

Robust Estimation of Loss Models for Truncated and Censored Severity Data

CHUDAMANI POUDYAL¹
University of Central Florida

VYTARAS BRAZAUSKAS²
University of Wisconsin-Milwaukee

SUBMITTED FOR PUBLICATION!

Abstract. In this paper, we consider robust estimation of claim severity models in insurance, when data are affected by truncation (due to deductibles), censoring (due to policy limits), and scaling (due to coinsurance). In particular, robust estimators based on the methods of *trimmed moments* (T -estimators) and *winsorized moments* (W -estimators) are pursued and fully developed. The general definitions of such estimators are formulated and their asymptotic properties are investigated. For illustrative purposes, specific formulas for T - and W -estimators of the tail parameter of a single-parameter Pareto distribution are derived. The practical performance of these estimators is then explored using the well-known Norwegian fire claims data. Our results demonstrate that T - and W -estimators offer a robust and computationally efficient alternative to the likelihood-based inference for models that are affected by deductibles, policy limits, and coinsurance.

Keywords. Insurance Payments; Loss Models; Robust Estimation; Trimmed and Winsorized Moments; Truncated and Censored Data.

¹CORRESPONDING AUTHOR: Chudamani Poudyal, Ph.D., is a Visiting Assistant Professor in the Department of Statistics and Data Science, University of Central Florida, Orlando, FL 32816, USA. *e-mail:* Chudamani.Poudyal@ucf.edu

²Vytaras Brazauskas, Ph.D., ASA, is a Professor in the Department of Mathematical Sciences, University of Wisconsin-Milwaukee, P.O. Box 413, Milwaukee, WI 53201, USA. *e-mail:* vytaras@uwm.edu

1 Introduction

Parametric statistical models for insurance claims severity are continuous, right-skewed, and frequently heavy-tailed (see Klugman *et al.*, 2019). The data sets that such models are usually fitted to contain outliers that are difficult to identify and separate from genuine data. Moreover, due to commonly used loss mitigation techniques, the random variables we observe and wish to model are affected by data truncation (due to deductibles), censoring (due to policy limits), and scaling (due to coinsurance). In the current practice, statistical inference for loss models is almost exclusively likelihood (MLE) based, which typically results in non-robust parameter estimators, pricing models, and risk measurements.

Construction of robust actuarial models includes many ideas from the mainstream robust statistics literature (see, e.g., Huber and Ronchetti, 2009), but there are additional nuances that need to be addressed. Namely, actuaries have to deal with heavy-tailed and skewed distributions, data truncation and censoring, identification and recycling of outliers, and aggregate loss, just to name a few. A number of specialized studies addressing some of these issues have been carried out in the actuarial literature; see, e.g., Künsch (1992), Gisler and Reinhard (1993), Brazauskas and Serfling (2003), Garrido and Pitselis (2000), Marceau and Rioux (2001), Serfling (2002), and Dornheim and Brazauskas (2007). Further, these and other actuarial studies motivated the development of two broad classes of robust estimators – the methods of *trimmed moments* (see, e.g., Brazauskas, 2009, Brazauskas *et al.*, 2009) and *winsorized moments* (see, e.g., Zhao *et al.*, 2018a,b). These two approaches, called T - and W -estimators for short, are sufficiently general and flexible for fitting continuous parametric models based on completely observed ground-up loss data. In Figure 1.1, we illustrate how T and W methods act on data and control the influence of extremes. First of all, notice that typical loss mitigation techniques employed in insurance practice (e.g., deductibles and policy limits) are closely related to data winsorizing or its variants. Secondly, we see that in order to taper the effects of rare but high severity claims on parameter estimates, data should be “pre-processed” using trimming or winsorizing. Thenceforth T and W estimates can be found by applying the classical *method of moments*. Note that these initial modifications of data have to be taken into account when deriving corresponding theoretical moments. This yields an additional benefit. Specifically, unlike the parameter estimators based on the standard method of moments, which may not exist for heavy-tailed models (due to the non-existence of finite moments), theoretical T and W moments are always finite. Finally, for trimmed or winsorized data, estimation of parameters via the method of moments is not the only option. Indeed, one might choose to apply another estimation procedure (e.g., properly constructed MLE) and gain similar robustness properties. In this paper, however, we focus on rigorous treatment of moment-type estimators.

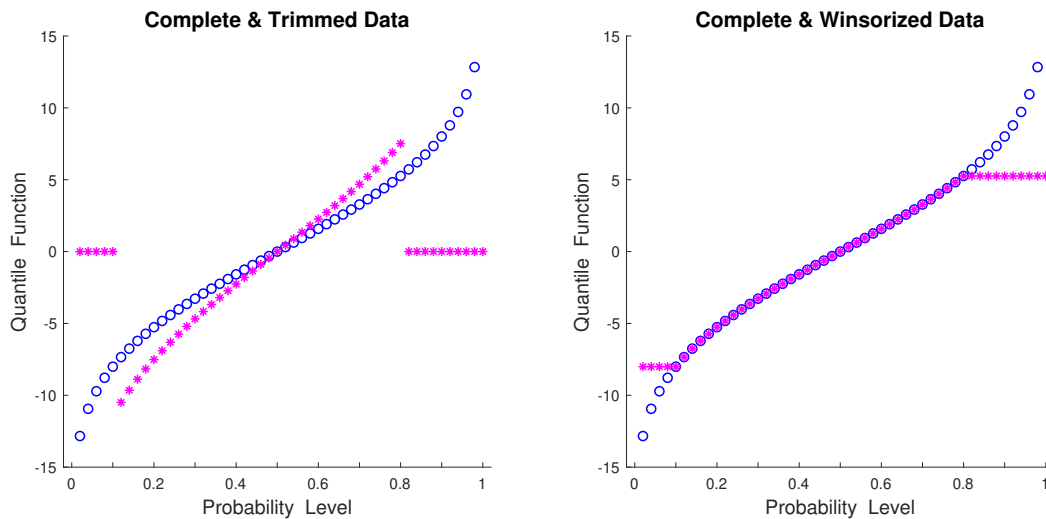


FIGURE 1.1: Quantile functions of complete data and its trimmed and winsorized versions. Sample size: $n = 50$. Trimming/winsorizing proportions: 10% (lower) and 20% (upper). Complete data marked by ‘o’ and trimmed/winsorized by ‘*’.

T -estimators have been discussed in the operational risk literature by [Opdyke and Cavallo \(2012\)](#), used in credibility studies by [Kim and Jeon \(2013\)](#), and further tested in risk measurement exercises by [Abu Bakar and Nadarajah \(2019\)](#). Also, the idea of trimming has been gaining popularity in modeling extremes (see [Bhattacharya et al., 2019](#); [Bladt et al., 2020](#)). Thus we anticipate the methodology developed in this paper will be useful and transferable to all these and other areas of research.

Moreover, besides typical non-robustness of MLE-based inference, implementation of such procedures on real data is also technically challenging (see discussions by [Frees, 2017](#); [Lee, 2017](#)). This issue is especially evident when one tries to fit complicated multi-parameter models such as finite mixtures of distributions (see [Verbelen et al., 2015](#); [Miljkovic and Grün, 2016](#); [Reynkens et al., 2017](#)). Thus, the primary objective of this paper is to go beyond the complete data scenario and develop T - and W -estimators for insurance data affected by the above-mentioned transformations. It will be shown that, when properly redesigned, T - and W -estimators can be a robust and computationally efficient alternative to the MLE-based inference for claim severity models that are affected by deductibles, policy limits, and coinsurance. In particular, we will provide the definitions of T - and W -estimators and derive their asymptotic properties such as normality and consistency. Specific formulas or estimating equations for a single-parameter Pareto (Pareto I) model will be provided. Finally, the practical performance of the estimators will be illustrated by fitting Pareto I to the well-known Norwegian fire claims data. We will use MLE, several T - and W -estimators, validate the fits, and apply the fitted models to price an insurance contract.

The remainder of the paper is organized as follows. In Section 2, we describe a series of loss variable (data) transformations, starting with complete data, then continuing with truncated and censored data, and finishing with two types of insurance payments. Section 3 uses the data scenarios and models of the previous section and derives T - and W -estimators for the parameters of those models. Then asymptotic properties of these estimators are established. In Section 4, we develop specific formulas of the estimators when the underlying loss distribution is Pareto I, and compare the asymptotic relative efficiency of T - and W -estimators with respect to MLE. Section 5 is devoted to practical applications of the Pareto I model; the effects of model fitting on insurance contract pricing are then investigated. Finally, concluding remarks are offered in Section 6.

2 Data and Models

In this section, we review typical transformations of continuous random variables that may be encountered in modeling claim severity. For each type of variable transformation, the resulting probability density function (pdf), cumulative distribution function (cdf) and quantile function (qf) are specified.

2.1 Complete Data

Let us start with the complete data scenario. Suppose the observable random variables

$$X_1, X_2, \dots, X_n \tag{2.1}$$

are independent and identically distributed (i.i.d.) and have the pdf $f(x)$, cdf $F(x)$, and qf $F^{-1}(v)$. Since loss random variables are nonnegative, the support of $f(x)$ is the set $\{x : x \geq 0\}$.

The complete data scenario is not common when claim severities are recorded, but it represents so-called “ground up” losses and thus important to consider. Statistical properties of the ground-up variable are of great interest in risk analysis, product design (for specifying insurance contract parameters), risk transfer considerations, and for other business decisions.

2.2 Truncated Data

Data truncation occurs when sample observations are restricted to some interval (not necessarily finite), say (t_1, t_2) with $t_1 < t_2$. Measurements and even a count of observations outside the interval are completely unknown. To formalize this discussion, we will say that we observe the i.i.d. data

$$X_1^*, X_2^*, \dots, X_n^*, \tag{2.2}$$

where each X^* is equal to the ground-up loss variable X , if X falls between t_1 and t_2 , and is undefined otherwise. That is, X^* satisfies the following conditional event relationship:

$$X^* \stackrel{d}{=} X \mid t_1 < X < t_2,$$

where $\stackrel{d}{=}$ denotes “equal in distribution”. Due to this relationship, the cdf F_* , pdf f_* , and qf F_*^{-1} of variables X^* are related to F , f , and F^{-1} (see Section 2.1) and given by:

$$F_*(x; t_1, t_2) = \mathbf{P} [X \leq x \mid t_1 < X < t_2] = \begin{cases} 0, & x \leq t_1; \\ \frac{F(x) - F(t_1)}{F(t_2) - F(t_1)}, & t_1 < x < t_2; \\ 1, & x \geq t_2, \end{cases} \quad (2.3)$$

$$f_*(x; t_1, t_2) = \frac{d}{dx} [F_*(x; t_1, t_2)] = \begin{cases} \frac{f(x)}{F(t_2) - F(t_1)}, & t_1 < x < t_2; \\ \text{undefined}, & x = t_1, x = t_2; \\ 0, & \text{elsewhere,} \end{cases} \quad (2.4)$$

and

$$F_*^{-1}(v; t_1, t_2) = F^{-1}(vF(t_2) + (1 - v)F(t_1)), \quad \text{for } 0 \leq v \leq 1. \quad (2.5)$$

In industry wide databases such as ORX Loss Data (managingrisktogether.orx.org), only losses above some pre-specified threshold, say d , are collected, which results in *left truncated* data at d . Thus, the observations available to the end-user can be viewed as a realization of random variables (2.2) with $t_1 = d$ and $t_2 \rightarrow \infty$. The latter condition slightly simplifies formulas (2.3)–(2.5); one just needs to replace $F(t_2)$ with 1.

2.3 Censored Data

There are several versions of data censoring that occur in statistical modeling: interval censoring (it includes left and right censoring depending on which end point of the interval is infinite), type I censoring, type II censoring, and random censoring. For actuarial work, the most relevant type is *interval censoring*. It occurs when complete observations are available within some interval, say (t_1, t_2) with $t_1 < t_2$, but data outside the interval is only partially known. That is, counts are available but actual values are not. To formalize this discussion, we will say that we observe the i.i.d. data

$$X_1^{**}, X_2^{**}, \dots, X_n^{**}, \quad (2.6)$$

where each X^{**} is equal to the ground-up variable X , if X falls between t_1 and t_2 , and is equal to the corresponding end-point of the interval if X is beyond that point. That is, X^{**} is given by

$$X^{**} = \min \{ \max(t_1, X), t_2 \} = \begin{cases} t_1, & X \leq t_1; \\ X, & t_1 < X < t_2; \\ t_2, & X \geq t_2. \end{cases}$$

Due to this relationship, the cdf F_{**} , pdf f_{**} , and qf F_{**}^{-1} of variables X^{**} are related to F , f , and F^{-1} and have the following expressions:

$$\begin{aligned} F_{**}(x; t_1, t_2) &= \mathbf{P} [\min \{ \max(t_1, X), t_2 \} \leq x] \\ &= \mathbf{P} [X \leq x] \mathbf{1} \{t_1 \leq x < t_2\} + \mathbf{1} \{t_2 \leq x\} = \begin{cases} 0, & x < t_1; \\ F(x), & t_1 \leq x < t_2; \\ 1, & x \geq t_2, \end{cases} \end{aligned} \quad (2.7)$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function. Further,

$$F_{**}^{-1}(v; t_1, t_2) = \begin{cases} t_1, & v < F(t_1); \\ F^{-1}(v), & F(t_1) \leq v < F(t_2); \\ t_2, & v \geq F(t_2). \end{cases} \quad (2.8)$$

Note that cdf (2.7) is a mixture of continuous cdf F and discrete probability mass at $x = t_1$ (with probability $F(t_1)$) and $x = t_2$ (with probability $1 - F(t_2^-)$). This results in a mixed pdf/pmf:

$$f_{**}(x; t_1, t_2) = \begin{cases} F(t_1), & x = t_1; \\ f(x), & t_1 < x < t_2; \\ 1 - F(t_2^-), & x = t_2; \\ 0, & \text{elsewhere.} \end{cases} \quad (2.9)$$

2.4 Insurance Payments

Insurance contracts have coverage modifications that need to be taken into account when modeling the underlying loss variable. Usually coverage modifications such as deductibles, policy limits, and coinsurance are introduced as loss control mechanisms so that unfavorable policyholder behavioral effects (e.g., adverse selection) can be minimized. There are also situations when certain features of the contract emerge naturally (e.g., the value of insured property in general insurance is a natural upper policy limit). Here we describe two common transformations of the loss variable along with the corresponding cdf's, pdf's, and qf's.

Suppose the insurance contract has ordinary deductible d , upper policy limit u , and coinsurance rate c ($0 \leq c \leq 1$). These coverage parameters imply that when a loss X is reported, the insurance company is responsible for a proportion c of X exceeding d , but no more than $c(u - d)$.

Next, if the loss severity X below the deductible d is completely unobservable (even its frequency is unknown), then the observed i.i.d. insurance payments Y_1, \dots, Y_n can be viewed as realizations of *left-truncated*, *right-censored*, and *linearly-transformed* X (called *payment-per-payment* variable):

$$Y \stackrel{d}{=} c(\min\{X, u\} - d) \mid X > d = \begin{cases} \text{undefined,} & X \leq d; \\ c(X - d), & d < X < u; \\ c(u - d), & u \leq X. \end{cases} \quad (2.10)$$

We can see that the payment variable Y is a linear transformation of a composition of variables X^* and X^{**} (see Sections 2.2 and 2.3). Thus, similar to variables X^* and X^{**} , its cdf G_Y , pdf g_Y , and qf G_Y^{-1} are also related to F , f , and F^{-1} and given by:

$$G_Y(y; c, d, u) = \mathbf{P}[c(\min\{X, u\} - d) \leq y \mid X > d] = \begin{cases} 0, & y \leq 0; \\ \frac{F(y/c+d) - F(d)}{1 - F(d)}, & 0 < y < c(u - d); \\ 1, & y \geq c(u - d), \end{cases} \quad (2.11)$$

$$g_Y(y; c, d, u) = \begin{cases} \frac{f(y/c+d)}{c[1 - F(d)]}, & 0 < y < c(u - d); \\ \frac{1 - F(u^-)}{1 - F(d)}, & y = c(u - d); \\ 0, & \text{elsewhere,} \end{cases} \quad (2.12)$$

and

$$G_Y^{-1}(v; c, d, u) = \begin{cases} c[F^{-1}(v + (1 - v)F(d)) - d], & 0 \leq v < \frac{F(u) - F(d)}{1 - F(d)}; \\ c(u - d), & \frac{F(u) - F(d)}{1 - F(d)} \leq v \leq 1. \end{cases} \quad (2.13)$$

The scenario that no information is available about X below d is likely to occur when modeling is done based on the data acquired from a third party (e.g., data vendor). For payment data collected in house, the information about the number of policies that did not report claims (equivalently, resulted in a payment of 0) would be available. This minor modification yields different payment variables, say Z_1, \dots, Z_n , which can be treated as i.i.d. realizations of *interval-censored* and *linearly-transformed* X (called *payment-per-loss* variable):

$$Z = c(\min\{X, u\} - \min\{X, d\}) = \begin{cases} 0, & X \leq d; \\ c(X - d), & d < X < u; \\ c(u - d), & u \leq X. \end{cases} \quad (2.14)$$

Again, its cdf G_Z , pdf g_Z , and qf G_Z^{-1} are related to F , f , and F^{-1} and given by:

$$G_Z(z; c, d, u) = \mathbf{P}[c(\min\{X, u\} - \min\{X, d\}) \leq z] = \begin{cases} 0, & z < 0; \\ F(z/c + d), & 0 \leq z < c(u - d); \\ 1, & z \geq c(u - d), \end{cases} \quad (2.15)$$

$$g_Z(z; c, d, u) = \begin{cases} F(d), & z = 0; \\ f(z/c + d)/c, & 0 < z < c(u - d); \\ 1 - F(u^-), & z = c(u - d); \\ 0, & \text{elsewhere,} \end{cases} \quad (2.16)$$

and

$$G_Z^{-1}(v; c, d, u) = \begin{cases} 0, & 0 \leq v \leq F(d); \\ c(F^{-1}(v) - d), & F(d) < v < F(u); \\ c(u - d), & F(u) \leq v \leq 1. \end{cases} \quad (2.17)$$

3 T - and W -Estimation

In this section, we first provide definitions of parameter estimators obtained by using the Method of Trimmed Moments (T -estimators; Section 3.1) and the Method of Winsorized Moments (W -estimators; Section 3.2) under the data scenarios of Sections 2.1–2.4. Then, in Section 3.3, we specify the asymptotic distribution of the resulting estimators. Also, throughout the section we assume that the ground-up losses follow a continuous parametric distribution with pdf $f(x | \boldsymbol{\theta})$ and cdf $F(x | \boldsymbol{\theta})$ which are indexed by $k \geq 1$ unknown parameters $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$. The goal is to estimate those parameters using T - and W -estimators by taking into account the probabilistic relationships between the cdf $F(x | \boldsymbol{\theta})$ and the distribution function of *observed* data.

3.1 T -Estimators

T -estimators are derived by following the standard method-of-moments approach, but instead of standard moments we match sample and population *trimmed* (T) moments (or their variants). The advantage of such an approach over the standard one is that the population T moments always exist irrespective of the tail-heaviness of the underlying distribution. The following definition lists the formulas of sample and population T moments for the data scenarios of Sections 2.1–2.4.

Definition 3.1. For data scenarios and models of Sections 2.1–2.4, let us denote the sample and population T moments as \widehat{T}_j and $T_j(\boldsymbol{\theta})$, respectively. If $w_{1:n} \leq \dots \leq w_{n:n}$ is an ordered realization of variables (2.1), (2.2), (2.6), (2.10), or (2.14) with qf denoted $F_V^{-1}(v | \boldsymbol{\theta})$ (which depending upon the data scenario equals to qf F^{-1} , (2.5), (2.8), (2.13), or (2.17)), then the sample and population T moments, with the trimming proportions a (lower) and b (upper), have the following expressions:

$$\widehat{T}_j = \frac{1}{n - m_n - m_n^*} \sum_{i=m_n+1}^{n-m_n^*} [h(w_{i:n})]^j, \quad j = 1, \dots, k, \quad (3.1)$$

$$T_j(\boldsymbol{\theta}) = \frac{1}{1-a-b} \int_a^{1-b} [h(F_V^{-1}(v|\boldsymbol{\theta}))]^j dv, \quad j = 1, \dots, k. \quad (3.2)$$

Under all the data scenarios, the trimming proportions a and b and function h are chosen by the researcher. Also, integers m_n and m_n^* ($0 \leq m_n < n - m_n^* \leq n$) are such that $m_n/n \rightarrow a$ and $m_n^*/n \rightarrow b$ when $n \rightarrow \infty$. In finite samples, the integers m_n and m_n^* are computed as $m_n = [na]$ and $m_n^* = [nb]$, where $[\cdot]$ denotes the greatest integer part.

Note 3.1. In the original formulation of MTM estimators for complete data (Brazauskas *et al.*, 2009), the trimming proportions a and b and function h were allowed to vary for different j , which makes the technique more flexible. On the other hand, for implementation of MTM estimators in practice, such flexibility requires more decisions to be made regarding the a and b interaction with each other and for different h . The follow-up research that used MTMs usually had not varied these constants and functions, which seems like a reasonable choice. Therefore, in this paper we choose to work with non-varying a , b , and h for all j . \square

Note 3.2. For incomplete data scenarios, possible permutations between a , b and $F(t_1)$, $F(t_2)$ have to be taken into account. For truncated data, there is only one possibility: $0 \leq F(t_1) \leq a < 1-b \leq F(t_2) \leq 1$. For censored data, however, it is possible to use part or all of censored data in estimation. Thus, we can have six arrangements:

- | | |
|--|--|
| 1. $0 \leq a < 1-b \leq F(t_1) < F(t_2) \leq 1$. | 4. $0 \leq F(t_1) < F(t_2) \leq a < 1-b \leq 1$. |
| 2. $0 \leq a \leq F(t_1) < 1-b \leq F(t_2) \leq 1$. | 5. $0 \leq F(t_1) \leq a < F(t_2) \leq 1-b \leq 1$. |
| 3. $0 \leq a \leq F(t_1) < F(t_2) \leq 1-b \leq 1$. | 6. $0 \leq F(t_1) \leq a < 1-b \leq F(t_2) \leq 1$. |

Among these, the case 6 ($0 \leq F(t_1) \leq a < 1-b \leq F(t_2) \leq 1$) makes most sense because it uses the available data in the most effective way. For the sake of completeness, however, we will investigate the other cases as well (see Section 4). Note that the insurance payments Y and Z are special (mixed) cases of truncated and censored data and thus will possess similar properties. Moreover, the T -estimators based on case 6 will be resistant to outliers, i.e., observations that are inconsistent with the assumed model and most likely appearing at the boundaries t_1 and t_2 . \square

Note 3.3. In view of Notes 3.1 and 3.2, the T -estimators with $a > 0$ and $b > 0$ ($0 \leq F(t_1) \leq a < 1-b \leq F(t_2) \leq 1$) are globally robust with the *lower* and *upper* breakdown points given by $LBP = a$ and $UBP = b$, respectively. The robustness of such estimators against small or large outliers comes from the fact that in the computation of estimates the influence of the order statistics with the index less than

$n \times \text{LBP}$ or higher than $n \times (1 - \text{UBP})$ is limited. For more details on LBP and UBP, see Brazauskas and Serfling (2000) and Serfling (2002). \square

3.2 W-Estimators

W -estimators are derived by following the standard method-of-moments approach, but instead of standard moments we match sample and population *winsorized* (W) moments (or their variants). Similar to T -estimators, the population W moments also always exist. The following definition lists the formulas of sample and population W moments for the data scenarios of Sections 2.1–2.4.

Definition 3.2. For data scenarios and models of Sections 2.1–2.4, let us denote the sample and population W moments as \widehat{W}_j and $W_j(\boldsymbol{\theta})$, respectively. If $w_{1:n} \leq \dots \leq w_{n:n}$ is an ordered realization of variables (2.1), (2.2), (2.6), (2.10), or (2.14) with qf denoted $F_V^{-1}(v | \boldsymbol{\theta})$ (which depending upon the data scenario equals to $qf F^{-1}$, (2.5), (2.8), (2.13), or (2.17)), then the sample and population W moments, with the winsorizing proportions a (lower) and b (upper), have the following expressions:

$$\widehat{W}_j = \frac{1}{n} \left[m_n [h(w_{m_n+1:n})]^j + \sum_{i=m_n+1}^{n-m_n^*} [h(w_{i:n})]^j + m_n^* [h(w_{n-m_n^*:n})]^j \right], \quad (3.3)$$

$$W_j(\boldsymbol{\theta}) = a [h(F_V^{-1}(a | \boldsymbol{\theta}))]^j + \int_a^{1-b} [h(F_V^{-1}(v | \boldsymbol{\theta}))]^j dv + b [h(F_V^{-1}(1-b | \boldsymbol{\theta}))]^j, \quad (3.4)$$

where $j = 1, \dots, k$, the winsorizing proportions a , b and function h are chosen by the researcher, and integers m_n, m_n^* are defined and computed the same way as in Definition 3.1.

Note 3.4. In the original formulation of MWM estimators for complete data (Zhao *et al.*, 2018a), the winsorizing proportions a and b and function h were allowed to vary for different j . Based on the arguments similar to those made in Note 3.1, in this paper we will choose the same a , b , and h for all j . Further, the focus will be on the case when a and $1 - b$ fall within the interval $[F(t_1); F(t_2)]$: $0 \leq F(t_1) \leq a < 1 - b \leq F(t_2) \leq 1$. Finally, the breakdown points of W -estimators are identical to those of T -estimators, i.e., $\text{LBP} = a$ and $\text{UBP} = b$. \square

3.3 Asymptotic Properties

In this section, the asymptotically normal distributions for the T - and W -estimators of Sections 3.1–3.2 are specified. It follows immediately from the parametric structure of those asymptotic distributions that all the estimators under consideration are *consistent*. Throughout the section the notation \mathcal{AN} is used to denote “asymptotically normal”.

3.3.1 T -Estimators

T -estimators are found by matching sample T -moments (3.1) with population T -moments (3.2) for $j = 1, \dots, k$, and then solving the system of equations with respect to $\theta_1, \dots, \theta_k$. The obtained solutions, which we denote by $\hat{\theta}_j = s_j(\hat{T}_1, \dots, \hat{T}_k)$, $1 \leq j \leq k$, are, by definition, the T -estimators of $\theta_1, \dots, \theta_k$. Note that the functions s_j are such that $\theta_j = s_j(T_1(\boldsymbol{\theta}), \dots, T_k(\boldsymbol{\theta}))$.

The asymptotic distribution of these estimators for complete data has been derived by Brazauskas *et al.* (2009). It also follows from a more general theorem established by Zhao *et al.* (2018a, Note 2.4), which relies on central limit theory of L -statistics (Chernoff *et al.*, 1967). The following theorem generalizes those results to all data scenarios of Sections 2.1–2.4.

Theorem 3.1. *Suppose an i.i.d. realization of variables (2.1), (2.2), (2.6), (2.10), or (2.14) has been generated by cdf $F_V(v | \boldsymbol{\theta})$ which depending upon the data scenario equals to cdf F , (2.3), (2.7), (2.11), or (2.15), respectively. Let $\hat{\boldsymbol{\theta}}_T = (\hat{\theta}_1, \dots, \hat{\theta}_k) = (s_1(\hat{T}_1, \dots, \hat{T}_k), \dots, s_k(\hat{T}_1, \dots, \hat{T}_k))$ denote a T -estimator of $\boldsymbol{\theta}$. Then*

$$\hat{\boldsymbol{\theta}}_T = (\hat{\theta}_1, \dots, \hat{\theta}_k) \text{ is } \mathcal{AN} \left((\theta_1, \dots, \theta_k), \frac{1}{n} \mathbf{D}_t \boldsymbol{\Sigma}_t \mathbf{D}_t' \right),$$

where $\mathbf{D}_t := [d_{ij}]_{i,j=1}^k$ is the Jacobian of the transformations s_1, \dots, s_k evaluated at $(T_1(\boldsymbol{\theta}), \dots, T_k(\boldsymbol{\theta}))$ and $\boldsymbol{\Sigma}_t := [\sigma_{ij}^2]_{i,j=1}^k$ is the covariance-variance matrix with the entries

$$\sigma_{ij}^2 = \frac{1}{(1-a-b)(1-a-b)} \int_a^{1-b} \int_a^{1-b} (\min\{v, w\} - vw) d[h(F_V^{-1}(v))]^j d[h(F_V^{-1}(w))]^i.$$

Proof. For complete data, generated by (2.1) and with the assumption that $F_V \equiv F$ is continuous, see Brazauskas *et al.* (2009) or Zhao *et al.* (2018a, Note 2.4).

For truncated data, generated by (2.2), the cdf F_* given by (2.3) is still continuous and hence the results established for complete data can be directly applied to F_* .

For the remaining data scenarios, generated by (2.6), (2.10), or (2.14), the qf F_V^{-1} is not smooth and the functions $H_j = [h \circ F_V^{-1}]^j$, $j = 1, 2, \dots, k$ have points of non-differentiability (see Lemma A.1 in Zhao *et al.*, 2018a). The set of such points, however, has probability zero, which means that the cdf's F_{**} , G_Y , and G_Z are *almost everywhere* continuous under the Borel probability measures induced by F_{**} , G_Y , and G_Z (see, e.g., Folland, 1999, Theorem 1.16). Therefore, H_j' shall be replaced with 0 whenever it is not defined; see Chernoff *et al.* (1967, Assumption A*). \square

Note 3.5. Theorem 3.1 states that T -estimators for the parameters of loss models considered in this paper are asymptotically unbiased with the entries of the covariance-variance matrix diminishing at the

rate $1/n$. Using these properties in conjunction with the multidimensional Chebyshev's inequality it is a straightforward exercise to establish the fact that T -estimators are consistent. \square

3.3.2 W -Estimators

W -estimators are found by matching sample W -moments (3.3) with population W -moments (3.4) for $j = 1, \dots, k$, and then solving the system of equations with respect to $\theta_1, \dots, \theta_k$. The obtained solutions, which we denote by $\widehat{\theta}_j = r_j(\widehat{W}_1, \dots, \widehat{W}_k)$, $1 \leq j \leq k$, are, by definition, the W -estimators of $\theta_1, \dots, \theta_k$. Note that the functions r_j are such that $\theta_j = r_j(W_1(\boldsymbol{\theta}), \dots, W_k(\boldsymbol{\theta}))$.

The asymptotic distribution of these estimators for complete data has been established by Zhao *et al.* (2018a, Theorem 2.1 and Lemma A.1.). The following theorem summarizes the asymptotic distribution of W -estimators to all data scenarios of Section 2.

Theorem 3.2. *Suppose an i.i.d. realization of variables (2.1), (2.2), (2.6), (2.10), or (2.14) has been generated by cdf $F_V(v | \boldsymbol{\theta})$ which depending upon the data scenario equals to cdf F , (2.3), (2.7), (2.11), or (2.15), respectively. Let $\widehat{\boldsymbol{\theta}}_w = (\widehat{\theta}_1, \dots, \widehat{\theta}_k) = (r_1(\widehat{W}_1, \dots, \widehat{W}_k), \dots, r_k(\widehat{W}_1, \dots, \widehat{W}_k))$ denote a W -estimator of $\boldsymbol{\theta}$. Then*

$$\widehat{\boldsymbol{\theta}}_w = (\widehat{\theta}_1, \dots, \widehat{\theta}_k) \text{ is } \mathcal{AN} \left((\theta_1, \dots, \theta_k), \frac{1}{n} \mathbf{D}_w \boldsymbol{\Sigma}_w \mathbf{D}'_w \right),$$

where $\mathbf{D}_w := [d_{ij}]_{i,j=1}^k$ is the Jacobian of the transformations r_1, \dots, r_k evaluated at $(W_1(\boldsymbol{\theta}), \dots, W_k(\boldsymbol{\theta}))$ and $\boldsymbol{\Sigma}_w := [\sigma_{ij}^2]_{i,j=1}^k$ is the covariance-variance matrix with the entries

$$\sigma_{ij}^2 = \widehat{A}_{i,j}^{(1)} + \widehat{A}_{i,j}^{(2)} + \widehat{A}_{i,j}^{(3)} + \widehat{A}_{i,j}^{(4)},$$

where the terms $\widehat{A}_{i,j}^{(m)}$, $m = 1, \dots, 4$, are specified in Zhao *et al.* (2018a, Lemma A.1).

Proof. The proof can be established by following the same arguments as in Theorem 3.1. \square

Note 3.6. Similar to the discussion of Note 3.5, the asymptotic normality statement of this theorem implies that W -estimators are consistent. \square

4 Analytic Examples: Pareto I

In this section, we first derive MLE, T - and W -estimators for the tail parameter of a single-parameter Pareto distribution, abbreviated as Pareto I, when the observed data is either in the form of insurance payments Y , defined by (2.10), or Z , defined by (2.14). The corresponding MLE and T -estimators for

lognormal distribution has recently been investigated by Poudyal (2021a). Note that Pareto I is the distribution of the ground-up variable X . The cdf, pdf, and qf of Pareto I are defined as follows:

$$\text{CDF: } F(x) = 1 - (x_0/x)^\alpha, \quad x > x_0, \quad (4.1)$$

$$\text{PDF: } f(x) = (\alpha/x_0)(x_0/x)^{\alpha+1}, \quad x > x_0, \quad (4.2)$$

$$\text{QF: } F^{-1}(v) = x_0(1-v)^{-1/\alpha}, \quad 0 \leq v \leq 1, \quad (4.3)$$

where $\alpha > 0$ is the shape (tail) parameter and $x_0 > 0$ is a known constant.

Then, the definitions of the estimators are complemented with their asymptotic distributions. Using the asymptotic normality results, we evaluate the asymptotic relative efficiency (ARE) of the T - and W -estimators with respect to the MLE:

$$\text{ARE(Q, MLE)} = \frac{\text{asymptotic variance of MLE estimator}}{\text{asymptotic variance of Q estimator}},$$

where Q represents T or W estimator. Since for Pareto I the asymptotic variance of MLE reaches the Cramér-Rao lower bound, the other estimators' efficiency will be between 0 and 1. Estimators with AREs close to 1 are preferred.

Also, for the complete data scenario, formulas of $\hat{\alpha}_{\text{MLE}}$ and $\hat{\alpha}_T$ are available in Brazauskas *et al.* (2009). Derivations for the other data scenarios of Section 2 (truncated and censored data) are analogous to the ones presented in this section and thus will be skipped.

4.1 MLE

4.1.1 Payments Y

If y_1, \dots, y_n is a realization of variables (2.10) with pdf (2.12) and cdf (2.11), where F and f are given by (4.1) and (4.2), respectively, then the log-likelihood function can be specified by following standard results presented in Klugman *et al.* (2019, Chapter 11):

$$\begin{aligned} \mathcal{L}_{P_Y}(\alpha | y_1, \dots, y_n) &= \sum_{i=1}^n \log [f(y_i/c + d)/c] \mathbf{1}\{0 < y_i < c(u-d)\} \\ &\quad - n \log [1 - F(d)] + \log [1 - F(u^-)] \sum_{i=1}^n \mathbf{1}\{y_i = c(u-d)\} \\ &= \sum_{i=1}^n \left[\log \left(\frac{\alpha}{cx_0} \right) - (\alpha + 1) \log \left(\frac{y_i/c + d}{x_0} \right) \right] \mathbf{1}\{0 < y_i < c(u-d)\} \\ &\quad - \alpha n \log(x_0/d) + \alpha \log(x_0/u) \sum_{i=1}^n \mathbf{1}\{y_i = c(u-d)\}, \end{aligned}$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function. Straightforward maximization of \mathcal{L}_{P_Y} yields an explicit formula of the MLE of α :

$$\hat{\alpha}_{\text{MLE}} = \frac{\sum_{i=1}^n \mathbf{1}\{0 < y_i < c(u-d)\}}{\sum_{i=1}^n \log(y_i/(cd) + 1) \mathbf{1}\{0 < y_i < c(u-d)\} + \log(u/d) \sum_{i=1}^n \mathbf{1}\{y_i = c(u-d)\}} \quad (4.4)$$

The asymptotic distribution of $\hat{\alpha}_{\text{MLE}}$ follows from standard results for MLEs (see, e.g., Serfling, 1980, Section 4.2). In this case, the Fisher information matrix has a single entry:

$$I_{11} = -\mathbf{E} \left[\frac{\partial^2 \log g_Y(Y | \alpha)}{\partial \alpha^2} \right] = -\mathbf{E} \left[-\frac{1}{\alpha^2} \mathbf{1}\{0 < Y < c(u-d)\} \right] = \frac{1}{\alpha^2} [1 - (d/u)^\alpha].$$

Hence, the estimator $\hat{\alpha}_{\text{MLE}}$, defined by (4.4), has the following asymptotic distribution:

$$\hat{\alpha}_{\text{MLE}} \text{ is } \mathcal{N} \left(\alpha, \frac{1}{n} \frac{\alpha^2}{1 - (d/u)^\alpha} \right). \quad (4.5)$$

A few observations can be made from this result. First, the coinsurance factor c has no effect on (4.5). Second, the corresponding result for the complete data scenario is obtained when there is no deductible (i.e., $d = x_0$) and no policy limit (i.e., $u \rightarrow \infty$). Third, if $u \rightarrow \infty$, then the asymptotic properties of $\hat{\alpha}_{\text{MLE}}$ remain equivalent to those of the complete data case irrespective of the choice of d ($d < \infty$). Also, notice that (4.5) implies that $\hat{\alpha}_{\text{MLE}}$ is a consistent and efficient estimator.

4.1.2 Payments Z

If z_1, \dots, z_n is a realization of variables (2.14) with pdf (2.16) and cdf (2.15), where F and f are given by (4.1) and (4.2), respectively, then the log-likelihood function can be specified by following standard results presented in Klugman *et al.* (2019, Chapter 11):

$$\begin{aligned} \mathcal{L}_{P_Z}(\alpha | z_1, \dots, z_n) &= \log [F(d)] \sum_{i=1}^n \mathbf{1}\{z_i = 0\} + \log [1 - F(u^-)] \sum_{i=1}^n \mathbf{1}\{z_i = c(u-d)\} \\ &\quad + \sum_{i=1}^n \log [f(z_i/c + d)/c] \mathbf{1}\{0 < z_i < c(u-d)\} \\ &= \log [1 - (x_0/d)^\alpha] \sum_{i=1}^n \mathbf{1}\{z_i = 0\} + \alpha \log(x_0/u) \sum_{i=1}^n \mathbf{1}\{z_i = c(u-d)\} \\ &\quad + \sum_{i=1}^n \left[\log \left(\frac{\alpha}{cx_0} \right) - (\alpha + 1) \log \left(\frac{z_i/c + d}{x_0} \right) \right] \mathbf{1}\{0 < z_i < c(u-d)\}. \end{aligned} \quad (4.6)$$

It is clear from the expression of \mathcal{L}_{P_Z} that it has to be maximized numerically. Suppose that a unique solution for maximization of (4.6) with respect to α is found, and let us denote it $\hat{\alpha}_{\text{MLE}}$.

Further, the asymptotic distribution of $\widehat{\alpha}_{\text{MLE}}$ follows from standard results for MLEs (see, e.g., Serfling, 1980, Section 4.2). In this case, the single entry of the Fisher information matrix is

$$\begin{aligned} I_{11} &= -\mathbf{E} \left[\frac{\partial^2 \log g_Z(Z|\alpha)}{\partial \alpha^2} \right] = -\mathbf{E} \left[-\frac{(x_0/d)^\alpha \log^2(x_0/d)}{(1 - (x_0/d)^\alpha)^2} \mathbf{1}\{Z = 0\} - \frac{1}{\alpha^2} \mathbf{1}\{0 < Z < c(u-d)\} \right] \\ &= \alpha^{-2} \left[\frac{(x_0/d)^\alpha}{1 - (x_0/d)^\alpha} \log^2 [(x_0/d)^\alpha] + (x_0/d)^\alpha - (x_0/u)^\alpha \right]. \end{aligned}$$

Hence, the estimator $\widehat{\alpha}_{\text{MLE}}$, found by numerically maximizing (4.6), has the following asymptotic distribution:

$$\widehat{\alpha}_{\text{MLE}} \text{ is } \mathcal{N} \left(\alpha, \frac{\alpha^2}{n} \left[\frac{(x_0/d)^\alpha}{1 - (x_0/d)^\alpha} \log^2 [(x_0/d)^\alpha] + (x_0/d)^\alpha - (x_0/u)^\alpha \right]^{-1} \right). \quad (4.7)$$

Here, we will again emphasize several points. First, as in Section 4.1.1, the coinsurance factor c has no effect on (4.7). Second, the corresponding result for the complete data scenario is obtained when there is no deductible (to eliminate d from (4.7), take the limit as $d \rightarrow x_0$) and no policy limit (i.e., $u \rightarrow \infty$). Third, (4.7) implies that $\widehat{\alpha}_{\text{MLE}}$ is a consistent and efficient estimator.

4.2 T-Estimators

4.2.1 Payments Y

Let $y_{1:n} \leq \dots \leq y_{n:n}$ denote an ordered realization of variables (2.10) with qf (2.13), where F and F^{-1} are given by (4.1) and (4.3), respectively. Since Pareto I has only one unknown parameter, we need only one T moment equation to estimate it. Also, since payments Y are left-truncated and right-censored, it follows from Note 3.2 that only the last three permutations between the trimming proportions a , b and $F(t_1)$, $F(t_2)$ are possible (i.e., a cannot be below $F(t_1)$). That is, after converting t_1 and t_2 into the notation involving c , d , and u , we get from (2.13) the following arrangements:

Case 1: $0 < \frac{F(u)-F(d)}{1-F(d)} \leq a < 1 - b \leq 1$ (estimation based on censored data only).

Case 2: $0 \leq a < \frac{F(u)-F(d)}{1-F(d)} \leq 1 - b \leq 1$ (estimation based on observed and censored data).

Case 3: $0 \leq a < 1 - b \leq \frac{F(u)-F(d)}{1-F(d)} \leq 1$ (estimation based on observed data only).

In all these cases, the sample T moments (3.1) can be easily computed by first estimating the probability $[F(u) - F(d)]/[1 - F(d)]$ as $n^{-1} \sum_{i=1}^n \mathbf{1}\{0 < y_i < c(u-d)\}$, then selecting a , b , and finally choosing $h_Y(y) = \log(y/(cd) + 1)$. Note that c and d are known constants, and the logarithmic transformation will linearize the qf F^{-1} in terms of α^{-1} (at least for the observed data part). With these choices in mind, let us examine what happens to the population T moments (3.2) under the cases 1–3. The following

steps can be easily verified:

$$\begin{aligned}
 (1-a-b)T_{1(y)}(\alpha) &= \int_a^{1-b} h_Y(G_Y^{-1}(v|\alpha)) dv = \int_a^{1-b} \log\left(\frac{G_Y^{-1}(v|\alpha)}{cd} + 1\right) dv \\
 &= \int_a^{1-b} \left[\log\left(\frac{1}{d}F^{-1}\left(v + (1-v)F(d)\right)\right) \mathbf{1}\left\{0 \leq v < \frac{F(u) - F(d)}{1 - F(d)}\right\} \right. \\
 &\quad \left. + \log(u/d) \mathbf{1}\left\{\frac{F(u) - F(d)}{1 - F(d)} \leq v \leq 1\right\} \right] dv \\
 &= \begin{cases} (1-a-b)\log(u/d), & \text{Case 1;} \\ \alpha^{-1}\left[(1-a)(1 - \log(1-a)) + b\log(d/u)^\alpha - (d/u)^\alpha\right], & \text{Case 2;} \\ \alpha^{-1}\left[(1-a)(1 - \log(1-a)) - b(1 - \log b)\right], & \text{Case 3.} \end{cases}
 \end{aligned}$$

It is clear from these expressions that estimation of α is impossible in Case 1 because there is no α in the formula of $T_{1(y)}(\alpha)$. In Case 2, α has to be estimated numerically by solving the following equation:

$$\alpha^{-1}\left[(1-a)(1 - \log(1-a)) + b\log(d/u)^\alpha - (d/u)^\alpha\right] = (1-a-b)\widehat{T}_{1(y)}, \quad (4.8)$$

where $\widehat{T}_{1(y)} = (n - m_n - m_n^*)^{-1} \sum_{i=m_n+1}^{n-m_n^*} \log(y_{i:n}/(cd) + 1)$. Suppose a unique solution of (4.8) with respect to α is found. Let us denote it $\widehat{\alpha}_T^{(2)}$ and remember that it is a function of $\widehat{T}_{1(y)}$, say $s_1^{(2)}(\widehat{T}_{1(y)})$.

Finally, if Case 3 is chosen, then we have an explicit formula for a T -estimator of α :

$$\widehat{\alpha}_T^{(3)} = \frac{I_t(a, 1-b)}{(1-a-b)\widehat{T}_{1(y)}} =: s_1^{(3)}(\widehat{T}_{1(y)}), \quad (4.9)$$

where $I_t(a, 1-b) := -\int_a^{1-b} \log(1-v) dv = (1-a)(1 - \log(1-a)) - b(1 - \log b)$ and the sample T moment $\widehat{T}_{1(y)}$ is computed as before; see (4.8).

Next, we will specify the asymptotic distributions and compute AREs of $\widehat{\alpha}_T^{(2)} = s_1^{(2)}(\widehat{T}_{1(y)})$ and $\widehat{\alpha}_T^{(3)} = s_1^{(3)}(\widehat{T}_{1(y)})$. The asymptotic distributions of $\widehat{\alpha}_T^{(2)}$ and $\widehat{\alpha}_T^{(3)}$ follow from Theorem 3.1. In both cases, the Jacobian \mathbf{D}_t and the covariance-variance matrix Σ_t are scalar. Denoting $d_{11}^{(2)}$ and $d_{11}^{(3)}$ the Jacobian entries for Cases 2 and 3, respectively, we get the following expressions:

$$\begin{aligned}
 d_{11}^{(2)} &= \left. \frac{\partial \widehat{\alpha}_T^{(2)}}{\partial \widehat{T}_{1(y)}} \right|_{\widehat{T}_{1(y)}=T_{1(y)}} = \left. \frac{\partial s_1^{(2)}(\widehat{T}_{1(y)})}{\partial \widehat{T}_{1(y)}} \right|_{\widehat{T}_{1(y)}=T_{1(y)}} \\
 &= \frac{(1-a-b)\alpha^2}{(d/u)^\alpha (1 - \log(d/u)^\alpha) - (1-a)(1 - \log(1-a))} = -\frac{(1-a-b)\alpha^2}{I_t(a, 1 - (d/u)^\alpha)}, \\
 d_{11}^{(3)} &= \left. \frac{\partial \widehat{\alpha}_T^{(3)}}{\partial \widehat{T}_{1(y)}} \right|_{\widehat{T}_{1(y)}=T_{1(y)}} = \left. \frac{\partial s_1^{(3)}(\widehat{T}_{1(y)})}{\partial \widehat{T}_{1(y)}} \right|_{\widehat{T}_{1(y)}=T_{1(y)}} = -\frac{(1-a-b)\alpha^2}{I_t(a, 1-b)}.
 \end{aligned}$$

Note that $d_{11}^{(2)}$ is found by implicitly differentiating (4.8). Further, denoting $\sigma_{11(2)}^2$ and $\sigma_{11(3)}^2$ the Σ_t entries for Cases 2 and 3, respectively, we get the following expressions:

$$\begin{aligned} (1-a-b)^2 \sigma_{11(2)}^2 &= \int_a^{1-b} \int_a^{1-b} (\min\{v, w\} - vw) dh_Y(G_Y^{-1}(v)) dh_Y(G_Y^{-1}(w)) \\ &= \alpha^{-2} \int_a^{1-(d/u)^\alpha} \int_a^{1-(d/u)^\alpha} (\min\{v, w\} - vw) d \log(1-v) d \log(1-w) \\ &=: \alpha^{-2} J_t(a, 1 - (d/u)^\alpha; a, 1 - (d/u)^\alpha) \end{aligned}$$

and

$$\begin{aligned} (1-a-b)^2 \sigma_{11(3)}^2 &= \int_a^{1-b} \int_a^{1-b} (\min\{v, w\} - vw) dh_Y(G_Y^{-1}(v)) dh_Y(G_Y^{-1}(w)) \\ &= \alpha^{-2} \int_a^{1-b} \int_a^{1-b} (\min\{v, w\} - vw) d \log(1-v) d \log(1-w) \\ &= \alpha^{-2} J_t(a, 1-b; a, 1-b). \end{aligned}$$

Now, as follows from Theorem 3.1, the asymptotic variances of these two estimators of α are equal to $n^{-1} d_{11}^{(k)} \sigma_{11(k)}^2 d_{11}^{(k)}$ for $k = 2, 3$. This implies that the estimators $\hat{\alpha}_T^{(2)}$, found by numerically solving (4.8), and $\hat{\alpha}_T^{(3)}$, given by (4.9), have the following asymptotic distributions:

$$\hat{\alpha}_T^{(2)} \text{ is } \mathcal{AN} \left(\alpha, \frac{\alpha^2}{n} \frac{J_t(a, 1 - (d/u)^\alpha; a, 1 - (d/u)^\alpha)}{I_t^2(a, 1 - (d/u)^\alpha)} \right) \quad (4.10)$$

and

$$\hat{\alpha}_T^{(3)} \text{ is } \mathcal{AN} \left(\alpha, \frac{\alpha^2}{n} \frac{J_t(a, 1-b; a, 1-b)}{I_t^2(a, 1-b)} \right). \quad (4.11)$$

From (4.10) we see that the asymptotic variance of $\hat{\alpha}_T^{(2)}$ does not depend on the upper trimming proportion b , where $\frac{F(u)-F(d)}{1-F(d)} \leq 1-b \leq 1$. As expected, both estimators and their asymptotic distributions coincide when $1-b = \frac{F(u)-F(d)}{1-F(d)} = 1 - (d/u)^\alpha$. Thus, for all practical purposes $\hat{\alpha}_T^{(3)}$ is a better estimator (i.e., it has explicit formula and it becomes equivalent to $\hat{\alpha}_T^{(2)}$ if one chooses $b = (d/u)^\alpha$); therefore $\hat{\alpha}_T^{(2)}$ (more generally, Case 2) will be discarded from further consideration.

As discussed in Note 3.3, the T -estimators are globally robust if $a > 0$ and $b > 0$. This is achieved by sacrificing the estimator's efficiency (i.e., the more robust the estimator the larger is its variance). From (4.5) and (4.11), we find that the asymptotic relative efficiency of $\hat{\alpha}_T^{(3)}$ with respect to $\hat{\alpha}_{\text{MLE}}$ is:

$$\text{ARE} \left(\hat{\alpha}_T^{(3)}, \hat{\alpha}_{\text{MLE}} \right) = \frac{\frac{\alpha^2}{n} \frac{1}{1-(d/u)^\alpha}}{\frac{\alpha^2}{n} \frac{J_t(a, 1-b; a, 1-b)}{I_t^2(a, 1-b)}} = \frac{I_t^2(a, 1-b)}{[1 - (d/u)^\alpha] J_t(a, 1-b; a, 1-b)}.$$

In this case the integrals I_t and J_t can be derived analytically, but in general it is easier and faster to approximate them numerically; see Appendix A.2 in Brazauskas and Kleefeld (2009) for specific approximation formulas of the bivariate integrals J_t . In Table 4.1, we present ARE computations.

TABLE 4.1. ARE $(\hat{\alpha}_T^{(3)}, \hat{\alpha}_{MLE})$ for selected a and b and various choices of right-censoring proportion $\delta = 1 - \frac{F(u)-F(d)}{1-F(d)} = (d/u)^\alpha$.

a	b (when $\delta = 0.01$)					b (when $\delta = 0.05$)				b (when $\delta = 0.10$)		
	0.01	0.05	0.10	0.15	0.25	0.05	0.10	0.15	0.25	0.10	0.15	0.25
0	0.992	0.927	0.856	0.791	0.673	0.966	0.892	0.824	0.701	0.941	0.870	0.740
0.05	0.992	0.927	0.856	0.791	0.674	0.966	0.892	0.825	0.702	0.942	0.871	0.741
0.10	0.991	0.927	0.857	0.793	0.678	0.966	0.893	0.826	0.704	0.943	0.872	0.744
0.15	0.991	0.928	0.858	0.795	0.679	0.967	0.894	0.828	0.708	0.944	0.874	0.747
0.25	0.988	0.927	0.860	0.798	0.686	0.966	0.896	0.832	0.715	0.946	0.878	0.755

It is obvious from the table that for a fixed b , the effect of the lower trimming proportion a on the ARE is negligible. As b increases, T -estimators become more robust but less efficient, yet their AREs are still sufficiently high (all at least 0.67; more than half above 0.85). Also, all estimators' efficiency improves as the proportion of right-censored data δ increases. Take, for example, $a = b = 0.10$: the T -estimator's efficiency grows from 0.857 (when $\delta = 0.01$) to 0.943 (when $\delta = 0.10$).

4.2.2 Payments Z

Let $z_{1:n} \leq \dots \leq z_{n:n}$ denote an ordered realization of variables (2.14) with qf (2.17), where F and F^{-1} are given by (4.1) and (4.3), respectively. Payments Z are left- and right-censored and it follows from Note 3.2 that there are six permutations possible between the trimming proportions a , b and $F(d)$, $F(u)$. However, analysis similar to the one done in Section 4.2.1 shows that two of those scenarios (estimation based on censored data only) have no α in the formulas of population T moments and three (estimation based on observed and censored data) are inferior to the estimation scenario based on fully observed data. (Due to space limitations those investigations will not be presented here.) Thus, from now on we will focus on the following arrangement:

$$0 \leq F(d) \leq a < 1 - b \leq F(u) \leq 1.$$

Similar to the previous section, standard empirical estimates of $F(d)$ and $F(u)$ provide guidance about the choice of a and $1 - b$. However, the function h is defined differently: $h_Z(z) = \log(z/c + d)$. For Pareto I only the first T moment is needed, and it is equal:

$$\begin{aligned} (1 - a - b) T_{1(z)}(\alpha) &= \int_a^{1-b} h_Z(G_Z^{-1}(v|\alpha)) dv = \int_a^{1-b} \log(F^{-1}(v)) dv \\ &= (1 - a - b) \log(x_0) + \alpha^{-1} I_t(a, 1 - b). \end{aligned}$$

Matching $T_{1(z)}(\alpha)$ expression with $\widehat{T}_{1(z)} = (n - m_n - m_n^*)^{-1} \sum_{i=m_n+1}^{n-m_n^*} \log(z_{i:n}/c + d)$ yields an explicit formula for a T -estimator of α :

$$\widehat{\alpha}_T = \frac{I_t(a, 1 - b)}{(1 - a - b)[\widehat{T}_{1(z)} - \log(x_0)]} =: s(\widehat{T}_{1(z)}). \quad (4.12)$$

To specify the asymptotic distribution and compute AREs of $\widehat{\alpha}_T$, we again rely on Theorem 3.1. The single Jacobian entry for estimator (4.12) is given by

$$d_{11} = \left. \frac{\partial \widehat{\alpha}_T}{\partial \widehat{T}_{1(z)}} \right|_{\widehat{T}_{1(z)}=T_{1(z)}} = \left. \frac{\partial s(\widehat{T}_{1(z)})}{\partial \widehat{T}_{1(z)}} \right|_{\widehat{T}_{1(z)}=T_{1(z)}} = -\frac{(1 - a - b)\alpha^2}{I_t(a, 1 - b)}.$$

The single covariance-variance matrix entry, σ_{11}^2 , is found as before:

$$(1 - a - b)^2 \sigma_{11}^2 = \alpha^{-2} J_t(a, 1 - b; a, 1 - b).$$

Hence, the estimator $\widehat{\alpha}_T$, given by (4.12), has the following asymptotic distribution:

$$\widehat{\alpha}_T \text{ is } \mathcal{AN} \left(\alpha, \frac{\alpha^2}{n} \frac{J_t(a, 1 - b; a, 1 - b)}{I_t^2(a, 1 - b)} \right). \quad (4.13)$$

Now, from (4.7) and (4.13) we find that ARE of $\widehat{\alpha}_T$ with respect to $\widehat{\alpha}_{\text{MLE}}$ is

$$\begin{aligned} \text{ARE} \left(\widehat{\alpha}_T, \widehat{\alpha}_{\text{MLE}} \right) &= \frac{\frac{\alpha^2}{n} \left[\frac{(x_0/d)^\alpha}{1-(x_0/d)^\alpha} \log^2 [(x_0/d)^\alpha] + (x_0/d)^\alpha - (x_0/u)^\alpha \right]^{-1}}{\frac{\alpha^2}{n} \frac{J_t(a, 1 - b; a, 1 - b)}{I_t^2(a, 1 - b)}} \\ &= \frac{I_t^2(a, 1 - b)}{\left[\frac{(x_0/d)^\alpha}{1-(x_0/d)^\alpha} \log^2 [(x_0/d)^\alpha] + (x_0/d)^\alpha - (x_0/u)^\alpha \right] J_t(a, 1 - b; a, 1 - b)}. \end{aligned}$$

In Table 4.2, we present ARE computations for selected scenarios of data censoring.

TABLE 4.2. ARE $\left(\widehat{\alpha}_T, \widehat{\alpha}_{\text{MLE}} \right)$ for selected a and b and various combinations of left- and right-censoring proportions (δ_l, δ_r) , where $\delta_l = F(d) = 1 - (x_0/d)^\alpha$ and $\delta_r = 1 - F(u) = (x_0/u)^\alpha$.

δ_l	a	b (when $\delta_r = 0.01$)					b (when $\delta_r = 0.05$)				b (when $\delta_r = 0.10$)		
		0.01	0.05	0.10	0.15	0.25	0.05	0.10	0.15	0.25	0.10	0.15	0.25
0.50	0.50	0.973	0.923	0.864	0.809	0.708	0.962	0.901	0.843	0.739	0.952	0.891	0.781
	0.60	0.939	0.896	0.843	0.793	0.700	0.934	0.879	0.827	0.730	0.929	0.874	0.772
	0.70	0.882	0.849	0.805	0.761	0.679	0.886	0.839	0.794	0.708	0.887	0.839	0.748
	0.80	0.787	0.770	0.737	0.702	–	0.803	0.768	0.732	–	0.812	0.774	–
0.75	0.75	0.927	0.898	0.855	0.811	–	0.941	0.895	0.850	–	0.952	0.903	–
	0.80	0.868	0.848	0.812	0.773	–	0.889	0.850	0.810	–	0.904	0.861	–
	0.85	0.789	0.781	0.753	–	–	0.818	0.789	–	–	0.839	–	–
0.85	0.85	0.896	0.887	0.856	–	–	0.936	0.902	–	–	0.968	–	–
	0.89	0.800	0.804	0.782	–	–	0.848	0.825	–	–	0.886	–	–

Patterns in Table 4.2 are similar to those in Table 4.1, but in this case we also observe that T -estimators become more efficient as one or both censoring proportions increase. Take, for example, $a = 0.80$ and $b = 0.10$: the T -estimator's efficiency grows from 0.737 ($\delta_l = 0.50$, $\delta_r = 0.01$) to 0.812 ($\delta_l = 0.50$, $\delta_r = 0.10$) or from 0.768 ($\delta_l = 0.50$, $\delta_r = 0.05$) to 0.850 ($\delta_l = 0.75$, $\delta_r = 0.05$).

4.3 W-Estimators

As is evident from (3.1) and (3.3), the “central” part of winsorized data is equal to trimmed data times $1 - a - b$. Therefore, W -estimators will be closely related to the corresponding T -estimators. Choosing the same h functions and trimming/winsorizing scenarios as in Section 4.2, we are able to derive W -estimators of α and their asymptotic distributions in a straightforward fashion.

4.3.1 Payments Y

Let $y_{1:n} \leq \dots \leq y_{n:n}$ denote an ordered realization of Y payments, $h_Y(y) = \log(y/(cd) + 1)$, and $0 \leq a < 1 - b \leq \frac{F(u) - F(d)}{1 - F(d)} \leq 1$. The population W -moment $W_{1(y)}(\alpha)$, given by equation (3.4), is related to $T_{1(y)}(\alpha)$ and equal to

$$\begin{aligned} W_{1(y)}(\alpha) &= a [h_Y(G_Y^{-1}(a|\alpha))] + \int_a^{1-b} h_Y(G_Y^{-1}(v|\alpha)) dv + b [h_Y(G_Y^{-1}(1-b|\alpha))] \\ &= a [-\alpha^{-1} \log(1-a)] + \alpha^{-1} I_t(a, 1-b) + b [-\alpha^{-1} \log b] \\ &= \alpha^{-1} [1 - a - b - \log(1-a)] =: \alpha^{-1} I_w(a, 1-b). \end{aligned}$$

Matching $W_{1(y)}(\alpha)$ with the empirical W -moment

$$\widehat{W}_{1(y)} = n^{-1} \left[m_n \log(y_{m_n+1:n}/(cd) + 1) + \sum_{i=m_n+1}^{n-m_n^*} \log(y_{i:n}/(cd) + 1) + m_n^* \log(y_{n-m_n^*:n}/(cd) + 1) \right]$$

yields an explicit formula for a W -estimator of α :

$$\widehat{\alpha}_W = \frac{I_w(a, 1-b)}{\widehat{W}_{1(y)}} =: r_y(\widehat{W}_{1(y)}). \quad (4.14)$$

The asymptotic distribution of $\widehat{\alpha}_W$ follows from Theorem 3.2. The single Jacobian entry for estimator (4.14) is given by

$$d_{11} = \frac{\partial \widehat{\alpha}_W}{\partial \widehat{W}_{1(y)}} \Big|_{\widehat{W}_{1(y)} = W_{1(y)}} = \frac{\partial r_y(\widehat{W}_{1(y)})}{\partial \widehat{W}_{1(y)}} \Big|_{\widehat{W}_{1(y)} = W_{1(y)}} = -\frac{\alpha^2}{I_w(a, 1-b)}.$$

The entry σ_{11}^2 is equal to $\widehat{A}_{1,1}^{(1)} + \dots + \widehat{A}_{1,1}^{(4)}$ (see Lemma A.1 in Zhao *et al.*, 2018a), where $\widehat{A}_{1,1}^{(1)}, \dots, \widehat{A}_{1,1}^{(4)}$ are derived as follows. Given that $\Delta_1 = W_{1(y)}(\alpha) = \alpha^{-1}(1 - a - b - \log(1 - a))$,

$$\begin{aligned} H_1(v) &= h_Y(G_Y^{-1}(v)) = \log\left(\frac{G_Y^{-1}(v|\alpha)}{cd} + 1\right) \\ &= -\alpha^{-1} \log(1 - v) \mathbf{1}\left\{0 \leq v < \frac{F(u) - F(d)}{1 - F(d)}\right\} + \log(u/d) \mathbf{1}\left\{\frac{F(u) - F(d)}{1 - F(d)} \leq v \leq 1\right\}, \end{aligned}$$

and $H_1'(v) = \alpha^{-1}(1 - v)^{-1} \mathbf{1}\left\{0 < v < \frac{F(u) - F(d)}{1 - F(d)}\right\}$, we have:

$$\begin{aligned} \widehat{A}_{1,1}^{(1)} &= \alpha^{-2} J_t(a, 1 - b; a, 1 - b), \\ \widehat{A}_{1,1}^{(2)} = \widehat{A}_{1,1}^{(3)} &= \alpha^{-2} \left[(1 - a - b) \left(\frac{a^2}{1 - a} - b \right) + b \log(1 - a) - b \log b \right], \\ \widehat{A}_{1,1}^{(4)} &= \alpha^{-2} \left[\frac{a^2}{1 - a} (a + 2b) + b(1 - b) \right]. \end{aligned}$$

This yields

$$\begin{aligned} \sigma_{11}^2 &= \alpha^{-2} \left[J_t(a, 1 - b; a, 1 - b) + \frac{a^2(2 - a)}{1 - a} - b[1 - 2a - b + 2 \log b - 2 \log(1 - a)] \right] \\ &=: \alpha^{-2} J_w(a, 1 - b; a, 1 - b). \end{aligned}$$

Putting it all together, $\widehat{\alpha}_W$, given by (4.14), has the following asymptotic distribution:

$$\widehat{\alpha}_W \text{ is } \mathcal{AN}\left(\alpha, \frac{\alpha^2}{n} \frac{J_w(a, 1 - b; a, 1 - b)}{I_w^2(a, 1 - b)}\right). \quad (4.15)$$

Consequently,

$$\text{ARE}(\widehat{\alpha}_W, \widehat{\alpha}_{\text{MLE}}) = \frac{\frac{\alpha^2}{n} \frac{1}{1 - (d/u)^\alpha}}{\frac{\alpha^2}{n} \frac{J_w(a, 1 - b; a, 1 - b)}{I_w^2(a, 1 - b)}} = \frac{I_w^2(a, 1 - b)}{[1 - (d/u)^\alpha] J_w(a, 1 - b; a, 1 - b)}.$$

In Table 4.3, we present ARE computations for selected scenarios of data censoring.

TABLE 4.3. ARE($\widehat{\alpha}_W, \widehat{\alpha}_{\text{MLE}}$) for selected a and b and various choices of right-censoring proportion $\delta = 1 - \frac{F(u) - F(d)}{1 - F(d)} = (d/u)^\alpha$.

a	b (when $\delta = 0.01$)					b (when $\delta = 0.05$)				b (when $\delta = 0.10$)		
	0.01	0.05	0.10	0.15	0.25	0.05	0.10	0.15	0.25	0.10	0.15	0.25
0	1.000	0.960	0.909	0.859	0.758	1.000	0.947	0.895	0.789	1.000	0.944	0.833
0.05	1.000	0.960	0.909	0.859	0.758	1.000	0.947	0.895	0.789	1.000	0.944	0.833
0.10	1.000	0.959	0.909	0.858	0.757	1.000	0.947	0.894	0.789	1.000	0.944	0.833
0.15	0.999	0.958	0.908	0.857	0.756	0.999	0.946	0.893	0.788	0.999	0.943	0.832
0.25	0.994	0.954	0.903	0.853	0.752	0.994	0.941	0.889	0.784	0.994	0.938	0.827

Patterns in Tables 4.1 and 4.3 are identical. However, it is worthwhile noting that for a fixed censoring scenario and fixed a and b , each W -estimator is slightly more efficient than its T counterpart.

4.3.2 Payments Z

Let $z_{1:n} \leq \dots \leq z_{n:n}$ denote an ordered realization of Z payments, $h_Z(z) = \log(z/c+d)$, and $0 \leq F(d) \leq a < 1 - b \leq F(u) \leq 1$. Then the population W -moment is equal to

$$\begin{aligned} W_{1(z)}(\alpha) &= a [h_Z(G_Z^{-1}(a|\alpha))] + \int_a^{1-b} h_Z(G_Z^{-1}(v|\alpha)) dv + b [h_Z(G_Z^{-1}(1-b|\alpha))] \\ &= a [\log x_0 - \alpha^{-1} \log(1-a)] + (1-a-b) \log x_0 + \alpha^{-1} I_t(a, 1-b) + b [\log x_0 - \alpha^{-1} \log b] \\ &= \log x_0 + \alpha^{-1} I_w(a, 1-b). \end{aligned}$$

Matching $W_{1(z)}(\alpha)$ with the empirical W -moment

$$\widehat{W}_{1(z)} = n^{-1} \left[m_n \log(z_{m_n+1:n}/c+d) + \sum_{i=m_n+1}^{n-m_n^*} \log(z_{i:n}/c+d) + m_n^* \log(z_{n-m_n^*:n}/c+d) \right]$$

yields an explicit formula for a W -estimator of α :

$$\widehat{\alpha}_W = \frac{I_w(a, 1-b)}{\widehat{W}_{1(z)} - \log x_0} =: r_z(\widehat{W}_{1(z)}). \quad (4.16)$$

The asymptotic distribution of $\widehat{\alpha}_W$ is derived by following the same steps as in Section 4.3.1. That is:

$$d_{11} = \frac{\partial \widehat{\alpha}_W}{\partial \widehat{W}_{1(z)}} \Big|_{\widehat{W}_{1(z)}=W_{1(z)}} = \frac{\partial r_z(\widehat{W}_{1(z)})}{\partial \widehat{W}_{1(z)}} \Big|_{\widehat{W}_{1(z)}=W_{1(z)}} = -\frac{\alpha^2}{I_w(a, 1-b)}.$$

Then, given that $\Delta_1 = W_{1(z)}(\alpha) = \log x_0 + \alpha^{-1} I_w(a, 1-b)$ and, for $0 \leq F(d) \leq a < 1 - b \leq F(u) \leq 1$, $H_1(v) = h_Z(G_Z^{-1}(v)) = \log x_0 - \alpha^{-1} \log(1-v)$, $H_1'(v) = \frac{1}{\alpha(1-v)}$, we have

$$\begin{aligned} \sigma_{11}^2 &= \alpha^{-2} \left[J_t(a, 1-b; a, 1-b) + \frac{\alpha^2(2-a)}{1-a} - b[1-2a-b+2\log b-2\log(1-a)] \right] \\ &= \alpha^{-2} J_w(a, 1-b; a, 1-b). \end{aligned}$$

Hence, $\widehat{\alpha}_W$, given by (4.16), has the following asymptotic distribution:

$$\widehat{\alpha}_W \text{ is } \mathcal{AN} \left(\alpha, \frac{\alpha^2}{n} \frac{J_w(a, 1-b; a, 1-b)}{I_w^2(a, 1-b)} \right). \quad (4.17)$$

Consequently,

$$\begin{aligned} \text{ARE}(\widehat{\alpha}_W, \widehat{\alpha}_{\text{MLE}}) &= \frac{\frac{\alpha^2}{n} \left[\frac{(x_0/d)^\alpha}{1-(x_0/d)^\alpha} \log^2 [(x_0/d)^\alpha] + (x_0/d)^\alpha - (x_0/u)^\alpha \right]^{-1}}{\frac{\alpha^2}{n} \frac{J_w(a, 1-b; a, 1-b)}{I_w^2(a, 1-b)}} \\ &= \frac{I_w^2(a, 1-b)}{\left[\frac{(x_0/d)^\alpha}{1-(x_0/d)^\alpha} \log^2 [(x_0/d)^\alpha] + (x_0/d)^\alpha - (x_0/u)^\alpha \right] J_w(a, 1-b; a, 1-b)}. \end{aligned}$$

In Table 4.4, we present ARE computations for selected scenarios of data censoring.

Patterns in Table 4.4 are similar to those in Table 4.2. However, unlike the ARE results in Tables 4.1 and 4.3, for payment Z comparison of the W -estimators versus the T -estimators shows that neither method outperforms the other all the time. Each type of estimators can have a better ARE than the competitor but that depends on the choice of a and b (which also depends on δ_l and δ_r).

TABLE 4.4. ARE $(\widehat{\alpha}_W, \widehat{\alpha}_{MLE})$ for selected a and b and various combinations of left- and right-censoring proportions (δ_l, δ_r) , where $\delta_l = F(d) = 1 - (x_0/d)^\alpha$ and $\delta_r = 1 - F(u) = (x_0/u)^\alpha$.

δ_l	a	b (when $\delta_r = 0.01$)					b (when $\delta_r = 0.05$)				b (when $\delta_r = 0.10$)		
		0.01	0.05	0.10	0.15	0.25	0.05	0.10	0.15	0.25	0.10	0.15	0.25
0.50	0.50	0.968	0.929	0.880	0.831	0.733	0.969	0.917	0.866	0.765	0.969	0.915	0.808
	0.60	0.930	0.893	0.847	0.801	0.710	0.932	0.883	0.835	0.741	0.933	0.883	0.783
	0.70	0.877	0.843	0.802	0.761	0.680	0.880	0.836	0.793	0.709	0.884	0.838	0.749
	0.80	0.796	0.769	0.734	0.701	–	0.802	0.766	0.731	–	0.809	0.772	–
0.75	0.75	0.927	0.893	0.851	0.809	–	0.935	0.891	0.848	–	0.948	0.901	–
	0.80	0.878	0.847	0.809	0.772	–	0.887	0.848	0.809	–	0.901	0.860	–
	0.85	0.812	0.785	0.753	–	–	0.823	0.789	–	–	0.839	–	–
0.85	0.85	0.922	0.892	0.856	–	–	0.941	0.902	–	–	0.968	–	–
	0.89	0.838	0.814	0.783	–	–	0.858	0.826	–	–	0.886	–	–

5 Real Data Example

In this section, we use MLE and several T and W estimators for fitting the Pareto I model to the well-studied Norwegian fire claims data (see Brazauskas and Serfling, 2003; Nadarajah and Bakar, 2015; Brazauskas and Kleefeld, 2016; Abu Bakar *et al.*, 2021), which are available at the following website:

<http://lstat.kuleuven.be/Wiley> (in Chapter 1, file NORWEGIANFIRE.TXT).

5.1 Data and Preliminary Diagnostics

The data represent the total damage done by fires in Norway for the years 1972 through 1992; only damages in excess of a priority of 500,000 Norwegian kroner (NOK) are available. We will analyze the data set for the year 1975, which has $n = 142$ observations with the most extreme loss of 52.6 million NOK. The data for this year was also modeled with Pareto I by Brazauskas and Serfling (2003). Table 5.1 provides a summary of the data set.

TABLE 5.1. Summary of *Norwegian Fire Claims* data for the year 1975.

<i>Severity</i> (millions NOK)	[0.5; 1.0)	[1.0; 2.0)	[2.0; 5.0)	[5.0; 10.0)	[10.0; 20.0)	[20.0; ∞)
<i>Relative Frequency</i>	0.54	0.28	0.12	0.03	0.02	0.01

Since no information is given below 500,000 and there is no policy limit and coinsurance, the random variable that generated the data is related to payment Y , i.e., it is $Y + d$ with $c = 1$, $d = 500,000$, and $u = \infty$. Moreover, as is evident from Table 5.1, the data are right-skewed and heavy-tailed suggesting that Pareto I, with cdf (4.1) and qf (4.3), might be an appropriate model in this case. To see how right-censoring changes the estimates of α , model fits, and ultimately premium estimates for a layer, we will consider two data scenarios: *Original Data* ($c = 1$, $d = 500,000$, $u = \infty$) and *Modified Data* ($c = 1$, $d = 500,000$, $u = 7,000,000$).

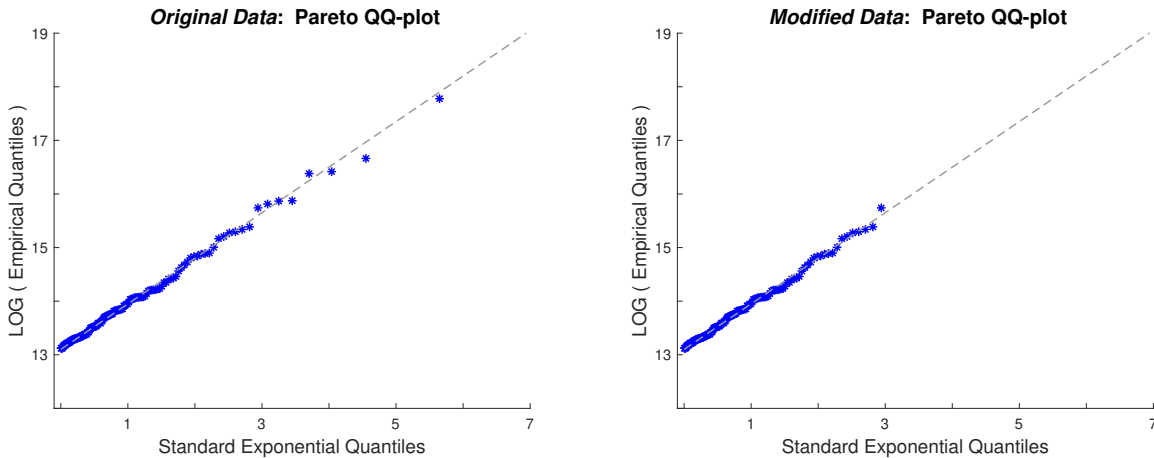


FIGURE 5.1: Pareto quantile-quantile plots for the original and modified data sets. The dashed line represents the “best” fit line (in both cases): $y = 13.1 + 0.85x$.

Further, we will fit Pareto I (x_0, α) under the original and modified data scenarios. Preliminary diagnostics – the quantile-quantile plots (QQ-plots) presented in Figure 5.1 – strongly suggests that the Pareto I assumption is reasonable. Note that the plots are parameter free. That is, since Pareto I is a log-location-scale family, its QQ-plot can be constructed without first estimating model parameters. Note also that only actual data can be used in these plots (i.e., no observations $u = 7,000,000$ under the modified data scenario).

5.2 Model Estimation and Validation

Compute parameter estimates $\hat{\alpha}$ we use the following formulas: (4.4) for MLE, (4.9) for T , and (4.14) for W . In order to match the specifications of fire claims data (denoted l_1, \dots, l_{142}), in (4.4) $c = 1$

and $y_i + d$ is replaced with l_i ; and in (4.9) and (4.14) function h_Y is now defined as $h_Y(l_i) = \log(l_i/d)$. Specifically, for *modified data* ($d = 0.5 \times 10^6$, $u = 7 \times 10^6$, claims l_1, \dots, l_n , $n = 142$), MLE is given by

$$\hat{\alpha}_{\text{MLE}} = \frac{\sum_{i=1}^n \mathbf{1}\{d < l_i < u\}}{\sum_{i=1}^n \log(l_i/d) \mathbf{1}\{d < l_i < u\} + \log(u/d) \sum_{i=1}^n \mathbf{1}\{l_i = u\}},$$

and for *original data* ($d = 0.5 \times 10^6$, $u = \infty$, claims l_1, \dots, l_n , $n = 142$), it becomes $\hat{\alpha}_{\text{MLE}} = \frac{n}{\sum_{i=1}^n \log(l_i/d)}$.

Computational formulas for the T and W estimators remain the same for both data scenarios:

$$\hat{\alpha}_T = \frac{(1-a)(1-\log(1-a)) - b(1-\log b)}{(1-a-b)\hat{T}_{1(y)}} \quad \text{and} \quad \hat{\alpha}_W = \frac{1-a-b-\log(1-a)}{\hat{W}_{1(y)}},$$

where $\hat{T}_{1(y)} = (n - m_n - m_n^*)^{-1} \sum_{i=m_n+1}^{n-m_n^*} \log(l_i/d)$ and

$$\hat{W}_{1(y)} = n^{-1} \left[m_n \log(l_{m_n+1}/d) + \sum_{i=m_n+1}^{n-m_n^*} \log(l_i/d) + m_n^* \log(l_{n-m_n^*}/d) \right],$$

with several choices of $m_n = [na]$ and $m_n^* = [nb]$. The corresponding asymptotic distributions are specified by (4.5), (4.11), and (4.15). They are used to construct the 90% confidence intervals for α . All computations are summarized in Table 5.2, where goodness-of-fit analysis is also provided; see Klugman *et al.* (2019) for how to perform Kolmogorov-Smirnov (KS) test for right-censored data (Section 15.4.1) and how to estimate its p -value using parametric bootstrap (Section 19.4.5).

TABLE 5.2. Pareto I ($x_0 = 7000, \alpha$) fitted to the original and modified data sets. Point and 90% confidence interval estimates of α , Kolmogorov-Smirnov (KS) statistics and their p -values.

Estimator	<i>Original Data</i>				<i>Modified Data</i>			
	$\hat{\alpha}$	90% CI	KS	p -value*	$\hat{\alpha}$	90% CI	KS	p -value*
MLE	1.22	[1.05; 1.39]	0.05	0.70	1.20	[1.03; 1.37]	0.05	0.71
$T, a = b = 0$	1.22	[1.05; 1.39]	0.05	0.70	–	–	–	–
$T, a = b = 0.10$	1.22	[1.04; 1.41]	0.05	0.61	1.22	[1.04; 1.41]	0.05	0.69
$T, a = 0.05, b = 0.15$	1.22	[1.03; 1.41]	0.05	0.60	1.22	[1.03; 1.41]	0.05	0.68
$W, a = b = 0$	1.22	[1.05; 1.39]	0.05	0.70	–	–	–	–
$W, a = b = 0.10$	1.22	[1.04; 1.40]	0.05	0.68	1.22	[1.04; 1.40]	0.05	0.74
$W, a = 0.05, b = 0.15$	1.21	[1.03; 1.39]	0.05	0.59	1.21	[1.03; 1.39]	0.05	0.68

* The p -values are computed using parametric bootstrap with 1000 simulation runs.

As is evident from Table 5.2, all estimators exhibit excellent goodness-of-fit performance, as one would expect after examining Figure 5.1. Irrespective of the method of estimation the fitted Pareto I model has very heavy right tail, i.e., for $1 < \alpha < 2$ all its moments are infinite except the mean. The

T and W estimators with $a = b = 0$ match the estimates of MLE under the original data scenario. As discussed in Section 4.2, this choice of a and b , however, would be inappropriate when data are censored at $u = 7,000,000$, which corresponds to about 4.9% of censoring. Clearly, this level of censoring has no effect whatsoever on T and W estimators with $a = b = 0.10$ and $a = 0.05, b = 0.15$, which demonstrates their robustness. The MLE, on the other hand, is affected by censoring. While the change in its estimated values of α and the corresponding confidence intervals seems minimal (less than 2%), it gets magnified when applied to calculation of premiums, as will be shown next.

5.3 Contract Pricing

Let us consider estimation of the loss severity component of the *pure premium* for an insurance benefit (B) that equals to the amount by which a fire loss damage (L) exceeds 7 million NOK with a maximum benefit of 28 million NOK. That is,

$$B = \begin{cases} 0, & \text{if } L \leq d^*; \\ L - d^*, & \text{if } d^* < L \leq u^*; \\ u^* - d^*, & \text{if } L > u^*, \end{cases} \quad (5.1)$$

and, if L follows the distribution function F , we seek

$$\Pi[F] = \mathbf{E}[B] = \int_{d^*}^{u^*} (x - d^*) dF(x) + (u^* - d^*)[1 - F(u^*)] = \int_{d^*}^{u^*} [1 - F(x)] dx,$$

where $d^* = 7 \cdot 10^6$ and $u^* = 35 \cdot 10^6$. (In U.S. dollars, this roughly corresponds to the layer from 1 to 5 million.) We will present premium estimates for two versions of L : *Observed Loss* (corresponds to $L \sim \text{Pareto I } (d = 5 \cdot 10^5, \alpha)$) and *Ground-Up Loss* (corresponds to $L \sim \text{Pareto I } (x_0 = 7 \cdot 10^3, \alpha)$). The second version shows how different the premium is if all – observed and unobserved – data were available. It also facilitates evaluation of various loss variable characteristics; for example, if one switches from a priority of 500,000 to 250,000, the change in loss elimination ratio could be estimated, but such computations are impossible under the first version of L .

Now, straightforward derivations yield the following expression for $\Pi[F]$:

$$\Pi[F] = C \times \frac{(u^*/C)^{1-\alpha} - (d^*/C)^{1-\alpha}}{1-\alpha}, \quad \alpha \neq 1, \quad (5.2)$$

where $C = d$ (for observed loss) or $= x_0$ (for ground-up loss). If $\alpha = 1$, then $\Pi[F] = C \log(u^*/d^*)$. To get point estimates $\Pi[\hat{F}]$, we plug in the estimates of α from Table 5.2 into (5.2). To construct interval estimators, we rely on the delta method (see Serfling, 1980, Section 3.3), which uses the asymptotic

distributions (4.5), (4.11), and (4.15) and transforms $\hat{\alpha}$ according to (5.2). Thus, we have that $\Pi[\hat{F}]$ is asymptotically normal with mean $\Pi[F]$ and variance $\mathbf{Var}(\hat{\alpha}) \times \left(\frac{\partial}{\partial \alpha} [\Pi[F]]\right)^2$, where

$$\frac{\partial}{\partial \alpha} [\Pi[F]] = \frac{C}{(1-\alpha)^2} \left\{ (1-\alpha) \left[\left(\frac{d^*}{C}\right)^{1-\alpha} \log\left(\frac{d^*}{C}\right) - \left(\frac{u^*}{C}\right)^{1-\alpha} \log\left(\frac{u^*}{C}\right) \right] + \left(\frac{u^*}{C}\right)^{1-\alpha} - \left(\frac{d^*}{C}\right)^{1-\alpha} \right\}$$

and $\mathbf{Var}(\hat{\alpha})$ is taken from (4.5), (4.11), or (4.15). To assure that the left endpoint of the confidence intervals is positive, we will construct log-transformed intervals which have the following structure: $[\Pi[\hat{F}] \cdot K^{-1}; \Pi[\hat{F}] \cdot K]$ for $K > 0$. Table 5.3 presents point and 90% log-transformed interval estimates of premiums for observed and ground-up losses under the original and modified data scenarios.

TABLE 5.3. Point and 90% log-transformed interval estimates of $\Pi[F]$ for observed and ground-up loss L under the original and modified data scenarios.

Estimator	<i>Observed Loss</i> ($\times 10^5$)				<i>Ground-up Loss</i> ($\times 10^3$)			
	<i>Original</i>		<i>Modified</i>		<i>Original</i>		<i>Modified</i>	
	$\Pi[\hat{F}]$	90% CI	$\Pi[\hat{F}]$	90% CI	$\Pi[\hat{F}]$	90% CI	$\Pi[\hat{F}]$	90% CI
MLE	3.82	[2.16; 6.77]	4.01	[2.25; 7.14]	2.11	[0.58; 7.67]	2.35	[0.64; 8.65]
$T, a = b = 0$	3.82	[2.16; 6.77]	–	–	2.11	[0.58; 7.67]	–	–
$T, a = b = 0.10$	3.77	[2.02; 7.01]	3.77	[2.02; 7.01]	2.04	[0.50; 8.32]	2.04	[0.50; 8.32]
$T, a = 0.05, b = 0.15$	3.75	[1.96; 7.17]	3.75	[1.96; 7.17]	2.03	[0.47; 8.75]	2.03	[0.47; 8.75]
$W, a = b = 0$	3.82	[2.16; 6.77]	–	–	2.11	[0.58; 7.67]	–	–
$W, a = b = 0.10$	3.77	[2.06; 6.89]	3.77	[2.06; 6.89]	2.05	[0.52; 8.00]	2.05	[0.52; 8.00]
$W, a = 0.05, b = 0.15$	3.92	[2.12; 7.26]	3.92	[2.12; 7.26]	2.24	[0.56; 8.99]	2.24	[0.56; 8.99]

As can be seen from Table 5.3, premiums for the ground-up loss are two orders of magnitude smaller than those for the observed loss. This was expected because the ground-up distribution automatically estimates that the number of losses below 500,000 is large while the observed loss distribution assumes that that number is zero. Further, as the data scenario changes from original to modified, the robust estimates of premiums (T and W with $a = b = 0.10$ and $a = 0.05, b = 0.15$) do not change, but those based on MLE increase by 5% (for observed loss) and 11% (for ground-up loss). Finally, note that these MLE-based premium changes occur albeit Pareto I fits the data exceptionally well (see Table 5.2). If the model fits were less impressive, the premium swings would be more pronounced.

5.4 Additional Illustrations

It was mentioned in Section 1 that robust model fits can be achieved by other methods of estimation; one just needs to apply them to trimmed or winsorized data. Since for the Pareto I distribution T and W estimators of α with $a = b = 0$ coincide with MLE (see Table 5.2), it is reasonable to expect that

left- and/or right-censored MLE should behave like a W estimator with similarly chosen winsorizing proportions. (This kind of strategy is sometimes used in data analysis practice to robustify MLE.) In what follows, we investigate how this idea works on Norwegian fire claims.

First of all, the asymptotic properties of MLE as stated in Section 4.1 are valid when the right-censoring threshold u is fixed, hence the probability to exceed it is random. Fixed thresholds method of moments and its some variants have been investigated by Poudyal (2021b). The corresponding properties for T and W estimators are established under the complete opposite scenario: data proportions are fixed but thresholds are random. To see what effect this difference has on actual estimates of α , we compute MLEs by matching its censoring points with those used for the T and W estimators in Table 5.2. In particular, for $a = b = 0.10$, we have $m_n = m_n^* = [14.2] = 14$ which implies that for observations from $l_{15} = 0.551 \cdot 10^6$ to $l_{128} = 3.289 \cdot 10^6$ their actual values are included in the computation of $\hat{\alpha}_W$ and for the remaining ones the minimum and maximum of actual observations are used, i.e., $l_1 = \dots = l_{14} = 0.551 \cdot 10^6$ and $l_{129} = \dots = l_{142} = 3.289 \cdot 10^6$. When computing censored MLE, this kind of effect on data can be achieved by choosing the left- and right-censoring levels \tilde{d} and \tilde{u} as follows: $\tilde{d} = 0.551 \cdot 10^6$ and $\tilde{u} = 3.289 \cdot 10^6$. Likewise, for $a = 0.05$ and $b = 0.15$, we have $m_n = [7.1] = 7$ and $m_n^* = [21.3] = 21$ and arrive at $\tilde{d} = 0.530 \cdot 10^6$ and $\tilde{u} = 2.497 \cdot 10^6$. Note that \tilde{d} and \tilde{u} are *not fixed*, which is required for derivations of asymptotic properties, rather they are *estimated* threshold levels. Rigorous theoretical treatment of MLEs with estimated threshold levels is beyond the scope of the current paper and thus is deferred to future research projects. For illustrative purposes, however, we can *assume* that the threshold levels \tilde{d} and \tilde{u} are fixed and apply the methodology of Section 4.1.

Due to the left-truncation of Norwegian fire claims at $d = 500,000$ and additional left- and right-censoring at \tilde{d} ($\tilde{d} > d$) and \tilde{u} , respectively, we are fitting Pareto I (d, α) to Payment Z data. Given these modifications, $\hat{\alpha}_{\text{MLE}}$ (censored at \tilde{d} and \tilde{u}) is found by maximizing (4.6) of the following form:

$$\begin{aligned} \mathcal{L}_{PZ}(\alpha | l_1, \dots, l_n) &= \log [1 - (d/\tilde{d})^\alpha] \sum_{i=1}^n \mathbf{1}\{l_i = \tilde{d}\} + \alpha \log(d/\tilde{u}) \sum_{i=1}^n \mathbf{1}\{l_i = \tilde{u}\} \\ &\quad + \sum_{i=1}^n [\log(\alpha/d) - (\alpha + 1) \log(l_i/d)] \mathbf{1}\{\tilde{d} < l_i < \tilde{u}\}. \end{aligned}$$

Similarly, the asymptotic distribution (4.7) should be of the following form:

$$\hat{\alpha}_{\text{MLE}} \text{ is } \mathcal{AN} \left(\alpha, \frac{\alpha^2}{n} \left[\frac{(d/\tilde{d})^\alpha}{1 - (d/\tilde{d})^\alpha} \log^2 \left[(d/\tilde{d})^\alpha \right] + (d/\tilde{d})^\alpha - (d/\tilde{u})^\alpha \right]^{-1} \right).$$

Numerical implementation of these formulas is provided in Table 5.4, where we compare censored MLEs with W estimators based on such a and b that act on data the same way as MLEs. It is clear from the

table that censored MLEs do achieve the same degree of robustness as the corresponding W estimators. Moreover, the point and interval estimates produced by these two methods are very close but not identical. Finally, it should be emphasized once again that the MLE-based intervals are constructed using the assumed asymptotic distribution which is not proven and may be incorrect.

TABLE 5.4. Comparison of W 's and censored MLEs of α of Pareto I ($d = 500,000, \alpha$) fitted to the original and modified data sets. Note that \tilde{d} and \tilde{u} are *assumed* to be fixed.

Estimator	Censoring Thresholds		<i>Original Data</i>		<i>Modified Data</i>	
	$\tilde{d} (\times 10^6)$	$\tilde{u} (\times 10^6)$	$\hat{\alpha}$	90% CI	$\hat{\alpha}$	90% CI
MLE	0.551	3.289	1.2155	[1.0385; 1.3925]	1.2155	[1.0385; 1.3925]
$W, a = b = 0.10$	--	--	1.2218	[1.0440; 1.3996]	1.2218	[1.0440; 1.3996]
MLE	0.530	2.497	1.2046	[1.0249; 1.3843]	1.2046	[1.0249; 1.3843]
$W, a = 0.05, b = 0.15$	--	--	1.2099	[1.0288; 1.3910]	1.2099	[1.0288; 1.3910]

6 Concluding Remarks

In this paper, we have developed the methods of *trimmed* (called T) and *winsorized* (called W) moments for robust estimation of claim severity models that are affected by deductibles, policy limits and coinsurance. The definitions and asymptotic properties of these estimators have been provided for various data scenarios, including *complete*, *truncated*, and *censored* data, and two types of *insurance payments*. Further, specific definitions and explicit asymptotic distributions of the maximum likelihood (MLE), T , and W estimators have been derived for insurance payments when the loss variable follows a single-parameter Pareto distribution. These analytic examples have clearly shown that T and W estimators sacrifice little efficiency with respect to MLE, but are robust and have explicit formulas (whereas finding MLE does require numerical optimization; see Section 4.1.2). These are highly desirable properties in practice. Finally, the practical performance of the estimators under consideration have been illustrated using the well-known Norwegian fire claims data.

The research presented in this paper invites follow-up studies in several directions. For example, the most obvious direction is to study small-sample properties of these estimators (for Pareto α) using simulations. Second, to derive specific formulas and investigate the estimators' efficiency properties for other loss models such as lognormal, gamma, log-logistic, folded- t , and GB2 distributions. Third, to consider robust estimation based on different influence functions such as Hampel's redescending or Tukey's biweight (bisquare) functions. Fourth, to compare practical performance of our models' robustness with that based on model distance and entropy. Note that the latter approach derives the

worst-case risk measurements, relative to measurements from a baseline model, and has been used by authors in the actuarial literature (e.g., Blanchet *et al.*, 2019) as well as in the financial risk management literature (see, e.g., Alexander and Sarabia, 2012; Glasserman and Xu, 2014). Fifth, it is also of interest to see how well future insurance claims can be predicted using the robust parametric approach of this paper versus more general predictive techniques that are designed to incorporate model uncertainty (see, e.g., Hong and Martin, 2017; Hong *et al.*, 2018).

References

- [1] Abu Bakar, S., Nadarajah, S., and Ngataman, N. (2021). A family of density-hazard distributions for insurance losses. *Communications in Statistics: Simulation and Computation (to appear)*, pages 1–19.
- [2] Abu Bakar, S.A. and Nadarajah, S. (2019). Risk measure estimation under two component mixture models with trimmed data. *J. Appl. Stat.*, **46**(5), 835–852.
- [3] Alexander, C. and Sarabia, J. (2012). Quantile uncertainty and value-at-risk model risk. *Risk Analysis*, **32**(8), 1293–1308.
- [4] Bhattacharya, S., Kallitsis, M., and Stoev, S. (2019). Data-adaptive trimming of the Hill estimator and detection of outliers in the extremes of heavy-tailed data. *Electronic Journal of Statistics*, **13**(1), 1872–1925.
- [5] Bladt, M., Albrecher, H., and Beirlant, J. (2020). Threshold selection and trimming in extremes. *Extremes*, **23**(4), 629–665.
- [6] Blanchet, J., Lam, H., Tang, Q., and Yuan, Z. (2019). Robust actuarial risk analysis. *North American Actuarial Journal*, **23**(1), 33–63.
- [7] Brazauskas, V. (2009). Robust and efficient fitting of loss models: diagnostic tools and insights. *North American Actuarial Journal*, **13**(3), 356–369.
- [8] Brazauskas, V., Jones, B.L., and Zitikis, R. (2009). Robust fitting of claim severity distributions and the method of trimmed moments. *Journal of Statistical Planning and Inference*, **139**(6), 2028–2043.
- [9] Brazauskas, V. and Kleefeld, A. (2009). Robust and efficient fitting of the generalized Pareto distribution with actuarial applications in view. *Insurance: Mathematics & Economics*, **45**(3), 424–435.
- [10] Brazauskas, V. and Kleefeld, A. (2016). Modeling severity and measuring tail risk of Norwegian fire claims. *North American Actuarial Journal*, **20**(1), 1–16.
- [11] Brazauskas, V. and Serfling, R. (2000). Robust and efficient estimation of the tail index of a single-parameter Pareto distribution. *North American Actuarial Journal*, **4**(4), 12–27.
- [12] Brazauskas, V. and Serfling, R. (2003). Favorable estimators for fitting Pareto models: a study using goodness-of-fit measures with actual data. *ASTIN Bulletin*, **33**(2), 365–381.
- [13] Chernoff, H., Gastwirth, J.L., and Johns, Jr., M.V. (1967). Asymptotic distribution of linear combinations of functions of order statistics with applications to estimation. *Annals of Mathematical Statistics*, **38**(1), 52–72.

- [14] Dornheim, H. and Brazauskas, V. (2007). Robust and efficient methods for credibility when claims are approximately gamma-distributed. *North American Actuarial Journal*, **11**(3), 138–158.
- [15] Folland, G.B. (1999). *Real Analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics (New York), second edition. John Wiley & Sons, Inc., New York.
- [16] Frees, E. (2017). Insurance portfolio risk retention. *North American Actuarial Journal*, **21**(4), 526–551.
- [17] Garrido, J. and Pitselis, G. (2000). On robust estimation in Bühlmann-Straub’s credibility model. *Journal of Statistical Research*, **34**(2), 113–132.
- [18] Gisler, A. and Reinhard, P. (1993). Robust credibility. *ASTIN Bulletin*, **23**(1), 118–143.
- [19] Glasserman, P. and Xu, X. (2014). Robust risk measurement and model risk. *Quantitative Finance*, **14**(1), 29–58.
- [20] Hong, L., Kuffner, T., and Martin, R. (2018). On prediction of future insurance claims when the model is uncertain. *Variance*, **12**(1), 90–99.
- [21] Hong, L. and Martin, R. (2017). A flexible Bayesian nonparametric model for predicting future insurance claims. *North American Actuarial Journal*, **21**(2), 228–241.
- [22] Huber, P.J. and Ronchetti, E.M. (2009). *Robust Statistics*. Second edition. John Wiley & Sons, Inc., Hoboken, NJ.
- [23] Kim, J.H.T. and Jeon, Y. (2013). Credibility theory based on trimming. *Insurance: Mathematics & Economics*, **53**(1), 36–47.
- [24] Klugman, S.A., Panjer, H.H., and Willmot, G.E. (2019). *Loss Models: From Data to Decisions*. Fifth edition. John Wiley & Sons, Hoboken, NJ.
- [25] Künsch, H. (1992). Robust methods for credibility. *ASTIN Bulletin*, **22**(1), 33–49.
- [26] Lee, G.Y. (2017). General insurance deductible ratemaking. *North American Actuarial Journal*, **21**(4), 620–638.
- [27] Marceau, E. and Rioux, J. (2001). On robustness in risk theory. *Insurance: Mathematics & Economics*, **29**(2), 167–185.
- [28] Miljkovic, T. and Grün, B. (2016). Modeling loss data using mixtures of distributions. *Insurance: Mathematics & Economics*, **70**, 387–396.
- [29] Nadarajah, S. and Bakar, S. (2015). New folded models for the log-transformed Norwegian fire claim data. *Communications in Statistics: Theory and Methods*, **44**(20), 4408–4440.
- [30] Opdyke, J. and Cavallo, A. (2012). Estimating operational risk capital: the challenges of truncation, the hazards of MLE, and the promise of robust statistics. *Journal of Operational Risk*, **7**(3), 3–90.
- [31] Poudyal, C. (2021a). Robust estimation of loss models for lognormal insurance payment severity data. *Astin Bulletin. The Journal of the International Actuarial Association*, **51**(2), 475–507.
- [32] Poudyal, C. (2021b). Truncated, censored, and actuarial payment-type moments for robust fitting of a single-parameter Pareto distribution. *Journal of Computational and Applied Mathematics*, **388**, 113310, 18.
- [33] Reynkens, T., Verbelen, R., Beirlant, J., and Antonio, K. (2017). Modelling censored losses using splicing: a global fit strategy with mixed Erlang and extreme value distributions. *Insurance:*

Mathematics & Economics, **77**, 65–77.

- [34] Serfling, R. (2002). Efficient and robust fitting of lognormal distributions. *North American Actuarial Journal*, **6**(4), 95–109.
- [35] Serfling, R.J. (1980). *Approximation Theorems of Mathematical Statistics*. John Wiley & Sons, New York.
- [36] Verbelen, R., Gong, L., Antonio, K., Badescu, A., and Lin, S. (2015). Fitting mixtures of Erlangs to censored and truncated data using the EM algorithm. *ASTIN Bulletin*, **45**(3), 729–758.
- [37] Zhao, Q., Brazauskas, V., and Ghorai, J. (2018a). Robust and efficient fitting of severity models and the method of Winsorized moments. *ASTIN Bulletin*, **48**(1), 275–309.
- [38] Zhao, Q., Brazauskas, V., and Ghorai, J. (2018b). Small-sample performance of the MTM and MWM estimators for the parameters of log-location-scale families. *J. Stat. Comput. Simul.*, **88**(4), 808–824.