \mathcal{X}$ for the main part of the paper.

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scenarios of interest and we shall not distinguish between the interpretations. If a risk (random loss) X and another risk Y have the same distribution under all relevant scenarios in $\mathcal{Q}$, then they should be assigned identical riskiness, whatever sense of riskiness we speak of. This leads to the following definition of $\mathcal{Q}$-based mappings.

Definition 1. For a non-empty collection of scenarios $\mathcal{Q} \subset \mathcal{P}$, a mapping $\rho : \mathcal{Y} \to (-\infty, \infty]$ is *$\mathcal{Q}$-based* if $\rho(X) = \rho(Y)$ for $X, Y \in \mathcal{Y}$ whenever $X \overset{\mathrm{d}}{=}_Q Y$ for all $Q \in \mathcal{Q}$.

To put the above concept into risk management, we focus on *$\mathcal{Q}$-based risk measures*. A risk measure is a mapping from $\mathcal{Y}$ to $(-\infty, \infty]$, with $\rho(X) < \infty$ for bounded X. We use the term *risk measure* in a broad sense, as it also includes deviation measures (such as variance) and other risk functionals[4] . In this paper, we adopt the sign convention as in McNeil et al. (2015): for a risk $X \in \mathcal{Y}$, losses are represented by positive values of X and profits are represented by negative values of X.

An immediate example of a $\mathcal{Q}$-based risk measure is one that depends on the joint law of a risk and an economic factor Θ as in Example 1. If $\rho(X)$ is determined by the joint distribution of (X, Θ), then ρ is $\mathcal{Q}$-based where $\mathcal{Q} = \{\mathbb{P}(\cdot|\Theta = \theta) : \theta \in K\}$. This includes the systemic risk measures CoVaR and CoES, which are evaluated based on conditional distributions of risks given events (see Adrian and Brunnermeier (2016)). For a fixed random variable S (the system) and $p \in (0,1)$, the systemic risk measure *CoVaR* is defined as:

$$\mathrm{CoVaR}_p^S(X) = \mathrm{VaR}_p^{\mathbb{P}}(S|X = \mathrm{VaR}_p^{\mathbb{P}}(X)), \quad X \in \mathcal{Y},$$

and the other systemic risk measure *CoES* is defined as:

$$\mathrm{CoES}_p^S(X) = \mathbb{E}^{\mathbb{P}}[S|S \geqslant \mathrm{CoVaR}_p^S(X)], \quad X \in \mathcal{Y}.$$

Since CoVaR and CoES are determined by the joint distribution of (X, S), they are $\mathcal{Q}$-based risk measures for $\mathcal{Q} = \{\mathbb{P}(\cdot|S = s) : s \in \mathbb{R}\}$.

Clearly, the $\mathcal{Q}$-based risk measures are generalizations of *law-based* (single-scenario-based) risk measures, which are determined by the law of random variables in a given probability space. Thus, $\mathcal{Q}$-based risk measures bridge law-based ones and generic ones, by noting the relationship (assuming $\mathbb{P} \in \mathcal{Q}$)

$$\underbrace{\{\mathbb{P}\}}_{\text{law-based}} \subset \underbrace{\mathcal{Q}}_{\mathcal{Q}\text{-based}} \subset \underbrace{\mathcal{P}}_{\text{generic}}.$$

Some immediate facts about $\mathcal{Q}$-based risk measures are summarized in the following.

[4] To keep things precise, our main examples are traditional risk measures such as VaR and ES, although our framework includes deviation measures. For the latter, see Rockafellar et al. (2006).

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(i) All risk measures on $\mathcal{Y}$ are $\mathcal{P}$-based. In fact, if $X \stackrel{\mathrm{d}}{=}_Q Y$ for all $Q \in \mathcal{P}$, then $X = Y$[5].

(ii) If $\mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \mathcal{P}$, then a $\mathcal{Q}_1$-based risk measure is also $\mathcal{Q}_2$-based.

(iii) For $\mathcal{Q}_1, \dots, \mathcal{Q}_n \subset \mathcal{P}$, let $\mathcal{Q} = \cup_{i=1}^n \mathcal{Q}_i$ and $\rho_i : \mathcal{Y} \to \mathbb{R}$ be $\mathcal{Q}_i$-based, $i = 1, \dots, n$. For any $f : \mathbb{R}^n \to \mathbb{R}$, the mapping $f \circ (\rho_1, \dots, \rho_n) : \mathcal{Y} \to \mathbb{R}$ is $\mathcal{Q}$-based.

Next we introduce a special type of collections of probability measures, which fits naturally into the context of Example 1.

Definition 2. A collection of probability measures $\mathcal{Q} \subset \mathcal{P}$ is *mutually singular* if there exist mutually disjoint sets $A_Q \in \mathcal{F}$, $Q \in \mathcal{Q}$, such that $Q(A_Q) = 1$ for $Q \in \mathcal{Q}$.

An example of this type would be to take $Q_i(B) = \mathbb{P}(B|A_i)$ for $B \in \mathcal{F}$ where $A_1, \dots, A_n$ is a partition of Ω with $\mathbb{P}(A_i) > 0$ for $i = 1, \dots, n$. That is, each Q_i amplifies the probability of the events A_i of interest, commonly seen e.g. in importance sampling. In Example 1, $\mathcal{Q} = \{Q_\theta : \theta \in K\}$ is mutually singular.

We also say that a tuple $(Q_1, \dots, Q_n) \in \mathcal{P}^n$ is mutually singular if $\{Q_1, \dots, Q_n\}$ is mutually singular, and any two of $Q_1, \dots, Q_n$ are non-identical.

Remark 1. In this paper, scenarios are treated in a generic sense. They may have different interpretations in different contexts. In a statistical context, they may represent different values of an estimated parameter in the model of the risk. In a simulation-based model, they may represent different parameters in the simulation dynamics, or a simply sets of different weights on the realized values of the risk in the simulation. In a regulatory framework, they may represent different economic situations that the regulator is concerned about. In a financial market, to assess a contingent payoff, one may need to incorporate its distribution under the pricing measure and under the physical measure, under multiple pricing measures, or with different heterogeneous opinions about the physical probability measure; these situations naturally require a risk measure determined by the distribution of the risk under different measures.

2.2 Preliminaries on risk measures

We adapt the terminology in Artzner et al. (1999), Kusuoka (2001) and Föllmer and Schied (2002). A risk measure ρ is *cash-invariant* if $\rho(X + c) = \rho(X) + c$ for $c \in \mathbb{R}$ and $X \in \mathcal{Y}$; ρ is *monotone* of $\rho(X) \leqslant \rho(Y)$ for $X, Y \in \mathcal{Y}$, $X \leqslant Y$; ρ is *positively homogeneous* if $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda \in (0, \infty)$ and $X \in \mathcal{Y}$, and ρ is *subadditive* if $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$ for $X, Y \in \mathcal{Y}$. A risk

[5] Let $\omega \in \Omega$ and define $Q : \mathcal{F} \to \mathbb{R}$, $A \mapsto \mathrm{I}_{\{\omega \in A\}}$. One can verify that Q defines a probability measure. The distributions of X and Y under Q are simply the point mass at $X(\omega)$ and $Y(\omega)$, respectively. Therefore, $X \stackrel{\mathrm{d}}{=}_Q Y$ implies that $X(\omega) = Y(\omega)$.

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measure is said to be *monetary* if it is monotone and cash-invariant. A risk measure is said to be *coherent* if it is monetary, positively homogeneous and subadditive.

Two random variables X and Y in $(\Omega, \mathcal{F})$ are *comonotonic* if

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geqslant 0 \quad \text{for all } \omega, \omega' \in \Omega.$$

A risk measure ρ is *comonotonic-additive* if $\rho(X+Y) = \rho(X) + \rho(Y)$ whenever X and Y are comonotonic.

Let us define some classic risk measures based on a single scenario $Q \in \mathcal{P}$. The most popular risk measures in banking and insurance regulation are the Value-at-Risk (VaR) and the Expected Shortfall (ES), calculated under a fixed probability measure $Q \in \mathcal{P}$. We shall refer to them as Q-VaR and Q-ES, respectively. For these risk measures, their domain $\mathcal{Y}$ can be chosen as any convex cone of random variables containing $\mathcal{X}$, possibly the entire set of random variables. For $p \in (0,1]$, $\mathrm{VaR}_p^Q : \mathcal{Y} \to (\infty, \infty]$ is defined as

$$\mathrm{VaR}_p^Q(X) = \inf\{x \in \mathbb{R} : Q(X \leqslant x) \geqslant p\} = F_{X,Q}^{-1}(p), \qquad X \in \mathcal{Y}, \tag{1}$$

and for $p \in (0,1)$, $\mathrm{ES}_p^Q : \mathcal{Y} \to (\infty, \infty]$ is defined as

$$\mathrm{ES}_p^Q(X) = \frac{1}{1-p} \int_p^1 \mathrm{VaR}_q^Q(X) \mathrm{d}q, \qquad X \in \mathcal{Y}. \tag{2}$$

In addition, we let $\mathrm{ES}_1^Q(X) = \mathrm{VaR}_1^Q(X)$[6].

For a specified scenario Q, Q-VaR and Q-ES belong to the class of distortion risk measures. Define the following sets of functions[7]

$$\mathcal{G} = \{g : g \text{ is an increasing function from } [0,1] \text{ to } [0,1] \text{ with } g(0) = 1 - g(1) = 0\},$$

and $\mathcal{G}_+ = \{g \in \mathcal{G} : g \text{ is concave}\}$. A Q*-distortion risk measure* is defined as

$$\rho_g^Q(X) = \int_{-\infty}^0 (g \circ Q(X \geqslant x) - 1) \mathrm{d}x + \int_0^\infty g \circ Q(X \geqslant x) dx, \quad X \in \mathcal{X}_g. \tag{3}$$

where $g \in \mathcal{G}$ is called the *distortion function* of ρ_g^Q, and $\mathcal{X}_g$ is the set of random variables such that (3) is well-defined[8]. A Q*-spectral risk measure* is a Q-distortion risk measure with a concave distortion function. A Q-distortion risk measure is always monetary, positively homogeneous and comonotonic-additive. A Q-spectral risk measure is, additionally, coherent. VaR_p^Q has a distortion function $g(x) = \mathrm{I}_{\{x > 1-p\}},\ x \in [0,1]$ and ES_p^Q has a distortion function $g(x) = \frac{1}{1-p} \min\{x, 1-p\},\ x \in [0,1]$.

[6] $\mathrm{VaR}_p(X)$ is always finite if $p \in (0,1)$. If X is not integrable, then $\mathrm{ES}_p(X)$ may be infinite.

[7] In this paper, terms "increasing", "decreasing" and "set inclusion" are in the non-strict sense.

[8] By "well-defined" we mean at least one of the two integrals in (3) is finite. $\mathcal{X}_g$ always contains $\mathcal{X}$.

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3 Scenario-based VaR and ES

Because of the prominent importance of VaR and ES in external regulatory capital calculation and internal risk management, in this section we investigate several versions of scenario-based risk measures which can be seen as the natural generalizations VaR and ES in a multi-scenario framework.

3.1 Max-type $\mathcal{Q}$-based risk measures

Inspired by the BIS ES formula, we introduce a class of $\mathcal{Q}$-based risk measures, which we refer to as max-type risk measures. We say that a $\mathcal{Q}$-based risk measure ρ is *max-type*, if

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \rho_Q(X), \quad X \in \mathcal{Y},$$

where for each $Q \in \mathcal{Q}$, ρ_Q is a $\{Q\}$-based risk measure. Max-type risk measures incorporate information evaluated under each scenario, and make a *conservative* capital calculation by taking the maximum value. Two major examples of max-type risk measures will be the classes of Max-ES and Max-VaR, which we introduce below. For all risk measures in this section, $\mathcal{Y}$ can be taken as any convex cone of random variables containing $\mathcal{X}$.

Definition 3 (Max-ES and Max-VaR)**.** For a collection of measures $\mathcal{Q}$ and $p \in (0,1)$, the *Max-ES (MES)* is defined as

$$\mathrm{MES}_p^{\mathcal{Q}}(X) = \sup_{Q \in \mathcal{Q}} \mathrm{ES}_p^Q(X), \quad X \in \mathcal{Y}, \tag{4}$$

and the *Max-VaR (MVaR)* is defined as

$$\mathrm{MVaR}_p^{\mathcal{Q}}(X) = \sup_{Q \in \mathcal{Q}} \mathrm{VaR}_p^Q(X), \quad X \in \mathcal{Y}. \tag{5}$$

For specific applications, a Max-ES or Max-VaR may be defined at multiple probability levels, as

$$\max_{i=1,\dots,n} \mathrm{ES}_{p_i}^{Q_i}(X) \quad \text{or} \quad \max_{i=1,\dots,n} \mathrm{VaR}_{p_i}^{Q_i}(X), \quad X \in \mathcal{Y}$$

for some $p_1, \dots, p_n \in (0,1)$ and $Q_1, \dots, Q_n \in \mathcal{P}$. In this paper, for the ease of presentation, and in view of the BIS formula, we focus on $\mathrm{MES}_p^{\mathcal{Q}}$ and $\mathrm{VaR}_p^{\mathcal{Q}}$ defined in (4) and (5).

Remark 2. If $\mathcal{Q}$ is chosen as a neighborhood (in the sense of some statistical distance) of a reference scenario $\mathbb{P}$, then (4) and (5) are known as the robust calculations of $\mathrm{ES}_p^{\mathbb{P}}(X)$ and $\mathrm{VaR}_p^{\mathbb{P}}(X)$.

The risk measures $\mathrm{MES}_p^{\mathcal{Q}}$ and $\mathrm{MVaR}_p^{\mathcal{Q}}$ are both max-type $\mathcal{Q}$-based risk measures. Similarly to the single-scenario-based ES and VaR in (1) and (2), MES and MVaR have different

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mathematical properties. Quite surprisingly, the risk measure $\mathrm{MVaR}_p^{\mathcal{Q}}$ satisfies comonotonic-additivity, whereas $\mathrm{MES}_p^{\mathcal{Q}}$ does not. This is sharp contrast to the case of single-scenario-based risk measures, in which both ES_p^{Q} and VaR_p^{Q} are comonotonic-additive.

Theorem 1. *For a collection of measures $\mathcal{Q}$ and $p \in (0,1)$, the following hold.*

(i) $\mathrm{MES}_p^{\mathcal{Q}}$ *is coherent, but generally not comonotonic-additive.*

(ii) $\mathrm{MVaR}_p^{\mathcal{Q}}$ *is comonotonic-additive, positively homogeneous and monetary, but generally not coherent.*

Remark 3. As a classic result (Delbaen (2002)), a coherent risk measure ρ on $\mathcal{X}$ with the Fatou property (see Appendix B.7 for details) has a dual representation

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q[X], \quad X \in \mathcal{X}, \tag{6}$$

for some set of probability measures $\mathcal{Q}$. Clearly, ρ is a max-type $\mathcal{Q}$-based risk measure. This includes, in particular, the methodology for margin requirement calculation developed by Chicago Mercantile Exchange; see p.63 of McNeil et al. (2015).

3.2 Various formulations of $\mathcal{Q}$-based Expected Shortfalls

Max-type risk measures are arguably one of the simplest $\mathcal{Q}$-based risk measures. Other than the max-type, using a finite or continuous mixture is also a convenient and simple way to construct $\mathcal{Q}$-based risk measures. As an example, we take $p \in (0,1)$ and a finite $\mathcal{Q} = \{Q_1, \dots, Q_n\}$, and define the *Average-ES* (AES) as the average of ES across different scenarios, that is,

$$\mathrm{AES}_p^{\mathcal{Q}}(X) = \frac{1}{n} \sum_{i=1}^{n} \mathrm{ES}_p^{Q_i}(X), \quad X \in \mathcal{Y}. \tag{7}$$

It is straightforward to see that $\mathrm{AES}_p^{\mathcal{Q}}$ is a coherent and comonotonic-additive risk measure. Certainly, one could choose different weights for each scenario (see Example 5). Here, we take an equally weighted version for simplicity.

Below we present a few other ways to formulate ES in the framework of $\mathcal{Q}$-based risk measures. Similarly, one may define the corresponding versions of VaR or any other law-based risk measure, but we take ES as an example in this section due to its relevance in Basel III & IV.

Recall that the single-scenario-based ES in (2) is an average of VaR of probability level beyond $p \in (0,1)$. Utilizing this connection, we define the *integral Max-ES* (iMES) as the integral of MVaR, that is,

$$\mathrm{iMES}_p^{\mathcal{Q}}(X) = \frac{1}{1-p} \int_p^1 \mathrm{MVaR}_q^{\mathcal{Q}}(X) \mathrm{d}q, \quad X \in \mathcal{Y}. \tag{8}$$

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One may equivalently write

$$\mathrm{iMES}_p^{\mathcal{Q}}(X) = \mathrm{ES}_p^{\mathbb{P}}\left(\sup_{Q\in\mathcal{Q}} F_{X,Q}^{-1}(U)\right), \quad X \in \mathcal{Y}, \tag{9}$$

where $U \sim_{\mathbb{P}} \mathrm{U}[0,1]$. Thus, $\mathrm{iMES}_p^{\mathcal{Q}}$ is constructed from an ES and a maximum operator on the individual quantile functions.

Another way to utilize the ES and a maximum operator is via independent replications of X under different scenarios. Let $p \in (0,1)$, $Q_1, \dots, Q_n$ be distinct scenarios and $\mathcal{Q} = \{Q_1, \dots, Q_n\}$. Define a *replicated Max-ES* (rMES) as the ES of a maximum of independent copies, that is,

$$\mathrm{rMES}_p^{\mathcal{Q}}(X) = \mathrm{ES}_p^{\mathbb{P}}\left(\max_{i=1,\dots,n} X_i\right), \quad X \in \mathcal{Y}, \tag{10}$$

where $X_i \sim_{\mathbb{P}} F_{X,Q_i}$, $i = 1, \dots, n$, and $X_1, \dots, X_n$ are independent under $\mathbb{P}$. The risk measure $\mathrm{rMES}_p^{\mathcal{Q}}$ is defined for a finite collection $\mathcal{Q}$ so that the maximum in (10) is well-posed[9]. Note that $\mathrm{iMES}_p^{\mathcal{Q}}$, and $\mathrm{rMES}_p^{\mathcal{Q}}$ are not max-type risk measures.

Each of the $\mathcal{Q}$-based risk measures $\mathrm{MES}_p^{\mathcal{Q}}$, $\mathrm{AES}_p^{\mathcal{Q}}$, $\mathrm{iMES}_p^{\mathcal{Q}}$ and $\mathrm{rMES}_p^{\mathcal{Q}}$ may be seen as a natural generalization of the single-scenario-based risk measure ES_p^{Q}. Although bearing similar ideas, these risk measures have different properties and values. If $\mathcal{Q} = \{Q\}$, then the above five risk measures are all equal. They are generally non-equivalent and satisfy an order summarized in the following theorem.

Theorem 2. *Let $\mathcal{Q}$ be a collection of n scenarios and $p \in (0,1)$.*

(i) $\mathrm{AES}_p^{\mathcal{Q}}$ *is comonotonic-additive and coherent.*

(ii) $\mathrm{iMES}_p^{\mathcal{Q}}$ *is comonotonic-additive, but generally not coherent.*

(iii) $\mathrm{rMES}_p^{\mathcal{Q}}$ *is comonotonic-additive and coherent.*

(iv) $\mathrm{AES}_p^{\mathcal{Q}}(X) \leqslant \mathrm{MES}_p^{\mathcal{Q}}(X) \leqslant \mathrm{iMES}_p^{\mathcal{Q}}(X) \leqslant \mathrm{rMES}_p^{\mathcal{Q}}(X)$ *for all $X \in \mathcal{Y}$.*

(v) If $n = 1$, then $\mathrm{AES}_p^{\mathcal{Q}}(X) = \mathrm{MES}_p^{\mathcal{Q}}(X) = \mathrm{iMES}_p^{\mathcal{Q}}(X) = \mathrm{rMES}_p^{\mathcal{Q}}(X)$ *for all $X \in \mathcal{Y}$.*

The above illustration suggests that the framework of $\mathcal{Q}$-based risk measures is generally flexible, and it allows for a great variety of risk measures to be formulated, even simply from the ES and a fixed p.

We note that there is a simple relationship between iMES (resp. MVaR) and ES (resp. VaR) when the collection of scenarios $\mathcal{Q}$ is the economic scenarios in Example 1.

Proposition 1. *Let $\mathcal{Q} = \{Q_\theta : \theta \in K\}$ as in Example 1. For $p \in (0,1)$,* $\mathrm{MVaR}_p^{\mathcal{Q}}(X) \geqslant \mathrm{VaR}_p^{\mathbb{P}}(X)$ *and* $\mathrm{iMES}_p^{\mathcal{Q}}(X) \geqslant \mathrm{ES}_p^{\mathbb{P}}(X)$ *for all $X \in \mathcal{Y}$.*

[9] The risk measure $\mathrm{rMES}_p^{\mathcal{Q}}$ finds some similarity to MINVAR in Cherny and Madan (2009); see Example 4 for more details.

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Proposition 1 suggests that when using the economic scenarios in Example 1, iMES (MVaR) is more conservative than ES (VaR) over the unconditional real world probability measure $\mathbb{P}$. Note that $\mathrm{MES}_p^{\mathcal{Q}}(X) \geqslant \mathrm{ES}_p^{\mathbb{P}}(X)$ does not hold in general (see Example 7 in the Appendix for a counter-example), although this inequality almost always holds empirically, as we shall see in the data analysis in Section 5.

Before closing this section, we remark that for a finite collection $\mathcal{Q}$, each of $\mathrm{MES}_p^{\mathcal{Q}}$, $\mathrm{iMES}_p^{\mathcal{Q}}$, $\mathrm{rMES}_p^{\mathcal{Q}}$ and $\mathrm{AES}_p^{\mathcal{Q}}$ is easy to (numerically) calculate if one has the distributions of X under each $Q \in \mathcal{Q}$, or one has simulated samples of X under each $Q \in \mathcal{Q}$.

Remark 4. The worst-case (e.g. MES) and the weighted risk measures (e.g. AES) were recently studied in Righi (2018), and they are special cases of $\mathcal{Q}$-based risk measures. For a finite collection of scenarios $\{Q_1, \dots, Q_n\}$, these risk measures take the form $f(\rho^{Q_1}, \dots, \rho^{Q_n})$ for some function f, where ρ^{Q_i} is $\{Q_i\}$-based, $i = 1, \dots, n$. As we can see from the example of iMES and rMES, the framework of scenario-based risk measures is much broader than risk measures of the above form.

Remark 5. Another example of scenario-based risk measure of the type $f(\rho^{Q_1}, \dots, \rho^{Q_n})$ is given by Kou and Peng (2016). Let $\rho_{h_i}^{Q_i}$, $i = 1, \dots, n$ be Q_i-distortion risk measures given by (3), and $\mathcal{W}$ be a subset of the standard simplex $\{(w_1, \dots, w_n) \in [0,1]^n : \sum_{i=1}^n w_i = 1\}$. The risk measure ρ, given by

$$\rho(X) = \sup_{(w_1,\dots,w_n)\in\mathcal{W}} \left\{ \sum_{i=1}^n w_i \rho_{h_i}^{Q_i}(X) \right\}, \quad X \in \mathcal{X},$$

is a combination of maximum and weighted averages of Q-distortion risk measures for $Q \in \{Q_1, \dots, Q_n\}$.

4 Axiomatic characterizations

In this section, we establish axiomatic characterizations of $\mathcal{Q}$-based comonotonic-additive risk measures as well as $\mathcal{Q}$-based coherent risk measures. For technical reasons, we focus on a finite collection $\mathcal{Q}$ and the set of bounded random variables, that is, $\mathcal{Y} = \mathcal{X}$.

Throughout this section, n is a positive integer, and let $\underline{Q} = (Q_1, \dots, Q_n)$ be a vector of measures, where $Q_1, \dots, Q_n \in \mathcal{P}$ are (pre-assigned) probability measures on $(\Omega, \mathcal{F})$, and $\mathcal{Q} = \{Q_1, \dots, Q_n\}$ is the set of these measures[10]. Write $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. We say that $P \in \mathcal{P}$ dominates $\mathcal{Q}$, if $Q \ll P$ for all $Q \in \mathcal{Q}$, that is, if for all $Q \in \mathcal{Q}$, Q is absolutely continuous with respect to P.

[10]The dimensionality of $\underline{Q}$ and the cardinality of $\mathcal{Q}$ may only differ if some of $Q_1, \dots, Q_n$ are identical. If $Q_1, \dots, Q_n$ are distinct, then the mutual singularity of $\underline{Q}$ is equivalent to that of $\mathcal{Q}$.

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4.1 Comonotonic-additive risk measures and Choquet integrals

As mentioned in Section 2.2, the most popular class of risk measures in practice are the ones that are additive for comonotonic risks. We choose this class as the starting point to establish an axiomatic theory of $\mathcal{Q}$-based risk measures. It is well-known that law-determined monetary risk measures are closely related to the notion of Choquet integrals; for instance Yaari's dual utility functionals (Yaari (1987)) and Kusuoka representations (Kusuoka (2001)) are based on Choquet integrals. First we recall the notions of Choquet integrals.

Definition 4. A set function $c : \mathcal{F} \to \mathbb{R}$, is *increasing* if $c(A) \leqslant c(B)$ for $A \subset B$, $A, B \in \mathcal{F}$, it is *standard* if c is increasing and satisfies $c(\varnothing) = 0$ and $c(\Omega) = 1$, and it is *submodular* if

$$c(A \cup B) + c(A \cap B) \leqslant c(A) + c(B), \quad A, B \in \mathcal{F}.$$

Definition 5. For a standard set function c and $X \in \mathcal{X}$, the *Choquet integral* $\int X \mathrm{d}c$ is defined as

$$\int X \mathrm{d}c = \int_{-\infty}^{0} (c(X > x) - 1)\mathrm{d}x + \int_{0}^{\infty} c(X > x)\mathrm{d}x. \tag{11}$$

The integral $\int X \mathrm{d}c$ in (11) might also be well-defined on sets larger than the set $\mathcal{X}$ of bounded random variables. Generally, depending on different choices of c, one may choose different domains for the Choquet integral. A Q-distortion risk measure in (3) is exactly a Choquet integral by choosing $c = g \circ Q$.

Now we are ready to present the characterization for comonotonic-additive $\mathcal{Q}$-based risk measures, which is based on a celebrated result dating back to Schmeidler (1986).

Theorem 3. *A risk measure ρ on $\mathcal{X}$ is monetary (resp. coherent), comonotonic-additive and $\mathcal{Q}$-based if and only if*

$$\rho(X) = \int X \mathrm{d}\psi \circ \underline{Q}, \quad X \in \mathcal{X} \tag{12}$$

for some function $\psi : [0,1]^n \to [0,1]$ such that $\psi \circ \underline{Q}$ is standard (resp. $\psi \circ \underline{Q}$ is standard and submodular).

We shall refer to a risk measure in (12) as a *$\underline{Q}$-distortion risk measure*, which is, by Theorem 3, precisely a monetary, comonotonic-additive and $\mathcal{Q}$-based risk measure. Coherent $\underline{Q}$-distortion risk measures are referred to as *$\underline{Q}$-spectral risk measures*. For the $\underline{Q}$-distortion risk measure ρ in (12), ψ is called its *$\underline{Q}$-distortion function*[11], and it is unique on the range of $\underline{Q}$, by noting that $\rho(\mathrm{I}_A) = \psi \circ \underline{Q}(A)$ for all $A \in \mathcal{F}$. The classes of $\underline{Q}$-distortion and $\underline{Q}$-spectral risk measures will be the building blocks of the theory of $\mathcal{Q}$-based risk measures.

[11] The reliance on $\underline{Q}$ is essential. For instance, taking $P, Q \in \mathcal{P}$, if $\rho(X) = \frac{1}{3}\mathbb{E}^P[X] + \frac{2}{3}\mathbb{E}^Q[X]$, $X \in \mathcal{X}$, then ρ has a (P, Q)-distortion function and a (Q, P)-distortion function, which are different.

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Clearly, if $n = 1$, then the concepts of a $\underline{Q}$-distortion risk measure, a $\underline{Q}$-spectral risk measure and a $\underline{Q}$-distortion function coincide with those defined for a single scenario in Section 2.2. In that case, the representation in (12) reduces to

$$\rho(X) = \int X \mathrm{d}\psi \circ Q_1, \;\; X \in \mathcal{X} \tag{13}$$

where $\psi \in \mathcal{G}$ (and $\psi \in \mathcal{G}_+$ if ρ is coherent).

The condition that $\psi \circ \underline{Q}$ is standard or $\psi \circ \underline{Q}$ is submodular may not be easy to verify in general, as it involves the joint properties of ψ and $\underline{Q}$. Below we establish some sufficient conditions based on solely ψ. Furthermore, these conditions are necessary and sufficient if $\underline{Q}$ is mutually singular.

Proposition 2. *Let $\psi : [0,1]^n \to [0,1]$ be a function satisfying $\psi(\mathbf{0}) = 1 - \psi(\mathbf{1}) = 0$, and ρ be defined as in (12).*

(i) If ψ is componentwise increasing, then ρ is a $\underline{Q}$-distortion risk measure.

(ii) If ψ is componentwise increasing, componentwise concave, and submodular, then ρ is a $\underline{Q}$-spectral risk measure.

(iii) The converse statements of (i) and (ii) are also true if $\underline{Q}$ is mutually singular.

Proposition 2 suggests that it is straightforward to design various comonotonic-additive $\mathcal{Q}$-based risk measures by choosing some componentwise increasing functions ψ. We remark that, if $\underline{Q}$ is not mutually singular, in order for $\psi \circ \underline{Q}$ to be standard (resp. submodular), it is generally not necessary for ψ to be componentwise increasing (resp. componentwise concave and submodular). See Example 8 in the Appendix for a counter-example.

4.2 Integral representation and examples

Recall that in Section 2.2, for a single scenario Q, a Q-distortion risk measure ρ_g^Q is defined as

$$\rho_g^Q(X) = \int_{-\infty}^0 (g \circ Q(X \geqslant x) - 1)\mathrm{d}x + \int_0^\infty g \circ Q(X \geqslant x) dx, \quad X \in \mathcal{X}. \tag{14}$$

If g is left-continuous, ρ_g^Q has a Lebesgue integral formulation via an argument of integration by parts (see e.g. Theorem 6 of Dhaene et al. (2012)), that is,

$$\rho_g^Q(X) = \int_0^1 \mathrm{VaR}_p^Q(X)\mathrm{d}\bar{g}(p), \quad X \in \mathcal{X}, \tag{15}$$

where $\bar{g}(t) = 1 - g(1-t)$ for $t \in [0,1]$. Note that in this case, $\bar{g}$ is right-continuous with $g(0) = 1 - g(1) = 0$; thus g is a distribution function on $[0,1]$. This property is key to the

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integral representation in (15). Next we establish an analogous integral formulation for the case of multiple scenarios under a similar assumption. For a function $\psi : [0,1]^n \to [0,1]$, denote by $\bar{\psi}(\mathbf{u}) = 1 - \psi(\mathbf{1} - \mathbf{u})$, $\mathbf{u} \in [0,1]^n$.

Proposition 3. *Let $\bar{\psi}$ be a distribution function on $[0,1]^n$, and $\rho_\psi : \mathcal{X} \to \mathbb{R}$ be given by*

$$\rho_\psi(X) = \int_{[0,1]^n} \max\{\mathrm{VaR}_{u_1}^{Q_1}(X), \dots, \mathrm{VaR}_{u_n}^{Q_n}(X)\} \mathrm{d}\bar{\psi}(u_1, \dots, u_n). \tag{16}$$

Then $\rho_\psi(X)$ is a $\underline{Q}$-distortion risk measure with $\underline{Q}$-distortion function ψ. Moreover, if $\bar{\psi}$ is componentwise convex, then ρ_ψ is a $\underline{Q}$-spectral risk measure.

Proposition 3 provides a convenient way to construct various $\underline{Q}$-distortion risk measures. For instance, one may choose $\bar{\psi}$ as an n-copula (see Joe (2014)). A direct consequence of Proposition 3 is that any $\underline{Q}$-distortion risk measure with $\underline{Q}$-distortion function ψ has a representation (16) if $\bar{\psi}$ is a distribution function. Below we present some examples, including $\mathrm{MVaR}_p^{\mathcal{Q}}$ and $\mathrm{iMES}_p^{\mathcal{Q}}$ in Section 3 and MINVAR in Cherny and Madan (2009) in the case of a single scenario.

Example 2 ($\mathrm{MVaR}_p^{\mathcal{Q}}$)**.** For $p \in (0,1)$, by choosing $\bar{\psi}$ as the distribution on the point mass $(p, \dots, p) \in (0,1)^n$, we obtain a special case of (16), defined as

$$\rho(X) = \max\{\mathrm{VaR}_p^{Q_1}(X), \dots, \mathrm{VaR}_p^{Q_n}(X)\}, \quad X \in \mathcal{X}. \tag{17}$$

The risk measure ρ is a $\underline{Q}$-distortion risk measure with $\underline{Q}$-distortion function $\psi(x_1, \dots, x_n) = 1 - \mathrm{I}_{\{x_1, \dots, x_n \leqslant 1-p\}}$, $(x_1, \dots, x_n) \in [0,1]^n$, and it is precisely $\mathrm{MVaR}_p^{\mathcal{Q}}$ in Section 3.1. Hence, this representation also verifies that $\mathrm{MVaR}_p^{\mathcal{Q}}$ is comonotonic-additive for a finite $\mathcal{Q}$. Note that, as we have seen previously, $\mathrm{MES}_p^{\mathcal{Q}}$ is not comonotonic-additive, and as such, it does not admit a representation as in Proposition 3.

Example 3 ($\mathrm{iMES}_p^{\mathcal{Q}}$)**.** For $p \in [0,1)$, by choosing $\bar{\psi}$ as a uniform distribution over the diagonal line segment $\{(u_1, \dots, u_n) \in [p,1]^n : u_1 = u_2 = \dots = u_n\}$, we obtain a special case of (16), defined as

$$\rho(X) = \frac{1}{1-p} \int_p^1 \max\{\mathrm{VaR}_u^{Q_1}(X), \dots, \mathrm{VaR}_u^{Q_n}(X)\} \mathrm{d}u, \quad X \in \mathcal{X}. \tag{18}$$

The risk measure ρ is a $\underline{Q}$-distortion risk measure with $\underline{Q}$-distortion function $\psi(x_1, \dots, x_n) = \min\{\frac{1}{1-p} \max\{x_1, \dots, x_n\}, 1\}$, $(x_1, \dots, x_n) \in [0,1]^n$. If $p \in (0,1)$, ρ is precisely $\mathrm{iMES}_p^{\mathcal{Q}}$ in Section 3.2. This also verifies that $\mathrm{iMES}_p^{\mathcal{Q}}$ is comonotonic-additive. However, ψ is not componentwise concave, which implies that $\mathrm{iMES}_p^{\mathcal{Q}}$ is not a coherent risk measure for mutually singular $\underline{Q}$ by Proposition 2.

Example 4 (Scenario-based MINVAR)**.** By choosing $\bar{\psi}(\mathbf{u}) = \prod_{i=1}^n u_i$ for $\mathbf{u} \in [0,1]^n$ in (16), we obtain

$$\rho(X) = \mathbb{E}^{\mathbb{P}}[\max\{X_1, \dots, X_n\}], \quad X \in \mathcal{X}, \tag{19}$$

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where for $i = 1, \ldots, n$, $F_{X_i,\mathbb{P}} = F_{X,Q_i}$, and $X_1, \ldots, X_n$ are independent under $\mathbb{P}$. Then ρ is a $\underline{Q}$-spectral risk measure with $\underline{Q}$-distortion function $\psi(x_1, \ldots, x_n) = 1 - \prod_{i=1}^n (1 - x_i)$, $(x_1, \ldots, x_n) \in [0,1]^n$. The risk measure ρ is coherent. The single-scenario-based risk measure MINVAR (Cherny and Madan (2009)), defined as

$$\mathrm{MINVAR}(X) = \mathbb{E}^{\mathbb{P}}[\max\{X_1, \ldots, X_n\}], \quad X \in \mathcal{X},$$

where $X_1, \ldots, X_n$ are iid copies of X under $\mathbb{P}$, is a special case of ρ by choosing $Q_1 = \cdots = Q_n = \mathbb{P}$.

For a single scenario Q, the distortion function g of a Q-spectral risk measure ρ_g^Q in (14) is concave, implying that $\bar{g}$ is automatically a distribution function, and hence ρ_g^Q always admits a representation in (15). This property does not carry through to the case of $\underline{Q}$-distortion risk measures in general. More precisely, the $\underline{Q}$-distortion function of a $\underline{Q}$-spectral risk measure is not necessarily always a distribution function, because all distribution functions on $[0,1]^n$ are supermodular but not vice versa. As a consequence, not all $\underline{Q}$-spectral risk measure have representation (16). This is sharp contrast to the case of a single scenario.

Example 5 (Average-ES)**.** For some $\mathbf{a} = (a_1, \ldots, a_n) \in [0,1]^n$ with $\mathbf{a} \cdot \mathbf{1} = 1$, let

$$\psi(x_1, \ldots, x_n) = \frac{1}{1-p} \sum_{i=1}^n a_i \min\{x_i, 1-p\}, \ (x_1, \ldots, x_n) \in [0,1]^n.$$

One can easily verify that ψ is componentwise increasing, componentwise concave and submodular. By Proposition 2, $\rho(X) = \int X \mathrm{d}\psi \circ \underline{Q}$, $X \in \mathcal{X}$ defines a $\underline{Q}$-spectral risk measure, which can be simplified as

$$\rho(X) = \sum_{i=1}^n a_i \mathrm{ES}_p^{Q_i}(X), \quad X \in \mathcal{X}. \tag{20}$$

If $a_1 = \cdots = a_n$, then the risk measure ρ is precisely $\mathrm{AES}_p^{\mathcal{Q}}$ in Section 3.2. Note that $\bar{\psi}$ is not a distribution function, and hence Proposition 3 does not apply.

4.3 Coherent risk measures

As a classic result in the theory of risk measures, the Kusuoka representation (Kusuoka (2001)) states that any single-scenario-based coherent risk measure admits a representation as the supremum over a collection of spectral risk measures.

It is of great interest to see whether a similar result holds true for $\mathcal{Q}$-based coherent risk measures. First, it is straightforward to notice that a supremum over a collection of $\underline{Q}$-spectral risk measure is always a $\mathcal{Q}$-based coherent risk measure. For the converse direction, we shall show that, if $\mathcal{Q}$ is mutually singular, then a $\mathcal{Q}$-based coherent risk measure admits a representation

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as the supremum of a collection of mixtures of Q-ES for $Q \in \mathcal{Q}$. More precisely, let $\mathcal{W}_0 = \{(w_1, \dots, w_n) \in [0,1]^n : \sum_{i=1}^n w_i = 1\}$ be the standard simplex. A mixture of Q-ES for $Q \in \mathcal{Q}$ is a risk measure $\rho_{\mathbf{w}}$ defined by

$$\rho_{\mathbf{w}}(X) = \sum_{i=1}^n w_i \int_0^1 \mathrm{ES}_p^{Q_i}(X) \mathrm{d}h_i^{\mathbf{w}}(p), \ X \in \mathcal{X}, \tag{21}$$

for some $\mathbf{w} = (w_1, \dots, w_n) \in \mathcal{W}_0$ and $h_1^{\mathbf{w}}, \dots, h_n^{\mathbf{w}}$ are distribution functions on $[0,1]$. Clearly, $\rho_{\mathbf{w}}$ is a $\underline{Q}$-spectral risk measure, as each Q-ES is a $\underline{Q}$-spectral risk measure. In the next theorem, we establish that, if $\underline{Q}$ is mutually singular, then any $\mathcal{Q}$-based coherent risk measure ρ can be written as

$$\rho(X) = \sup_{\mathbf{w} \in \mathcal{W}} \rho_{\mathbf{w}}(X), \quad X \in \mathcal{X}, \tag{22}$$

for some set $\mathcal{W} \subset \mathcal{W}_0$ and $\rho^{\mathbf{w}}$ is a mixture of Q-ES given by (21).

Theorem 4. *(i) If ρ is the supremum of some $\underline{Q}$-spectral risk measures, then it is a $\mathcal{Q}$-based coherent risk measure.*

(ii) If $\underline{Q}$ is mutually singular, then a $\mathcal{Q}$-based coherent risk measure admits a representation as the supremum of mixtures of Q-ES for $Q \in \mathcal{Q}$ as in (22).

Remark 6. Theorem 4 is one of the most technical results of this paper. The mutual singularity of $\underline{Q}$ is used repetitively in the proof (see Appendix B.7) and it does not seem to be dispensable.

An immediate example of risk measure of the type (22) is $\mathrm{MES}_p^{\mathcal{Q}}$ in Section 3.1, where each $h_i^{\mathbf{w}}$ for $i = 1, \dots, n$ and $\mathbf{w} \in \mathcal{W}$ is a point mass at $p \in (0,1)$, and $\mathcal{W} = \mathcal{W}_0$.

Theorem 4 resembles the ideas in the Basel formula; see Section 1. Indeed, it is remarkable that only using maximum and linear combinations of Q-ES, as done in BCBS (2016), one arrives at all possible $\mathcal{Q}$-based coherent risk measures, if $\mathcal{Q}$ is mutually singular.

5 Data analysis for $\mathcal{Q}$-based risk measures

In this section, we discuss two examples of data analysis for $\mathcal{Q}$-based risk measures. The two examples are conceptually different with the aim to illustrate the broad spectrum of possible interpretations for the collection $\mathcal{Q}$ of scenarios; cf. Remark 1. Many interesting questions can arise from the following subsections, as the examples we consider cover fundamentally different possible applications of $\mathcal{Q}$-based risk measures. Various versions of the $\mathcal{Q}$-based Expected Shortfalls as in Section 3 are chosen to illustrate the main ideas; clearly the analysis may be applied to other scenario-based risk measures.

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5.1 $\mathcal{Q}$-based Expected Shortfalls for economic scenarios

Taking up Example 1, we consider $Q_i = \mathbb{P}(\cdot|\Theta = \theta_i)$, $i = 1, \ldots, n$, where Θ is an economic factor taking values in a finite set $\{\theta_1, \ldots, \theta_n\}$ of cardinality n, where $\mathbb{P}$ can be interpreted as the real-world probability measure. While the $\mathcal{Q}$-based Expected Shortfalls of X are clearly defined mathematical quantities, it is not completely obvious how to estimate them. The approach we describe can be justified under suitable assumptions on the data generating processes. However, we leave a detailed study of the proposed estimator for future work.

In order to estimate $\mathcal{Q}$-based Expected Shortfalls of X, we assume that we have n sequences of data $D_1 = \{X_1^{Q_1}, \ldots, X_{N_1}^{Q_1}\}, \ldots, D_n = \{X_1^{Q_n}, \ldots, X_{N_n}^{Q_n}\}$ such that the empirical distribution of D_i is a reasonable estimate of F_{X,Q_i}. Then, we estimate the risk measures ES_p, MES_p, iMES_p, and rMES_p given at (4), (7), (8), and (10), respectively, by their empirical counterparts.

Given a series of returns $(X_t)_{t\in\mathbb{N}}$, for each trading day, we would like to compute $\mathcal{Q}$-based expected shortfalls of X_t. We will use data on a rolling window of length $w \in \{250, 500\}$ for the estimation. We considered $n = 4$ scenarios which can be interpreted as

$$\{\theta_1, \ldots, \theta_4\} = \{\text{high volatility}, \text{low volatility}\} \times \{\text{good economy}, \text{bad economy}\}.$$

The value of Θ is based on the values of VIX (high volatility/low volatility) and S&P 500 (good economy/bad economy). To be precise, for day t_0 we use the time window $t_0 - w, \ldots, t_0 - 1$ of length $w \in \{250, 500\}$. Then, we use the VIX to split the time period into two categories depending on whether the VIX is higher or lower than its empirical median in the time window. We removed a log-linear trend from the S&P 500 since 1950, and then we subdivide the $w/2$ days with high volatility in the current time window into two categories of (almost) equal size according to whether the S&P 500 residuals are above or below their median during those $w/2$ days. The same is then done for the $w/2$ days with low volatility. This results in a split of the time window into four scenarios of (almost) equal size $w/4$.

The sets D_1, $\ldots$, D_4 now consist of the values of X_t for $t = t_0 - w, \ldots, t_0 - 1$ depending on which scenario the respective day has been assigned. We considered return data from the NASDAQ Composite Index, the DAX Index, Apple Inc. stock, Walmart stock, BMW stock and Siemens stock[12]. The considered time periods are 1991–2018. We do not consider data from before 1990 because there is no VIX data available. We chose the confidence level $p = 0.9$ for simplicity. For each series of return data, we also computed the empirical ES_p using a rolling windows the same size w. The results of the analysis are summarized in Figures 1 and 2.

The risk measures MES_p and iMES_p generally yield similar values. One can observe that during times of financial stress, the risk measures MES_p and ES_p deviate substantially, whereas

[12]The data were obtained from https://finance.yahoo.com.

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they are close during an economically stable period. For the indices (NASDAQ and DAX) MES_p and rMES_p are closer than for the stock returns (Apple, Walmart, BMW and Siemens). This may be explained by the fact that the indices are more closely related to the quantities defining the economic scenarios (VIX and S&P 500). During economically stable periods, the ratio between rMES_p and MES_p is generally larger than during financial stress. The ratio between MES_p and ES_p qualitatively distinguishes the early 2000s recession from the 2008 financial crisis being larger during the latter event, except for the Apple stock. Apple seems to have been more influenced by the dot-com crash in 2000 than the other stocks and indices. The results are qualitatively similar for both considered lengths w for the time windows.

5.2 The Basel stress-adjustment for Expected Shortfall

In this section, we calculate the stress-adjustment for Expected Shortfall in the Basel market risk evaluation as outlined in Section 1.2. Suppose that there are n securities in a portfolio, and let P_t^i, $i = 1, \dots, n$, $t \in \mathbb{N}$ denote the time-t price of security i. Let $X_t^i = -(P_t^i / P_{t-1}^i - 1)$ be its daily negative return. Construct a portfolio with price process $V_t = \sum_{i=1}^n \alpha_i P_t^i$ where α_i is the unit of shares invested in security i, which we assume to be fixed throughout the investment period. At time $t-1$, we need to to calculate the empirical ES of the next day loss of this portfolio. Note that the daily loss is

$$V_{t-1} - V_t = \sum_{i=1}^n \alpha_i (P_{t-1} - P_t) = \sum_{i=1}^n X_t^i \alpha_i P_{t-1}^i.$$

At time $t-1$, the numbers α_i and P_{t-1}^i are known, and the random risk factors are $(X_t^1, \dots, X_t^n)$. To calculate the ES over the data of the past 12-month of data, we need to evaluate the quantity, given the number $\alpha_i P_{t-1}^i$,

$$\mathrm{ES}_p^P (V_{t-1} - V_t) = \mathrm{ES}_p^P \left(\sum_{i=1}^n X_t^i \alpha_i P_{t-1}^i \right),$$

where $p = 0.975$ as specified by BCBS (2016). For this purpose, the scenario P is modelled such that the distribution of $(X_t^1, \dots, X_t^n)$ is according to its empirical version over the past 250 observations, i.e. over the period $[t-250, t-1]$.

ES should be calibrated to the most severe 12-month period of stress over a long observation horizon, which has to span back to 2007, specified by BCBS (2016). To mimic this adjustment for the period before the introduction of Basel III, it seems fair for everyday evaluation to look back 10 years, and find the maximum ES over a 12-month period. For this purpose, we evaluate, while treating $\alpha_i P_{t-1}^i$ as a constant,

$$\mathrm{MES}_p^{\mathcal{Q}} (V_{t-1} - V_t) = \mathrm{MES}_p^{\mathcal{Q}} \left(\sum_{i=1}^n X_t^i \alpha_i P_{t-1}^i \right) = \max_{j=1,\dots,N} \mathrm{ES}_p^{Q_j} \left(\sum_{i=1}^n X_t^i \alpha_i P_{t-1}^i \right),$$

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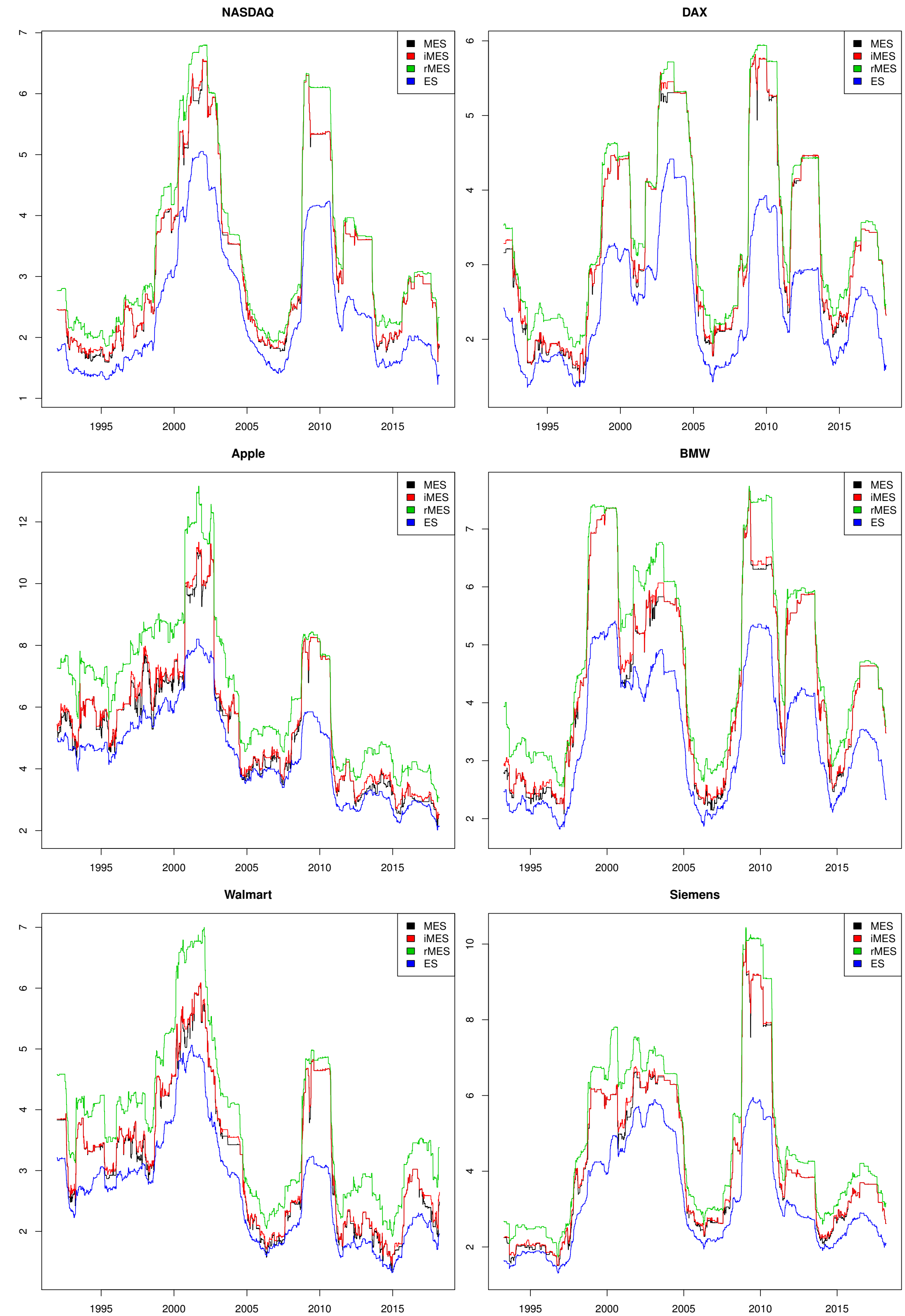

Figure 1: $\mathcal{Q}$-based risk measures estimated for data based on economic scenarios with $w = 500$.

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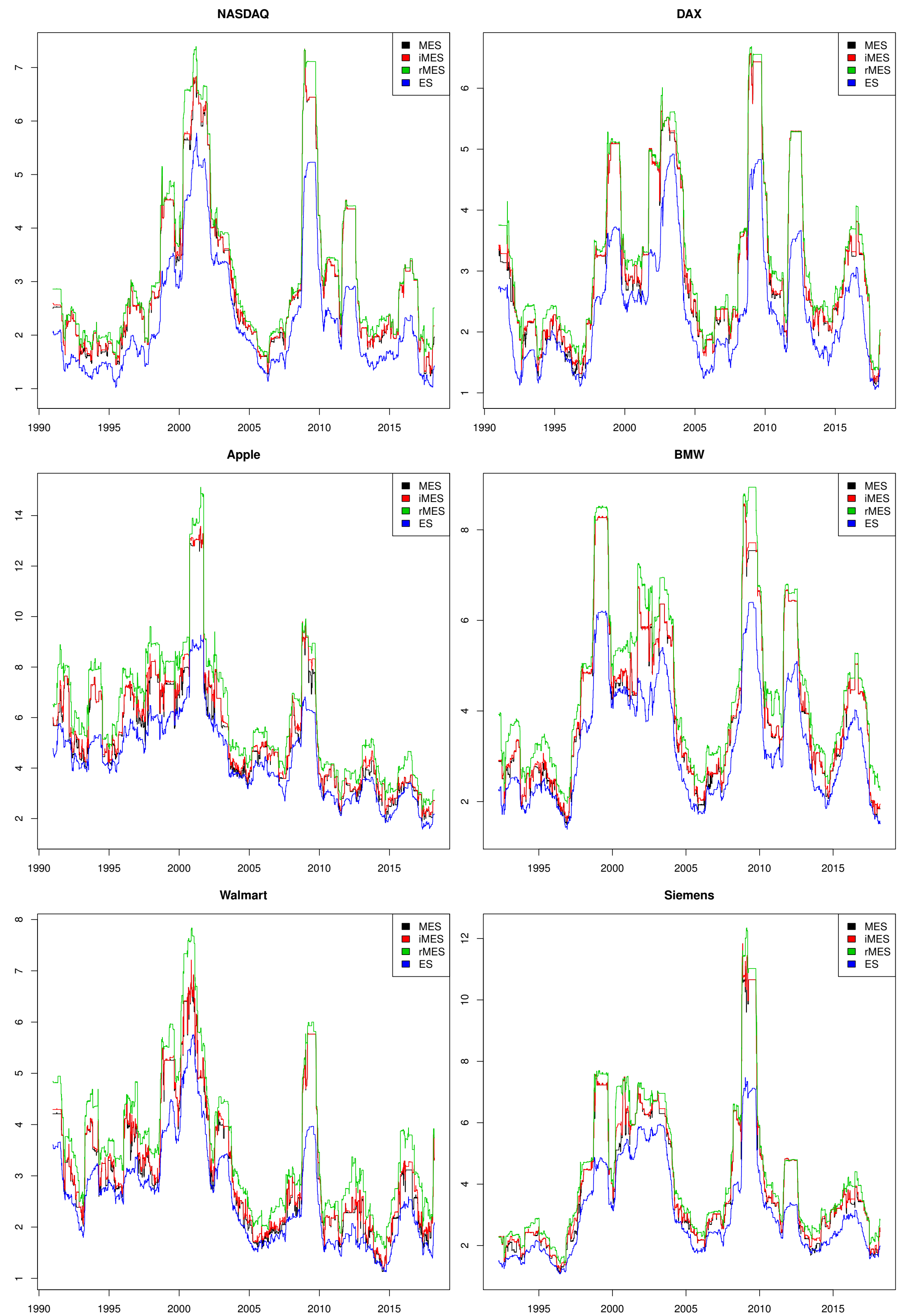

Figure 2: $\mathcal{Q}$-based risk measures estimated for data based on economic scenarios with $w = 250$.

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where $N = 2251$, $\mathcal{Q} = \{Q_j\}_{j=1,\ldots,N}$, and under Q_j, $(X_t^1, \ldots, X_t^n)$ is distributed according to its empirical distribution over the time period $[t-j-249, t-j]$. We choose $\alpha_1, \ldots, \alpha_n$ such that each $\alpha_i P_i^t$ starts from \$1. We construct a US stocks portfolio (Apple and Walmart) and a German stocks portfolio (BMW and Siemens).

In Figure 3, we report for both portfolios, the regular ES ($\mathrm{ES}_p^P(V_{t-1} - V_t)$), the stress-adjusted ES ($\mathrm{MES}_p^{\mathcal{Q}}(V_{t-1} - V_t)$), the percentage of ES ($\frac{\mathrm{ES}_p^P(V_{t-1}-V_t)}{V_{t-1}}$), and the percentage of stress-adjusted ES ($\frac{\mathrm{MES}_p^{\mathcal{Q}}(V_{t-1}-V_t)}{V_{t-1}}$). We can see from the results that the percentage MES is relatively stable (always between 6% and 9%), and the ES is changing drastically (between 2% and 9%), very much depending on the performance of the individual stocks over the past year. This suggests that MES have the advantage of being more robust since it uses data for a much longer period of time. Moreover, the US portfolio has a quite high percentage MES till 1998 and this is due to the effect of the Black Monday (Oct 19, 1987) wears out after 10 years. If regulatory capital for the market risk is calculated via ES, then both portfolio exhibit serious under-capitalization right before the 2007 financial crisis, and their ES values increased drastically when the financial crisis took place. On the other hand, if MES is used for regulatory capital calculation, then the requirement of capital for both portfolios only increased moderately during the financial crisis.

6 Concluding remarks

In this paper, we proposed a framework for scenario-based risk evaluation, where different scenarios (probability measures or models) are incorporated into the procedure of risk calculation. Our framework allows for flexible interpretation of the scenarios and it is in particular motivated by the Basel calculation procedures for the Expected Shortfall, the Chicago board, and the credit ratings. Several theoretical contributions are made. We introduced the new classes of risk measures including Max-ES, Max-VaR and their variants, and studied their theoretical properties. Axiomatic characterization of scenario-based comonotonic additive and coherent classes of risk measures are obtained, and they are well connected to the Basel formulas for market risk. Finally, we presented data analyses to illustrate how scenario-based risk measures can be estimated, computed, and interpreted.

Given the pivotal importance of model uncertainty and scenario analysis in modern risk management, scenario-based risk measures can be useful in many disciplines of risk assessment, not limited to financial risk management.

We remark that for various interpretations of the scenarios, the estimation procedures of a scenario-based risk measure may exhibit different properties, as illustrated in Section 5. This

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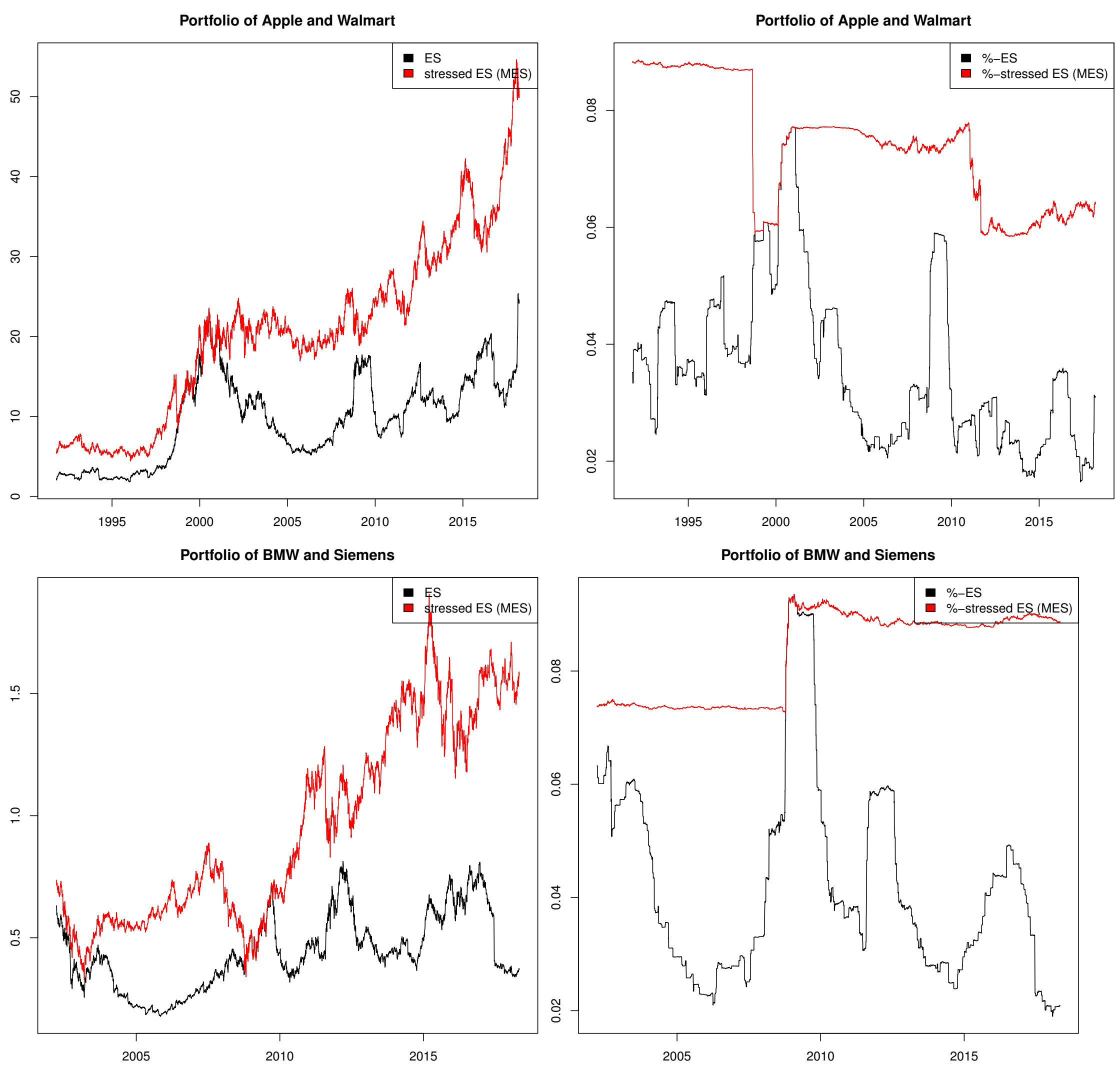

Figure 3: The MES and ES of the US and German portfolios. Left panel: MES and ES of the portfolio. Right panel: the percentage of MES and ES in the value of the portfolio.

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calls for future research in statistical theory for scenario-based risk functionals.

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Appendix

A Examples and counter-examples

Example 6 ($\mathrm{MES}_p^{\mathcal{Q}}$ is not comonotonic-additive)**.** Take $p \in (0,1)$, $Q_1, Q_2 \in \mathcal{P}$ and $A_1, A_2 \in \mathcal{F}$ such that $A_1 \subset A_2$, $Q_1(A_1) > Q_2(A_1)$ and $Q_1(A_2) < Q_2(A_2) < 1-p$. The existence of such Q_1, Q_2, A_1, A_2 can be justified by taking $(\Omega, \mathcal{F}, Q_1)$ and $(\Omega, \mathcal{F}, Q_2)$ as atomless probability spaces. Let $\mathcal{Q} = \{Q_1, Q_2\}$, $X = \mathrm{I}_{A_1}$ and $Y = \mathrm{I}_{A_2}$. It is clear that X and Y are comonotonic. Note that

$$\begin{aligned}\mathrm{ES}_p^{Q_1}(X+Y) &= \mathrm{ES}_p^{Q_1}(X) + \mathrm{ES}_p^{Q_1}(Y)\\ &= \frac{1}{1-p}(Q_1(A_1) + Q_1(A_2))\\ &< \frac{1}{1-p}(Q_1(A_1) + Q_2(A_2)) = \max_{Q \in \mathcal{Q}} \mathrm{ES}_p^{Q}(X) + \max_{Q \in \mathcal{Q}} \mathrm{ES}_p^{Q}(Y),\end{aligned}$$

and similarly,

$$\mathrm{ES}_p^{Q_2}(X+Y) < \max_{Q \in \mathcal{Q}} \mathrm{ES}_p^{Q}(X) + \max_{Q \in \mathcal{Q}} \mathrm{ES}_p^{Q}(Y) = \mathrm{MES}_p^{\mathcal{Q}}(X) + \mathrm{MES}_p^{\mathcal{Q}}(Y).$$

Then we have

$$\mathrm{MES}_p^{\mathcal{Q}}(X+Y) = \max\{\mathrm{ES}_p^{Q_1}(X+Y), \mathrm{ES}_p^{Q_2}(X+Y)\} < \mathrm{MES}_p^{\mathcal{Q}}(X) + \mathrm{MES}_p^{\mathcal{Q}}(Y).$$

Thus, $\mathrm{MES}_p^{\mathcal{Q}}$ is not comonotonic-additive.

Example 7 ($\mathrm{MES}_p^{\mathcal{Q}}(X) < \mathrm{ES}_p^{\mathbb{P}}(X)$ for $\mathcal{Q}$ in Example 1)**.** Let $\Omega = \{\omega_1, \dots, \omega_8\}$ and $\mathbb{P}$ be a uniform probability measure on Ω. Write $\Omega_1 = \{\omega_1, \dots, \omega_4\}$ and $\Theta = \mathrm{I}_{\Omega_1}$. Let $Q_1(\cdot) = \mathbb{P}(\cdot|\Theta = 1)$, $Q_2(\cdot) = \mathbb{P}(\cdot|\Theta = 0)$ and $X = \mathrm{I}_{\Omega_1} + 2 \times \mathrm{I}_{\{\omega_8\}}$. It is easy to see that $\mathrm{ES}_p^{\mathbb{P}}(X) = 1.25$ and $\mathrm{ES}_p^{Q_1}(X) = \mathrm{ES}_p^{Q_2}(X) = 1$. Thus, $\mathrm{MES}_p^{\mathcal{Q}}(X) < \mathrm{ES}_p^{\mathbb{P}}(X)$.

Example 8 (A $\underline{Q}$-increasing function that is not component-wise increasing)**.** Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$ and λ be the Lebesgue measure. Define measures Q_1 and Q_2 via

$$\frac{\mathrm{d}Q_1}{\mathrm{d}\lambda}(t) = \frac{2}{3}(1 + \mathrm{I}_{\{t \geqslant 1/2\}}) \quad \text{and} \quad \frac{\mathrm{d}Q_2}{\mathrm{d}\lambda}(t) = \frac{2}{3}(1 + \mathrm{I}_{\{t < 1/2\}}),$$

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for $t \in [0,1]$. One can easily verify $1/2 \leqslant \mathrm{d}Q_2/\mathrm{d}Q_1 \leqslant 2$, and

$$\{(Q_1(A), Q_2(A)) : A \in \mathcal{F}\} = \left\{(s,t) \in [0,1]^2 : \frac{1}{2} \leqslant \frac{s}{t} \leqslant 2,\ \frac{1}{2} \leqslant \frac{1-s}{1-t} \leqslant 2\right\}.$$

Let

$$\psi(s,t) = 2s - t \quad s,t \in [0,1].$$

Clearly, ψ is not component-wise increasing on the convex set $R_{\underline{Q}} = \{(Q_1(A), Q_2(A)) : A \in \mathcal{F}\}$, and hence $\psi|_{R_{\underline{Q}}}$ cannot be extended to a componentwise increasing function on $[0,1]^2$. However, $\psi \circ \underline{Q}$ is increasing. Indeed, take $A, B \in \mathcal{F}$ such that $A \subset B$. Write $Q_1(A) = p_1$, $Q_2(A) = p_2$, $Q_1(B) = q_1$ and $Q_2(B) = q_2$. Note that

$$\frac{q_2 - p_2}{q_1 - p_1} = \frac{Q_2(B \setminus A)}{Q_1(B \setminus A)} \leqslant 2,$$

implying

$$\psi \circ (Q_1, Q_2)(A) = 2p_1 - p_2 \leqslant 2q_1 - q_2 = \psi \circ (Q_1, Q_2)(B).$$

Thus $\psi \circ \underline{Q}$ is increasing. One also easily see that $\psi \circ \underline{Q}$ is standard.

B Proofs of the main results

B.1 Proof of Theorem 1

Proof. (i) Note that ES_p^Q is coherent for $Q \in \mathcal{Q}$. Since MES_p^{Q} can be written as a supremum of coherent risk measures, and taking a supremum preserves all properties of coherent risk measures, $\mathrm{MES}_p^{\mathcal{Q}}$ is also coherent. An example showing that $\mathrm{MES}_p^{\mathcal{Q}}$ is not comonotonic-additive is given in Example 6.

(ii) Note that VaR_p^Q is monetary for $Q \in \mathcal{Q}$, and hence $\mathrm{MVaR}_p^{\mathcal{Q}}$, as a supremum of monetary risk measures, is monetary. It remains to show that $\mathrm{MVaR}_p^{\mathcal{Q}}$ is a comonotonic-additive risk measure. Using Denneberg's lemma (Denneberg (1994)), for comonotonic random variables X and Y, there exist increasing continuous functions f and g such that $X = f(X+Y)$ and $Y = g(X+Y)$. Therefore, for any $Q \in \mathcal{Q}$, we have

$$\mathrm{MVaR}_p^{\mathcal{Q}}(X) = \sup_{Q \in \mathcal{Q}} \mathrm{VaR}_p^Q(f(X+Y)) = \sup_{Q \in \mathcal{Q}} f(\mathrm{VaR}_p^Q(X+Y)) = f\left(\sup_{Q \in \mathcal{Q}} \mathrm{VaR}_p^Q(X+Y))\right),$$

and similarly,

$$\mathrm{MVaR}_p^{\mathcal{Q}}(Y) = g\left(\sup_{Q \in \mathcal{Q}} \mathrm{VaR}_p^Q(X+Y))\right).$$

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Noting that $f(z)+g(z)=z$ for z in the range of $X+Y$, we have

$$\begin{aligned}\mathrm{MVaR}_p^{\mathcal{Q}}(X+Y) &= \sup_{Q\in\mathcal{Q}} \mathrm{VaR}_p^{Q}(X+Y)\\ &= f\left(\sup_{Q\in\mathcal{Q}} \mathrm{VaR}_p^{Q}(X+Y)\right) + g\left(\sup_{Q\in\mathcal{Q}} \mathrm{VaR}_p^{Q}(X+Y)\right)\\ &= \mathrm{MVaR}_p^{\mathcal{Q}}(X) + \mathrm{MVaR}_p^{\mathcal{Q}}(Y).\end{aligned}$$

The statement that $\mathrm{MVaR}_p^{\mathcal{Q}}$ is not necessarily coherent comes from the well-known fact that VaR_p^{Q} is not coherent for any $Q\in\mathcal{P}$ such that $(\Omega,\mathcal{F},Q)$ is atomless. □

B.2 Proof of Theorem 2

Proof. (i) This is straightforward as an average of coherent and comonotonic-additive risk measures is still coherent and comonotonic-additive.

(ii) By (8), $\mathrm{iMES}_p^{\mathcal{Q}}$ is a mixture of comonotonic-additive risk measures, and hence it is comonotonic-additive. The fact that $\mathrm{iMES}_p^{\mathcal{Q}}$ is not a coherent risk measure in general is shown in Example 3.

(iii) For each $i=1,\dots,n$, let the distribution (X_i,Y_i) under $\mathbb{P}$ be that of (X,Y) under Q_i, and $(X_1,Y_1),\dots,(X_n,Y_n)$ are independent. We have

$$\begin{aligned}\mathrm{rMES}_p^{\mathcal{Q}}(X+Y) &= \mathrm{ES}_p^{\mathbb{P}}\left(\max_{i=1,\dots,n}(X_i+Y_i)\right)\\ &\leqslant \mathrm{ES}_p^{\mathbb{P}}\left(\max_{i=1,\dots,n}X_i+\max_{i=1,\dots,n}Y_i\right) \leqslant \mathrm{rMES}_p^{\mathcal{Q}}(X)+\mathrm{rMES}_p^{\mathcal{Q}}(Y). \qquad (23)\end{aligned}$$

Therefore, $\mathrm{rMES}_p^{\mathcal{Q}}$ is subadditive. It is straightforward to verify that $\mathrm{rMES}_p^{\mathcal{Q}}$ is monetary and positively homogeneous, and hence a coherent risk measure. Moreover, if X and Y are comonotonic, then the two inequalities in (23) are equalities. As a consequence, $\mathrm{rMES}_p^{\mathcal{Q}}$ is also comonotonic-additive.

(iv) Take an arbitrary $X\in\mathcal{Y}$. $\mathrm{AES}_p^{\mathcal{Q}}(X)\leqslant\mathrm{MES}_p^{\mathcal{Q}}(X)$ is trivial. For U which is uniform$[0,1]$ under $\mathbb{P}$, we have

$$\mathrm{ES}_p^{Q}(X)=\mathrm{ES}_p^{\mathbb{P}}(F_{X,Q}^{-1}(U)). \qquad (24)$$

By (24), for each $Q\in\mathcal{Q}$, $\mathrm{ES}_p^{Q}(X)\leqslant\mathrm{iMES}_p^{\mathcal{Q}}(X)$. Consequently, $\mathrm{MES}_p^{\mathcal{Q}}(X)\leqslant\mathrm{iMES}_p^{\mathcal{Q}}(X)$. To show $\mathrm{iMES}_p^{\mathcal{Q}}(X)\leqslant\mathrm{rMES}_p^{\mathcal{Q}}(X)$, note that, for $x\in\mathbb{R}$,

$$\mathbb{P}\left(\max_{i=1,\dots,n}F_{X,Q_i}^{-1}(U)\leqslant x\right)=\min_{i=1,\dots,n}F_{X,Q_i}(x)\geqslant\prod_{i=1}^{n}F_{X,Q_i}(x)=\mathbb{P}\left(\max_{i=1,\dots,n}X_i\leqslant x\right).$$

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Therefore, $\max_{i=1,\dots,n} F^{-1}_{X,Q_i}(U)$ is first-order stochastically dominated by $\max_{i=1,\dots,n} X_i$ under $\mathbb{P}$ (see e.g. Müller and Stoyan (2002)). As a consequence,

$$\mathrm{ES}_p^{\mathbb{P}}\left(\max_{i=1,\dots,n} F^{-1}_{X,Q_i}(U)\right) \leqslant \mathrm{ES}_p^{\mathbb{P}}\left(\max_{i=1,\dots,n} X_i\right).$$

In summary, $\mathrm{AES}_p^{\mathcal{Q}}(X) \leqslant \mathrm{MES}_p^{\mathcal{Q}}(X) \leqslant \mathrm{iMES}_p^{\mathcal{Q}}(X) \leqslant \mathrm{rMES}_p^{\mathcal{Q}}(X)$ for all $X \in \mathcal{X}$.

(v) If $\mathcal{Q} = \{Q\}$, we can check directly by (24) that $\mathrm{ES}_p^{Q}(X) = \mathrm{AES}_p^{\mathcal{Q}}(X) = \mathrm{MES}_p^{\mathcal{Q}}(X) = \mathrm{iMES}_p^{\mathcal{Q}}(X) = \mathrm{rMES}_p^{\mathcal{Q}}(X)$. □

B.3 Proof of Proposition 1

Proof. We only need to show $\mathrm{MVaR}_p^{\mathcal{Q}}(X) \geqslant \mathrm{VaR}_p^{\mathbb{P}}(X)$, which implies $\mathrm{iMES}_p^{\mathcal{Q}}(X) \geqslant \mathrm{ES}_p^{\mathbb{P}}(X)$. Take $x < \mathrm{VaR}_p^{\mathbb{P}}(X)$, which implies $\mathbb{P}(X \leqslant x) < p$. As $\mathbb{P}$ is a convex combination of Q_θ, $\theta \in K$, for some $\theta \in K$, we have $Q_\theta(X \leqslant x) < p$, implying $x \leqslant \mathrm{VaR}_p^{Q_\theta}(X) \leqslant \mathrm{MVaR}_p^{\mathcal{Q}}(X)$. Therefore, $\mathrm{MVaR}_p^{\mathcal{Q}}(X) \geqslant \sup\{x \in \mathbb{R} : x < \mathrm{VaR}_p^{\mathbb{P}}(X)\} = \mathrm{VaR}_p^{\mathbb{P}}(X)$. □

B.4 Proof of Theorem 3

Proof. Summarizing Theorems 4.88 and 4.94 of Föllmer and Schied (2011), a risk measure ρ on $\mathcal{X}$ is monetary and comonotonic-additive if and only if

$$\rho(X) = \int X \mathrm{d}c,\ \ X \in \mathcal{X} \tag{25}$$

for a standard set function c. In addition, ρ in (25) is coherent if and only if c is submodular. This result resembles the Choquet-integral representation of Schmeidler (1986) in the framework of risk measures. First we discuss the case in which ρ is not necessarily coherent.

(i) $\Leftarrow$: Note that for a $\underline{Q}$-standard function ψ, $\psi \circ \underline{Q} : \mathcal{F} \to \mathbb{R}$ is a standard set function, and hence $X \mapsto \int X \mathrm{d}\psi \circ \underline{Q}$ is a Choquet integral. As a consequence of the above representation result, ρ is comonotonic-additive and monetary. From the definition of $\int X \mathrm{d}\psi \circ \underline{Q}$,

$$\rho(X) = \int_{-\infty}^{0} (\psi \circ \underline{Q}(X > x) - 1)\mathrm{d}x + \int_0^\infty \psi \circ \underline{Q}(X > x)\mathrm{d}x$$

and hence ρ is $\mathcal{Q}$-based.

(ii) $\Rightarrow$: By the above representation result, ρ can be written as a Choquet integral. There exists a standard set function c such that $\rho(X) = \int X \mathrm{d}c$ for $X \in \mathcal{X}$. By taking $X = \mathrm{I}_A$, $A \in \mathcal{F}$, we have $c(A) = \rho(\mathrm{I}_A)$. Since ρ is $\mathcal{Q}$-based, $\rho(\mathrm{I}_A)$ is determined by the distribution of I_A under $Q_1, \dots, Q_n$, namely, there exists a function $\psi : [0,1]^n \to \mathbb{R}$ such that $c(A) = \rho(\mathrm{I}_A) = \psi \circ \underline{Q}(A)$ for all $A \in \mathcal{F}$. As $c = \psi \circ \underline{Q}$ is standard, we have that ψ is $\underline{Q}$-standard.

To show that coherence of ρ is equivalent to $\underline{Q}$-submodular of ψ, one uses again the representation result, and noting that, by definition, $\psi \circ \underline{Q}$ is submodular if and only if ψ is $\underline{Q}$-submodular. □

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B.5 Proof of Proposition 2

In order to show Proposition 2, we collect some auxiliary results below, which might are of independent interest. The proof of Proposition 2 follows directly from Theorem 3 and Propositions 4 and 5 below.

Lemma 1. *A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is componentwise concave and submodular if and only if for all $\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z} \in \mathbb{R}^n$ with $\mathbf{w} \leqslant \mathbf{x}, \mathbf{y} \leqslant \mathbf{z}$ and $\mathbf{w} + \mathbf{z} = \mathbf{x} + \mathbf{y}$, we have*

$$f(\mathbf{x}) + f(\mathbf{y}) \geqslant f(\mathbf{w}) + f(\mathbf{z}). \tag{26}$$

In addition, if f is two times continuously differentiable, then (26) holds if and only if the entries of its Hessian are all non-positive.

Proof. It is not difficult to see that the stated property (26) implies both the componentwise concavity and the submodularity of f. We prove the converse by induction over n. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is concave if and only if for all $x, y, w, z \in \mathbb{R}$ with $w \leqslant x, y \leqslant z$ and $x + y = w + z$ we have

$$f(x) + f(y) \geqslant f(w) + f(z). \tag{27}$$

Let $\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z} \in \mathbb{R}^n$ with $\mathbf{w} \leqslant \mathbf{x}, \mathbf{y} \leqslant \mathbf{z}$ and $\mathbf{w} + \mathbf{z} = \mathbf{x} + \mathbf{y}$. Applying (26) to the first $n-1$ components and (27) to the last component, we obtain

$$\begin{aligned} f(y_1, \ldots, y_{n-1}, x_n) &\geqslant \frac{1}{2}\big(f(w_1, \ldots, w_{n-1}, x_n) + f(z_1, \ldots, z_{n-1}, x_n) - f(x_1, \ldots, x_{n-1}, x_n) \\ &\qquad + f(y_1, \ldots, y_{n-1}, w_n) + f(y_1, \ldots, y_{n-1}, z_n) - f(y_1, \ldots, y_{n-1}, y_n)\big) \\ &\geqslant \frac{1}{2}\big(f(\mathbf{w}) + 2f(y_1, \ldots, y_{n-1}, x_n) + f(\mathbf{z}) - f(\mathbf{x}) - f(\mathbf{y})\big), \end{aligned}$$

where the second step follows from the submodularity of f. Therefore (26) holds. The conclusion on the Hessian matrix is an elementary exercise. $\square$

Proposition 4. *If $\psi : [0,1]^n \to [0,1]$ is componentwise increasing, then $\psi \circ \underline{Q}$ is increasing. If ψ is componentwise concave and submodular, then $\psi \circ \underline{Q}$ is submodular.*

Proof. The first statement is trivial, and we only show the second statement. For $A, B \in \mathcal{F}$, we have $Q(A \cup B) + Q(A \cap B) = Q(A) + Q(B)$ for $Q \in \mathcal{Q}$. Therefore, from (26), we have

$$\psi \circ \underline{Q}(A \cup B) + \psi \circ \underline{Q}(A \cap B) \leqslant \psi \circ \underline{Q}(A) + \psi \circ \underline{Q}(B)$$

which gives the submodularity of $\psi \circ \underline{Q}$. $\square$

Proposition 5. *Suppose that $\underline{Q}$ is mutually singular and $\psi : [0,1]^n \to [0,1]$.*

(i) The mapping $\underline{Q} : \mathcal{F} \to [0,1]^n$ is surjective.

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(ii) $\psi \circ \underline{Q}$ *is increasing if and only if* ψ *is componentwise increasing.*

(iii) $\psi \circ \underline{Q}$ *is submodular if and only if* ψ *is componentwise concave and submodular.*

Proof. Let $A_1, \dots, A_n \in \mathcal{F}$ be disjoint sets such that $Q_i(A_i) = 1$ for each $i = 1, \dots, n$. The conclusion (i) is straightforward. To show (ii) and (iii), we only need to show that the $\mathcal{Q}$-specific conditions in (ii) and (iii) imply the non-$\mathcal{Q}$-specific conditions, respectively. For (ii), suppose that $x, y \in [0,1]^n$ with $x_1 \leqslant y_1$ and $x_2 = y_2, \dots, x_n = y_n$. Let $B \in \mathcal{F}$ with $x = (Q_1(B), \dots, Q_n(B))$. As $(A_1, \mathcal{F}, Q_1)$ is an atomless probability space, there exists a set C with $(B \cap A_1) \subset C \subset A_1$ and $Q_1(C) = y_1$ (Delbaen, 2002, Theorem 1). We have $y = (Q_1(C \cup B), \dots, Q_n(C \cup B))$, which yields the claim. Next we show (iii) for the case $n = 1$, and the general case follows easily due to the fact that $\underline{Q}$ is mutually singular. Let $x, y, w, z \in \mathbb{R}$ with $w \leqslant x, y \leqslant z$ and $w + z = x + y$. Take $B, C \in \mathcal{F}|_{A_1}$ with $Q_1(B) = x$ and $Q_1(C) = y$. If $Q_1(B \cap C) > w$, take $B' \subset (B \backslash C)$ with $Q_1(B') = Q_1(B \cap C) - w$ and $C' \subset (C \backslash B)$ with $Q_1(C') = Q_1(B \cap C) - w$. Then, $\bar{C} = (C \backslash C') \cup B'$ fulfils $Q_1(\bar{C}) = y$ and $Q_1(B \cap \bar{C}) = w$. If $Q_1(B \cap C) < w$, take $B' \subset (B \cup C)^c$ with $Q_1(B') = w - Q_1(B \cap C)$ and $C' \subset C \cap B$ with $Q_1(C') = w - Q_1(C \cap B)$. Then, $\bar{C} = (C \backslash C') \cup B'$ fulfills $Q_1(\bar{C}) = y$ and $Q_1(B \cap \bar{C}) = w$. The equation $w + z = x + y = Q_1(B) + Q_1(\bar{C}) = Q_1(B \cap \bar{C}) + Q_1(B \cup \bar{C})$, hence $z = Q_1(B \cup \bar{C})$. Now, the submodularity of $\psi \circ \underline{Q}$ implies (26). $\square$

B.6 Proof of Proposition 3

Proof. Let $Y = \max\{F^{-1}_{X,Q_1}(U_1), \dots, F^{-1}_{X,Q_2}(U_n)\}$ where $(U_1, \dots, U_n) \sim_{\mathbb{P}} \bar{\psi}$. By definition,

$$\rho_\psi(X) = \int_{[0,1]^n} \max\{F^{-1}_{X,Q_1}(u_1), \dots, F^{-1}_{X,Q_2}(u_n)\} \mathrm{d}\bar{\psi}(u_1, \dots, u_n) = \mathbb{E}^{\mathbb{P}}[Y].$$

For almost every $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}(Y \leqslant x) &= \mathbb{P}(F^{-1}_{X,Q_1}(U_1) \leqslant x, \dots, F^{-1}_{X,Q_2}(U_n) \leqslant x) \\
&= \mathbb{P}(U_1 \leqslant F_{X,Q_1}(x), \dots, U_n \leqslant F_{X,Q_2}(x)) \\
&= \bar{\psi}(Q_1(X \leqslant x), \dots, Q_n(X \leqslant x)) = 1 - \psi \circ \underline{Q}(X > x).
\end{aligned}$$

It follows that

$$\begin{aligned}
\rho_\psi(X) = \mathbb{E}^{\mathbb{P}}[Y] &= \int_{-\infty}^{0} (\mathbb{P}(Y > x) - 1)\mathrm{d}x + \int_0^\infty \mathbb{P}(Y > x)\mathrm{d}x \\
&= \int_{-\infty}^{0} \big(\psi \circ \underline{Q}(X > x) - 1\big)\,\mathrm{d}x + \int_0^\infty \psi \circ \underline{Q}(X > x)\mathrm{d}x = \int X \mathrm{d}\psi \circ \underline{Q}.
\end{aligned}$$

Note that any distribution function $\bar{\psi}$ is componentwise increasing and supermodular. Hence, ψ is componentwise increasing and submodular, and further by Theorem 3 and Proposition 2 we obtain the desired results. $\square$

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B.7 Proof of Theorem 4

Before proving Theorem 4, we need to establish some auxiliary results, which might be of independent interest.

First, we discuss a popular property related to coherent risk measures, the Fatou property (see Delbaen (2002, 2012)), which we shall define with respect to a scenario dominating $\mathcal{Q}$. Such a dominating scenario may be conveniently chosen as $\overline{Q} = \frac{1}{n}\sum_{i=1}^{n} Q_i$. Formally, a risk measure ρ is said to satisfy the *$\mathcal{Q}$-Fatou property* if for a uniformly bounded sequence $X_1, X_2, \dots \in \mathcal{X}$, $X_k \overset{\overline{Q}}{\to} X \in \mathcal{X}$ implies $\rho(X) \leqslant \liminf_{k\to\infty} \rho(X_k)$. We also introduce a norm $||\cdot||_{\mathcal{Q}}$ on the $\overline{Q}$-equivalent classes of $\mathcal{X}$, defined as $||\cdot||_{\mathcal{Q}} = \sup\{x > 0 : \overline{Q}(|X| > x) > 0\}$, which is the usual L^∞ norm for essentially bounded random variables in $(\Omega, \mathcal{F}, \overline{Q})$. Note that in the definitions of the $\mathcal{Q}$-Fatou property and the norm $||\cdot||_{\mathcal{Q}}$, the dominating measure $\overline{Q}$ can be chosen equivalently as any probability measure dominating $\mathcal{Q}$. It is straightforward to check that all $\mathcal{Q}$-based monetary risk measures are continuous with respect to $||\cdot||_{\mathcal{Q}}$. In what follows, a quasi-convex risk measure ρ is one that satisfies $\rho(\lambda X + (1-\lambda)Y) \leqslant \max\{\rho(X), \rho(Y)\}$ for all $\lambda \in [0,1]$ and $X, Y \in \mathcal{X}$.

Lemma 2. *If $\underline{Q}$ is mutually singular, then a $\mathcal{Q}$-based quasi-convex risk measure that is continuous with respect to $||\cdot||_{\mathcal{Q}}$ satisfies the $\mathcal{Q}$-Fatou property.*

Proof. Write $\overline{Q} = \frac{1}{n}\sum_{i=1}^{n} Q_i$, and note that $X_k \overset{\overline{Q}}{\to} X \in \mathcal{X}$ implies $X_k \overset{Q_i}{\to} X$ for each $i = 1, \dots, n$. We shall show the lemma in a similar way to Theorem 30 of Delbaen (2012), which states that a $\{\overline{Q}\}$-based, $||\cdot||_{\{\overline{Q}\}}$-continuous and quasi-convex functional satisfies the $\{\overline{Q}\}$-Fatou property (first shown by Jouini et al. (2006) with a minor extra condition). The main difference here is that our $\mathcal{Q}$-based risk measure is not necessarily $\{\overline{Q}\}$-based, and hence the above result does not directly apply. Nevertheless, we shall utilize Lemma 11 of Delbaen (2012), which gives that for each $i = 1, \dots, n$ and $k \in \mathbb{N}$, there exist a natural number N_k and random variables $Z^i_{k,1}, Z^i_{k,2}, \dots, Z^i_{k,N_k}$ having the same distribution as X_k under Q_i, such that

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{j=1}^{N_k} Z^i_{k,j} = X \quad \text{in } ||\cdot||_{\{Q_i\}}.$$

The numbers N_k can be chosen independently of i, as explained in Remark 40 of Delbaen (2012). Define for $k \in \mathbb{N}$ and $j = 1, \dots, N_k$, let $Y_{k,j} = \sum_{i=1}^{n} Z^i_{k,j} \mathrm{I}_{A_i}$ where $A_1, \dots, A_n \in \mathcal{F}$ are disjoint sets such that $Q_i(A_i) = 1$ for each $i = 1, \dots, n$. It is clear that for each choice of (i, j, k), $Y_{k,j}$ has the same distribution as X_k under Q_i, and

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{j=1}^{N_k} Y_{k,j} = X \quad \text{in } ||\cdot||_{\mathcal{Q}}.$$

Therefore, $\rho(Y_{k,j}) = \rho(X_k)$. Finally, as ρ is $||\cdot||_{\mathcal{Q}}$-continuous, quasi-convex and $\mathcal{Q}$-based, we

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have

$$\rho(X) = \lim_{k\to\infty} \rho\left(\frac{1}{N_k}\sum_{j=1}^{N_k} Y_{k,j}\right) \leqslant \liminf_{k\to\infty} \max_{j=1,\dots,N_k} \{\rho(Y_{k,j})\} = \liminf_{k\to\infty} \rho(X_k).$$

Thus, ρ satisfies the $\mathcal{Q}$-Fatou property. $\square$

A direct consequence of Lemma 2 is that, if $\mathcal{Q}$ is mutually singular, then any $\mathcal{Q}$-based coherent risk measure, such as a $\underline{Q}$-spectral risk measure, satisfies the $\mathcal{Q}$-Fatou property.

Next we present lemma which serves as a building block for the proof of Theorem 4. For $X \in \mathcal{X}$, let

$$L_X(\mathcal{Q}) = \{Y \in \mathcal{X} : Y \overset{\mathrm{d}}{=}_Q X \text{ for all } Q \in \mathcal{Q}\}.$$

That is, $L_X(\mathcal{Q})$ is the set of all random variables identically distributed as X under each measure in $\mathcal{Q}$. Clearly $X \in L_X(\mathcal{Q})$ and hence $L_X(\mathcal{Q})$ is not empty.

Lemma 3. *Suppose that $\underline{Q}$ is mutually singular, and the probability measure $P \ll \frac{1}{n}\sum_{i=1}^n Q_i$. The functional $\rho : \mathcal{X} \to \mathbb{R}$, $\rho(X) = \sup_{Y\in L_X(\mathcal{Q})} \mathbb{E}^P[Y]$ is a mixture of Q-ES for $Q \in \mathcal{Q}$.*

Proof. Let $A_1, \dots, A_n \in \mathcal{F}$ be disjoint sets such that $Q_i(A_i) = 1$ for each $i = 1, \dots, n$. Write $\overline{Q} = \frac{1}{n}\sum_{i=1}^n Q_i$ and $Z = \mathrm{d}P/\mathrm{d}\overline{Q}$. For each $i = 1, \dots, n$, let U_i be, under Q_i, a uniform random variable on $[0,1]$ such that $Z = F^{-1}_{Z,Q_i}(U_i)$ Q_i-almost surely. The existence of such a random variable U_i can be guaranteed by, for instance, Lemma A.32 of Föllmer and Schied (2016). By the Fréchet-Hoeffding inequality (see e.g. Remark 3.25 of Rüschendorf (2013)), for $Y \in \mathcal{X}$, we have $\mathbb{E}^{Q_i}[ZY] \leqslant \mathbb{E}^{Q_i}[ZF^{-1}_{Y,Q_i}(U_i)]$. It follows that, for $Y \in L_X(\mathcal{Q})$,

$$\begin{aligned}\mathbb{E}^P[Y] = \sum_{i=1}^n \mathbb{E}^P[Y\mathrm{I}_{A_i}] &= \sum_{i=1}^n \mathbb{E}^{\overline{Q}}\left[\frac{\mathrm{d}P}{\mathrm{d}\overline{Q}} Y\mathrm{I}_{A_i}\right] \\ &= \sum_{i=1}^n \frac{1}{n}\mathbb{E}^{Q_i}\left[\frac{\mathrm{d}P}{\mathrm{d}\overline{Q}} Y\right] \leqslant \sum_{i=1}^n \frac{1}{n}\mathbb{E}^{Q_i}\left[ZF^{-1}_{X,Q_i}(U_i)\right].\end{aligned}$$

On the other hand, it is easy to verify that $\sum_{i=1}^n F^{-1}_{X,Q_i}(U_i)\mathrm{I}_{A_i} \in L_X(\mathcal{Q})$, and

$$\mathbb{E}^P\left[\sum_{i=1}^n F^{-1}_{X,Q_i}(U_i)\mathrm{I}_{A_i}\right] = \sum_{i=1}^n \frac{1}{n}\mathbb{E}^{Q_i}\left[ZF^{-1}_{X,Q_i}(U_i)\right].$$

Therefore,

$$\sup_{Y\in L_X(\mathcal{Q})} \mathbb{E}^P[Y] = \sum_{i=1}^n \frac{1}{n}\mathbb{E}^{Q_i}\left[ZF^{-1}_{X,Q_i}(U_i)\right].$$

Note that

$$\mathbb{E}^{Q_i}\left[ZF^{-1}_{X,Q_i}(U_i)\right] = \int_0^1 F^{-1}_{Z,Q_i}(u)F^{-1}_{X,Q_i}(u)\mathrm{d}u,$$

and the function $\bar{g} : [0,1] \to [0,1]$, $t \mapsto \int_0^t F^{-1}_{Z,Q_i}(u)\mathrm{d}u$ is in $\mathcal{G}$ and is convex. It follows that the mapping $X \mapsto \mathbb{E}^{Q_i}[ZF^{-1}_{X,Q_i}(U_i)]$ is a spectral risk measure in the form of (15). Therefore, ρ is a

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linear combination of Q-spectral risk measures, $Q \in \mathcal{Q}$. Note that each Q-spectral risk measure is a mixture of Q-ES (Theorem 4 of Kusuoka (2001)), and hence ρ is a mixture of Q-ES for $Q \in \mathcal{Q}$. □

Proof of Theorem 4. (i) $\underline{Q}$-spectral risk measures are coherent. It is straightforward that a supremum of $\mathcal{Q}$-based coherent risk measures is also a $\mathcal{Q}$-based coherent risk measure.

(ii) Since ρ is coherent, by Lemma 2, it has the $\mathcal{Q}$-Fatou property. From the classic coherent risk measure representation (e.g. Delbaen (2002)), there exists a set $\mathcal{R} \subset \mathcal{P}$ of probability measures which are absolutely continuous with respect to $\overline{Q}$, such that

$$\rho(X) = \sup_{P \in \mathcal{R}} \mathbb{E}^P[X], \quad X \in \mathcal{X}. \tag{28}$$

Now fix $X \in \mathcal{X}$. As ρ is $\mathcal{Q}$-based, $\rho(Y) = \rho(X)$ for all $Y \in L_X(\mathcal{Q})$. It follows that

$$\rho(X) = \sup_{Y \in L_X(\mathcal{Q})} \sup_{P \in \mathcal{R}} \mathbb{E}^P[Y] = \sup_{P \in \mathcal{R}} \sup_{Y \in L_X(\mathcal{Q})} \mathbb{E}^P[Y].$$

By Lemma 3, for each $P \in \mathcal{R}$, the mapping $\mathcal{X} \to \mathbb{R}$, $X \mapsto \sup_{Y \in L_X(\mathcal{Q})} \mathbb{E}^P[Y]$ is a mixture of Q-ES for $Q \in \mathcal{Q}$. Therefore, ρ is the supremum of mixtures of Q-ES for $Q \in \mathcal{Q}$. □

References

Acharya, V., Engle, R. and Richardson, M. (2012). Capital shortfall: A new approach to ranking and regulating systemic risks. *American Economic Review*, **102**(3), 59-64.

Adrian, T. and Brunnermeier, M. K. (2016). CoVaR. *American Economic Review*, **106**(7), 1705–1741.

Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, **9**(3), 203–228.

BCBS (Basel Committee on Banking Supervision) (2016). *Standards. Minimum capital requirements for Market Risk. January 2016.* Available at `www.bis.org`.

BCBS (Basel Committee on Banking Supervision) (2017). *Frequently asked questions on market risk capital requirements. January 2017.* Available at `www.bis.org`.

Cambou, M. and Filipovic, D. (2017). Model uncertainty and scenario aggregation. *Mathematical Finance*, **27**(2), 534–567.

CME (Chicago Mercantile Exchange) (2010). *CME SPAN: standard portfolio analysis of risk*. Available at `www.cmegroup.com`.

Cherny, A. S. and Madan, D. (2009). New measures for performance evaluation. *Review of Financial Studies*, **22**(7), 2571–2606.

Delbaen, F. (2002). Coherent risk measures on general probability spaces. In: *Advances in Finance and Stochastics*. Springer, Berlin, Heidelberg, 1-37.

Electronic copy available at: https://ssrn.com/abstract=3235450

Delbaen, F. (2012). *Monetary Utility Functions.* Osaka University Press, Osaka.

Denneberg, D. (1994). *Non-additive Measure and Integral.* Springer Science & Business Media.

Dhaene, J. and Kukush, A., Linders, D. and Tang, Q. (2012). Remarks on quantiles and distortion risk measures. *European Actuarial Journal*, **2**(2), 319–328.

Du, Z. and Escanciano, J. C. (2017). Backtesting expected shortfall: Accounting for tail risk. *Management Science*, **63**(4), 940–958.

Embrechts, P., Liu, H. and Wang, R. (2018). Quantile-based risk sharing. *Operations Research*, published online.

Embrechts, P., Puccetti, G., Rüschendorf, L., Wang, R. and Beleraj, A. (2014). An academic response to Basel 3.5. *Risks*, **2**(1), 25-48.

Embrechts, P., Wang, B. and Wang, R. (2015). Aggregation-robustness and model uncertainty of regulatory risk measures. *Finance and Stochastics*, **19**(4), 763–790.

Fissler, T. and Ziegel, J. F. (2016). Higher order elicitability and Osband's principle. *Annals of Statistics*, **44**, 1680–1707.

Föllmer, H. and Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics*, **6**(4), 429–447.

Föllmer, H. and Schied, A. (2011). *Stochastic Finance: An Introduction in Discrete Time.* Walter de Gruyter, Berlin, Third Edition.

Föllmer, H. and Schied, A. (2016). *Stochastic Finance. An Introduction in Discrete Time.* Walter de Gruyter, Berlin, Fourth Edition.

IAIS (2014). *Consultation Document December 2014. Risk-based global insurance capital standard.* International Association of Insurance Supervisors. Available at `www.iaisweb.org`.

Joe, H. (2014). *Dependence Modeling with Copulas.* London: Chapman & Hall.

Jouini, E., Schachermayer, W. and Touzi, N. (2006). Law invariant risk measures have the Fatou property. *Advances in Mathematical Economics*, **9**, 49–71.

Kou, S. and Peng, X. (2016). On the measurement of economic tail risk. *Operations Research*, **64**(5), 1056–1072.

Krätschmer, V., Schied, A. and Zähle, H. (2017). Domains of weak continuity of statistical functionals with a view toward robust statistics. *Journal of Multivariate Analysis*, **158**, 1–19.

Kusuoka, S. (2001). On law invariant coherent risk measures. *Advances in Mathematical Economics*, **3**, 83–95.

McNeil, A. J., Frey, R. and Embrechts, P. (2015). *Quantitative Risk Management: Concepts, Techniques and Tools.* Revised Edition. Princeton, NJ: Princeton University Press.

Moody's (2010). *Moody's structured finance rating scale.* Available at `www.moodys.com`.

Müller, A. and Stoyan, D. (2002). *Comparison Methods for Statistical Models and Risks.* Wiley, England.

OCC (Office of the Comptroller of the Currency) (2011). *Supervisory guidance on model risk manage-*

Electronic copy available at: https://ssrn.com/abstract=3235450

ment. OCC 2011-12, Board of Governors of the Federal Reserve System.

Righi M. B. (2018). A theory for robust risk measures. `arXiv:1807.01977v1`.

Rockafellar, R. T., Uryasev, S. and Zabarankin, M. (2006). Generalized deviation in risk analysis. *Finance and Stochastics*, **10**, 51–74.

Rüschendorf, L. (2013). *Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios*. Springer, Heidelberg.

Schmeidler, D. (1986). Integral representation without additivity. *Proceedings of the American Mathematical Society*, **97**(2), 255–261.

Standard and Poor's (2009). *Understanding Standard and Poor's rating definitions*. Available at `standardandpoors.com`.

Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica*, **55**(1), 95–115.

Electronic copy available at: https://ssrn.com/abstract=3235450