The Time Varying Bayesian Beta*

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Abstract

The paper proposes a generalized version of the Vasicek's beta, able to accommodate the time varying dynamic of the weights that determine the indicator. The empirical analysis applied to the 10 Fama-French value weighted U.S. industry portfolios is constructed above an econometric framework that relies on the Diagonal BEKK(1,1) specification for jointly estimating the dynamics of the conditional variances and covariances. The estimation results rely on the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm that is an iterative method for solving unconstrained nonlinear optimization problems. The empirical results also discuss the short and long term predictions of the generalized Vasicek's beta.

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1. Introduction

Vasicek's (1973) Bayesian approach for estimating security beta has been extensively accepted in industry and in the empirical finance literature. Fisher and Kamin (1978) have argued that Vasicek's Bayesian estimator of security beta is generally not unbiased and is a static estimator. Therefore, proposing a time-varying estimator of the Bayesian beta is a solution for investigating the non-stationarity. The Vasicek adjustment can mitigate estimation error and thereby increase the reliability of beta (and ultimately cost of capital) estimates.

In the static version, Vasicek (1973) demonstrated that the adjustment to eliminate the effect of estimation error depends upon the standard error of the beta estimate. So the Vasicek-adjusted estimate places some weight on the OLS estimate, and some weight on a prior estimate formed prior to analysing the stock returns, and the weights depend on the standard error of the beta estimate.

This paper proposes a generalized version of the Vasicek's beta able to accommodate the time varying dynamic of the weights that determine the indicator. The theoretical framework is constructed above the Diagonal BEKK(1,1) specification for jointly estimating the dynamics of the conditional variances and covariances, with a multivariate *t-student* distribution of the innovations. The estimation results rely on the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm that is an iterative method for solving unconstrained nonlinear optimization problems. Further, the conditional variance of the beta stocks in the sample and the conditional variance of each stock beta are modelled with an EGARCH(1,1) methodology (Nelson 1991). This specification leads to the determination of the time varying weights able to construct the generalized Vasicek's Bayesian beta.

The empirical results also discuss the short and the long term predictions of the estimator, constructed under the so called Heterogeneous Market Hypothesis, which recognizes the presence of heterogeneity across traders, with a financial market composed of participants having a large spectrum of trading frequency. The main idea is that agents with different time horizons perceive, react to, and cause different types of price and beta components. Therefore, there are short-term traders with daily trading frequency, the medium-term investors who typically rebalance their positions weekly, and the long-term agents with a characteristic time of one, three and six months.

The short and the long term predictions are based on robust regressions that rely on the Huber-Bisquare objective function for the residuals, with a scale function based on the Median Absolute Deviation, centered around the Median (MADMED).

The rest of this paper is organized as follows: Section 2 describes the theoretical framework; Section 3 discusses the data and provide some summary statistics. Section 4 proposes the econometric methodology. In Section 5 are reported the empirical results; whereas, Section 6 concludes the paper.

2. The theoretical framework

The equation of the Conditional Capital Asset Pricing Model (Bollerslev et al. 1988, Bali and Engle 2010) that relies on the conditions of market equilibrium and finds its fundamental pillars on the theory of portfolio selection proposed by Markowitz (1952, 1959) can be expressed as follows:

$$(r_{i,t} - \mu_{f,t}) = \alpha_{i,t} + \beta_{i,t} \cdot (r_{m,t} - \mu_{f,t}) + \varepsilon_{i,t}, \tag{1}$$

with, $r_{i,t}$ that is equal to the stock return i at time t, $r_{m,t}$ the market portfolio return at time t, $\mu_{f,t}$ is the benchmark interest rate at time t, $\alpha_{i,t}$ and $b_{i,t}$ are the coefficient parameters at time t,with $b_{i,t}$ computed in the following way:

$$b_{i,t} = \frac{\sigma_{mi,t}}{\sigma^2_{m,t}}, \quad \left(i = 1, ..., N \, and \, N \, number \, of \, stocks\right), \tag{2}$$

where, $\sigma_{mi,t}$ depicts the co-movements between the return of the market portfolio and the stock return and $\sigma^2_{m,t}$ is the conditional variance of the market portfolio. Now, if $\bar{\beta}_{1,t}$ represents the average conditional beta values of the shares in the sample, and $\beta_{i1,t}$ is the conditional stock beta, the bayesian procedure is a weighted average of both betas. The weight used for the weighted average is based on the amount of conditional variance. If $\sigma^2_{\beta 1,t}$ is the conditional variance of the beta stocks in the sample and $\sigma^2_{\beta i1,t}$ is the conditional variance of the stock beta i, then the weight suggested is $\sigma^2_{\beta 1,t}/\left(\sigma^2_{\beta 1,t}+\sigma^2_{\beta i1,t}\right)$ for $\beta_{i1,t}$ and $\sigma^2_{\beta i1,t}/\left(\sigma^2_{\beta 1,t}+\sigma^2_{\beta i1,t}\right)$ for $\bar{\beta}_{1,t}$. So, the predicted beta is adjusted for stocks i in the second period as follows:

$$beta_{VAS,t} = \frac{\sigma^2_{\beta 1,t}}{\left(\sigma^2_{\beta 1,t}+\sigma^2_{\beta i1,t}\right)}\beta_{i1,t} + \frac{\sigma^2_{\beta i1,t}}{\left(\sigma^2_{\beta 1,t}+\sigma^2_{\beta i1,t}\right)}\bar{\beta}_{1,t}. \tag{3}$$

This procedure will lead to the average for observations that have greater variance than those with smaller variance.

3. Data and descriptive statistics

The empirical analysis considers the U.S. data downloaded from Kenneth French's website, based on daily returns for the 10 Fama-French value weighted U.S. industry portfolios, with the aim to study the relationship between each industry portfolio and the market portfolio, that is the value weighted return for all CRSP firms incorporated in the U.S. and listed on the NYSE, AMEX and NASDAQ stock exchanges. Table 1 reports the descriptive statistics for the 10 Fama-French industry portfolios and the 5 Fama-French factors, considering the period from July 1st, 1963 to April 30th, 2021.

[Please insert Table 1 around here]

The average return across industry portfolios is above 0.0473%, with the business equipment industry (HITEC) reaching a level of 0.0522% and healthcare, medical equipment and drugs industry (HLTH) providing an average of 0.0516%. The median return across industry portfolios is above 0.030%, with the industry portfolio called OTHER reaching a level of 0.080%. The standard deviation is equal to 1.425% for HITEC industry portfolio and declines to 0.884% for the portfolio UTILS. The level of the kurtosis increases from 12.308 for HITEC industry portfolio, reaching a level of 20.945 for the manufacturing industry (MANUF), to 29.435 for the portfolio UTILS.

4. The Econometric methodology

This section proposes the econometric framework able to estimate the parameters that determine the construction of the indicator. The framework is applied to the 10 Fama-French value weighted U.S. industry portfolios and consider the Diagonal BEKK(1,1) specification for jointly estimating the dynamic of the portfolios and the co-movements, with a multivariate *t-student* distribution of the innovations. Therefore,

$$R_{DURBL,t} - \mu_{f,t} = c_0 + \varepsilon_{DURBL,t} \tag{4}$$
$$R_{ENRGY,t} - \mu_{f,t} = c_1 + \varepsilon_{ENRGY,t} \tag{5}$$
$$R_{HITEC,t} - \mu_{f,t} = c_2 + \varepsilon_{HITEC,t} \tag{6}$$
$$R_{HLTH,t} - \mu_{f,t} = c_3 + \varepsilon_{HLTH,t} \tag{7}$$
$$R_{MANUF,t} - \mu_{f,t} = c_4 + \varepsilon_{MANUF,t} \tag{8}$$
$$R_{NODUR,t} - \mu_{f,t} = c_5 + \varepsilon_{NODUR,t} \tag{9}$$
$$R_{OTHER,t} - \mu_{f,t} = c_6 + \varepsilon_{OTHER,t} \tag{10}$$
$$R_{SHOPS,t} - \mu_{f,t} = c_7 + \varepsilon_{SHOPS,t} \tag{11}$$
$$R_{TELCM,t} - \mu_{f,t} = c_8 + \varepsilon_{TELCM,t} \tag{12}$$
$$R_{UTILS,t} - \mu_{f,t} = c_9 + \varepsilon_{UTILS,t} \tag{13}$$
$$R_{M,t} - \mu_{f,t} = c_{10} + \varepsilon_{M,t} \tag{14}$$

where, $R_{DURBL,t}$, $R_{ENRGY,t}$, $R_{HITEC,t}$, $R_{HLTH,t}$, $R_{MANUF,t}$, $R_{NODUR,t}$, $R_{OTHER,t}$, $R_{SHOPS,t}$,

$R_{TELCM,t}, R_{UTILS,t}, R_{M,t}$ are respectively the returns on the DURBL, ENRGY, HITEC, HLTH, MANUF, NODUR, OTHER, SHOPS, TELCM, UTILS and market portfolios.

The residuals for the equations (4)-(14) are respectively represented with $\varepsilon_{DURBL,t}, \varepsilon_{ENRGY,t}, \varepsilon_{HITEC,t}, \varepsilon_{HLTH,t}, \varepsilon_{MANUF,t}, \varepsilon_{NODUR,t}, \varepsilon_{OTHER,t}, \varepsilon_{SHOPS,t}, \varepsilon_{TELCM,t}, \varepsilon_{UTILS,t}, \varepsilon_{M,t}$. Therefore, the conditional variance processes for the U.S. industry portfolios, provided the information set at time $t-1$, are computed in the following way:

$$E\left[\varepsilon^2_{DURBL,t} \,|\, F_{t-1}\right] = \sigma^2_{DURBL,t} = \alpha_0 + \beta_0^2 \cdot \varepsilon^2_{DURBL,t-1} + \delta_0^2 \cdot \sigma^2_{DURBL,t-1} \tag{15}$$

$$E\left[\varepsilon^2_{ENRGY,t} \,|\, F_{t-1}\right] = \sigma^2_{ENRGY,t} = \alpha_1 + \beta_1^2 \cdot \varepsilon^2_{ENRGY,t-1} + \delta_1^2 \cdot \sigma^2_{ENRGY,t-1} \tag{16}$$

$$E\left[\varepsilon^2_{HITEC,t} \,|\, F_{t-1}\right] = \sigma^2_{HITEC,t} = \alpha_2 + \beta_2^2 \cdot \varepsilon^2_{HITEC,t-1} + \delta_2^2 \cdot \sigma^2_{HITEC,t-1} \tag{17}$$

$$E\left[\varepsilon^2_{HLTH,t} \,|\, F_{t-1}\right] = \sigma^2_{HLTH,t} = \alpha_3 + \beta_3^2 \cdot \varepsilon^2_{HLTH,t-1} + \delta_3^2 \cdot \sigma^2_{HLTH,t-1} \tag{18}$$

$$E\left[\varepsilon^2_{MANUF,t} \,|\, F_{t-1}\right] = \sigma^2_{MANUF,t} = \alpha_4 + \beta_4^2 \cdot \varepsilon^2_{MANUF,t-1} + \delta_4^2 \cdot \sigma^2_{MANUF,t-1} \tag{19}$$

$$E\left[\varepsilon^2_{NODUR,t} \,|\, F_{t-1}\right] = \sigma^2_{NODUR,t} = \alpha_5 + \beta_5^2 \cdot \varepsilon^2_{NODUR,t-1} + \delta_5^2 \cdot \sigma^2_{NODUR,t-1} \tag{20}$$

$$E\left[\varepsilon^2_{OTHER,t} \,|\, F_{t-1}\right] = \sigma^2_{OTHER,t} = \alpha_6 + \beta_6^2 \cdot \varepsilon^2_{OTHER,t-1} + \delta_6^2 \cdot \sigma^2_{OTHER,t-1} \tag{21}$$

$$E\left[\varepsilon^2_{SHOPS,t} \,|\, F_{t-1}\right] = \sigma^2_{SHOPS,t} = \alpha_7 + \beta_7^2 \cdot \varepsilon^2_{SHOPS,t-1} + \delta_7^2 \cdot \sigma^2_{SHOPS,t-1} \tag{22}$$

$$E\left[\varepsilon^2_{TELCM,t} \,|\, F_{t-1}\right] = \sigma^2_{TELCM,t} = \alpha_8 + \beta_8^2 \cdot \varepsilon^2_{TELCM,t-1} + \delta_8^2 \cdot \sigma^2_{TELCM,t-1} \tag{23}$$

$$E\left[\varepsilon^2_{UTILS,t} \,|\, F_{t-1}\right] = \sigma^2_{UTILS,t} = \alpha_9 + \beta_9^2 \cdot \varepsilon^2_{UTILS,t-1} + \delta_9^2 \cdot \sigma^2_{UTILS,t-1} \tag{24}$$

$$E\left[\varepsilon^2_{M,t} \,|\, F_{t-1}\right] = \sigma^2_{M,t} = \alpha_{10} + \beta_{10}^2 \cdot \varepsilon^2_{M,t-1} + \delta_{10}^2 \cdot \sigma^2_{M,t-1}, \tag{25}$$

where, the quantities $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}$ are the diagonal coefficients that depict the long term components of the conditional variances and the conditional covariances; $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}$ are the diagonal coefficients that depict the influence of the squared residuals at time $t-1$; whereas, $\delta_0, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}$ are the diagonal coefficients that depict the persistence of the conditional variances. The conditional betas can be computed in the following way:

$$b_{DURBL,t} = \frac{E\left[\varepsilon_{DURBL,t} \cdot \varepsilon_{M,t} \mid F_{t-1}\right]}{E\left[\varepsilon_{M,t}^2 \mid F_{t-1}\right]} = \frac{\rho_{DURBL\,M,t} \cdot \sigma_{DURBL,t} \cdot \sigma_{M,t}}{\sigma_{M,t}^2} = \frac{\rho_{DURBL\,M,t} \cdot \sigma_{DURBL,t}}{\sigma_{M,t}} \tag{26}$$

$$b_{ENERGY,t} = \frac{E\left[\varepsilon_{ENRGY,t} \cdot \varepsilon_{M,t} \mid F_{t-1}\right]}{E\left[\varepsilon_{M,t}^2 \mid F_{t-1}\right]} = \frac{\rho_{ENRGY\,M,t} \cdot \sigma_{ENRGY,t} \cdot \sigma_{M,t}}{\sigma_{M,t}^2} = \frac{\rho_{ENRGY\,M,t} \cdot \sigma_{ENRGY,t}}{\sigma_{M,t}} \tag{27}$$

$$b_{HITEC,t} = \frac{E\left[\varepsilon_{HITEC,t} \cdot \varepsilon_{M,t} \mid F_{t-1}\right]}{E\left[\varepsilon_{M,t}^2 \mid F_{t-1}\right]} = \frac{\rho_{HITEC\,M,t} \cdot \sigma_{HITEC,t} \cdot \sigma_{M,t}}{\sigma_{M,t}^2} = \frac{\rho_{HITEC\,M,t} \cdot \sigma_{HITEC,t}}{\sigma_{M,t}} \tag{28}$$

$$b_{HLTH,t} = \frac{E\left[\varepsilon_{HLTH,t} \cdot \varepsilon_{M,t} \mid F_{t-1}\right]}{E\left[\varepsilon_{M,t}^2 \mid F_{t-1}\right]} = \frac{\rho_{HLTH\,M,t} \cdot \sigma_{HLTH,t} \cdot \sigma_{M,t}}{\sigma_{M,t}^2} = \frac{\rho_{HLTH\,M,t} \cdot \sigma_{HLTH,t}}{\sigma_{M,t}} \tag{29}$$

$$b_{MANUF,t} = \frac{E\left[\varepsilon_{MANUF,t} \cdot \varepsilon_{M,t} \mid F_{t-1}\right]}{E\left[\varepsilon_{M,t}^2 \mid F_{t-1}\right]} = \frac{\rho_{MANUF\,M,t} \cdot \sigma_{MANUF,t} \cdot \sigma_{M,t}}{\sigma_{M,t}^2} = \frac{\rho_{MANUF\,M,t} \cdot \sigma_{MANUF,t}}{\sigma_{M,t}} \tag{30}$$

$$b_{NODUR,t} = \frac{E\left[\varepsilon_{NODUR,t} \cdot \varepsilon_{M,t} \mid F_{t-1}\right]}{E\left[\varepsilon_{M,t}^2 \mid F_{t-1}\right]} = \frac{\rho_{NODUR\,M,t} \cdot \sigma_{NODUR,t} \cdot \sigma_{M,t}}{\sigma_{M,t}^2} = \frac{\rho_{NODUR\,M,t} \cdot \sigma_{NODUR,t}}{\sigma_{M,t}} \tag{31}$$

$$b_{OTHER,t} = \frac{E\left[\varepsilon_{OTHER,t} \cdot \varepsilon_{M,t} \mid F_{t-1}\right]}{E\left[\varepsilon_{M,t}^2 \mid F_{t-1}\right]} = \frac{\rho_{OTHER\,M,t} \cdot \sigma_{DURBL,t} \cdot \sigma_{M,t}}{\sigma_{M,t}^2} = \frac{\rho_{OTHER\,M,t} \cdot \sigma_{DURBL,t}}{\sigma_{M,t}} \tag{32}$$

$$b_{SHOPS,t} = \frac{E\left[\varepsilon_{SHOPS,t} \cdot \varepsilon_{M,t} \mid F_{t-1}\right]}{E\left[\varepsilon_{M,t}^2 \mid F_{t-1}\right]} = \frac{\rho_{SHOPS\,M,t} \cdot \sigma_{SHOPS,t} \cdot \sigma_{M,t}}{\sigma_{M,t}^2} = \frac{\rho_{SHOPS\,M,t} \cdot \sigma_{SHOPS,t}}{\sigma_{M,t}} \tag{33}$$

$$b_{TELCM,t} = \frac{E\left[\varepsilon_{TELCM,t} \cdot \varepsilon_{M,t} \mid F_{t-1}\right]}{E\left[\varepsilon_{M,t}^2 \mid F_{t-1}\right]} = \frac{\rho_{TELCM\,M,t} \cdot \sigma_{TELCM,t} \cdot \sigma_{M,t}}{\sigma_{M,t}^2} = \frac{\rho_{TELCM\,M,t} \cdot \sigma_{TELCM,t}}{\sigma_{M,t}} \tag{34}$$

$$b_{UTILS,t} = \frac{E\left[\varepsilon_{UTILS,t} \cdot \varepsilon_{M,t} \mid F_{t-1}\right]}{E\left[\varepsilon_{M,t}^2 \mid F_{t-1}\right]} = \frac{\rho_{UTILS\,M,t} \cdot \sigma_{UTILS,t} \cdot \sigma_{M,t}}{\sigma_{M,t}^2} = \frac{\rho_{UTILS\,M,t} \cdot \sigma_{UTILS,t}}{\sigma_{M,t}} \tag{35}$$

where, $\rho_{DURBL\,M,t}$, $\rho_{ENRGY\,M,t}$, $\rho_{HITEC\,M,t}$, $\rho_{HLTH\,M,t}$, $\rho_{MANUF\,M,t}$, $\rho_{NODUR\,M,t}$, $\rho_{OTHER\,M,t}$, $\rho_{SHOPS\,M,t}$, $\rho_{TELCM\,M,t}$, $\rho_{UTILS\,M,t}$ respectively represent the conditional correlations of each industry portfolio with the market portfolio, jointly estimated with the Diagonal BEKK(1,1) specification, with a multivariate *t-student* distribution of the innovations. Now, the average conditional beta across industry portfolios is computed in the following way:

$$\bar{\beta}_{1,t} = \frac{b_{DURBL,t} + b_{ENRGY,t} + b_{HITEC,t} + b_{HLTH,t} + b_{MANUF,t} + b_{NODUR,t} + b_{OTHER,t} + b_{SHOPS,t} + b_{TELCM,t} + b_{UTILS,t}}{10}. \tag{36}$$

The conditional variances of the average beta and each conditional stock beta follow an exponential GARCH(1,1) specification. Therefore, the conditional Bayesian beta derives by the application of the formula (3).

5. Empirical results

This section discusses the estimates and the empirical results of the econometric methodology proposed in Section 4 for estimating the components that determine the Bayesian conditional beta. The estimation results rely on the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm that is an iterative method for solving unconstrained nonlinear optimization problems. It belongs to quasi-Newton methods and seeks a stationary point of a function, reachable when the gradient is zero. The optimization algorithm begins at an initial estimate for the optimal values and proceeds iteratively to get better estimates at each stage, till when there is a convergence for finding the solutions. For simplicity, the maximum number of iterations is fixed to n. 5,000 and the convergence rate to 1e-06.

The step method is based on the Levenberg-Marquardt algorithm that is more robust than the Gauss-Newton algorithm, since it allows to derive solutions even if the algorithm starts very far off from the final minimum. In cases with multiple minima, the algorithm converges to the global minimum only if the initial guess is already somewhat close to the final solution. The estimation procedure also accommodates the Huber-White estimator that allows to derive the variance/covariance matrix considering the heteroscedasticity of the residuals.

[Please Insert Table 2 around here]

The results of the estimates are reported in Table 2. The constants of the mean equations for the industry portfolios have values that range from 0.000171 to 0.003070; whereas, the constant for the market portfolio is equal to 0.001772. The coefficients that depict the persistence of the conditional variances are positive and statistically significant with a value of the t-student equals to 9.963841. Therefore, an increase of the conditional variances for the previous trading day allows to increment the actual conditional variances.

[Please Insert Figure 1 around here]

Figure 1 reports the dynamic of the conditional betas across industry portfolios. For eight out of ten sub-plots, the figure show a minimum around the third quarter of the year 2000 and a monotonic increase of the conditional betas from that period of time. The variances of the conditional betas are modelled with EGARCH(1,1) specifications (Nelson 1991) and the estimated coefficients are reported in Table 3. The coefficient that depicts the asymmetry component is negative for six out of ten industry portfolios; whereas it is positive for four out of ten industry portfolios. The coefficient that depicts the persistence of the variance component has values between 0.91359 (MANUF) and 0.93752 (NODUR).

[Please Insert Table 3 around here]

The dynamic of the variance for the conditional betas and the dynamic of the variance for all conditional betas are reported in Figure 2. The variance of the conditional betas spikes around the third/fourth quarter of the year 2000, due to a dramatic decline of the conditional betas, implying a decline of the sensitivities of the industry portfolios with respect to the market portfolio.

[Please Insert Figure 2 around here]

The estimated components allow to determine the time varying Bayesian beta also called as generalized version of the Vasicek's beta. Figure 3 shows the average dynamic Vasicek's betas and the average dynamic conditional betas, from July 1st, 1963 to April 30th, 2021. The mean value of the difference between the average dynamic Vasicek's betas and the average dynamic conditional betas is equal to 0.001640 and the difference reports a minimum value equals to -0.076891 with a standard deviation equals to 0.021312. Therefore, the average dynamic time varying Bayesian beta tends to produce higher values than the average dynamic conditional betas, due to the weights used for weighting the betas in the computation of the time varying Bayesian beta.

[Please Insert Figure 3 around here]

The estimated coefficients related to the variance of the time varying Bayesian betas are reported in Table 4. The coefficient that depicts the asymmetry component is negative for six out of ten industry portfolios; whereas it is positive for four out of ten industry portfolios. The coefficient that depicts the persistence of the variance component has values between 0.88811 (HITEC) and 0.95761 (ENRGY).

[Please Insert Table 4 around here]

5.1 The short term and the long term predictions of the time varying Bayesian beta

This subsection discusses the short and the long term predictions of the time varying Bayesian beta at 5 days, 22 days, 66 days and 132 days ahead. Table 5 and Table 6 report the results of the robust regressions able to predict the values of the time varying Bayesian beta at several horizons ahead. The robust regressions rely on the Huber-Bisquare objective function for the residuals, with a scale function based on the Median Absolute Deviation, centered around the Median (MADMED).

[Please Insert Table 5 and Table 6 around here]

The presence of heterogeneity across traders is relevant in the estimation of the coefficients that depict this effect. The estimated coefficient beta_vas(-1) of the robust regressions is positive and statistically significant. It tends to decline along the horizons of predictability and across the industry portfolios; whereas, the coefficient beta_vas(-66) and beta_vas(-132) tend to increase along the predicted periods of time.

6. Conclusions

Modern finance relies heavily on estimates of systematic risk beta for testing assets pricing theory, for testing trading strategies and conducting event studies.

This paper proposes a generalized version of the Vasicek's beta able to accommodate the time varying dynamic of the weights that determine the sensitivity of an asset with respect to the market portfolio. The theoretical framework is constructed above the Diagonal BEKK(1,1) specification for jointly estimating the dynamics of the conditional variances and covariances, with a multivariate *t-student* distribution of the innovations.

The empirical results also discuss the short and the long term predictions of the estimator, constructed under the so called Heterogeneous Market Hypothesis. The short and the long term predictions are based on robust regressions that rely on the Huber-Bisquare objective function for the residuals, with a scale function based on the Median Absolute Deviation, centered around the Median (MADMED).

The estimator can be used by market participants and portfolio managers for better estimating the sensitivity of an asset with respect to the market portfolio.

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Table 1.
Descriptive statistics

Table 1 reports the summary descriptive statistics (mean, median, max., min., std. dev., skewness and kurtosis) for the 10 Fama-French industry portfolios, considering the period July 1st, 1963 to April 30th, 2021. The Fama-French industry portfolios are respectively: DURBL, ENRGY, HITEC, HLTH, MANUF, NODUR, OTHER, SHOPS, TELCM, UTILS.

	DURBL	ENRGY	HITEC	HLTH	MANUF	NODUR	OTHER	SHOPS	TELCM	UTILS
Mean	0.000470	0.000464	0.000522	0.000516	0.000470	0.000499	0.000458	0.000511	0.000424	0.000394
Median	0.000400	0.000500	0.000750	0.000600	0.000700	0.000700	0.000800	0.000700	0.000300	0.000500
Maximum	0.150300	0.193300	0.160400	0.111000	0.108300	0.102400	0.122400	0.109900	0.144700	0.144300
Minimum	-0.183500	-0.197300	-0.199800	-0.178900	-0.200100	-0.170300	-0.152500	-0.167400	-0.166900	-0.128600
Std. Dev.	0.013886	0.013896	0.014247	0.010830	0.010674	0.008854	0.011692	0.010643	0.011001	0.008845
Skewness	-0.214765	-0.247318	-0.027068	-0.384500	-0.669522	-0.642826	-0.312265	-0.319099	-0.088189	-0.009154
Kurtosis	13.27463	20.92763	12.30751	13.83414	20.94455	22.20735	18.67376	14.19476	16.66231	29.43476

Table 2.
Diagonal BEKK(1,1) model

Table 2 reports the coefficient values, the standard errors and the *z*-statistics for the mean equations and the variance equations of the Diagonal BEKK(1,1) model. The estimation period is from July 1st, 1963 to April 30th, 2021.

	Coefficient	Std. Error	z-Statistic
c_0	0.001030	0.000140	7.357186
c_1	0.000495	0.000127	3.896538
c_2	0.003070	0.000116	26.56187
c_3	0.001480	0.000113	13.05389
c_4	0.000912	0.000109	8.360437
c_5	0.000761	9.18E-05	8.293408
c_6	0.000831	0.000101	8.242770
c_7	0.001476	0.000109	13.56077
c_8	0.000954	0.000109	8.782445
c_9	0.000171	7.31E-05	2.339889
c_{10}	0.001772	9.68E-05	18.30508
	Variance Equation Coefficients		
α_0	4.21E-08	4.98E-09	8.455090
α_1	1.97E-08	2.36E-09	8.333966
α_2	5.26E-09	1.12E-09	4.713438
α_3	3.50E-08	3.45E-09	10.12747
α_4	1.06E-08	9.47E-10	11.18561
α_5	2.69E-08	2.17E-09	12.39670
α_6	1.05E-08	1.05E-09	10.08094
α_7	3.20E-08	2.90E-09	11.00883
α_8	3.93E-08	4.24E-09	9.268613
α_9	2.62E-08	2.47E-09	10.58124
α_{10}	-1.92E-09	1.53E-10	-12.54536
β_0	0.139024	0.002966	46.86561
β_1	0.144964	0.002902	49.94892
β_2	0.142852	0.002883	49.54626
β_3	0.140587	0.003381	41.57736
β_4	0.142133	0.002957	48.06314
β_5	0.136068	0.003001	45.34476
β_6	0.145009	0.003000	48.34060
β_7	0.139099	0.003116	44.64391
β_8	0.134616	0.003227	41.71846
β_9	0.143749	0.003608	39.84194
β_{10}	0.141570	0.002988	47.37869
δ_0	0.990837	0.000368	2692.893
δ_1	0.990168	0.000368	2691.431
δ_2	0.990510	0.000348	2843.425
δ_3	0.990460	0.000429	2309.900
δ_4	0.990449	0.000372	2665.704
δ_5	0.990927	0.000373	2657.109
δ_6	0.990000	0.000388	2553.132
δ_7	0.990644	0.000390	2539.860
δ_8	0.991188	0.000397	2494.128
δ_9	0.989826	0.000467	2118.740
δ_{10}	0.990499	0.000379	2616.318
t	9.963841	0.238582	41.76282

Table 3.
Variance of the Dynamic Betas

Table 3 reports the coefficients related to the estimation of the variance processes for the dynamic betas of the Fama-French industry portfolios. The estimation period is from July 1st, 1963 to April 30th, 2021. The significance levels at 1%, 5% and 10% are respectively represented in the following way: ***, **, *.

	DURBL	**ENRGY**	**HITEC**	**HLTH**	**MANUF**	**NODUR**	**OTHER**	**SHOPS**	**TELCM**	**UTILS**
ϕ_0	1.16703***	0.98577***	1.32214***	0.98548***	1.02273***	0.73661***	0.99220***	1.00735***	0.83266***	0.51092***
ϕ_1	-1.42945***	-1.47314***	-1.44083***	-1.44215***	-1.61896***	-1.52560***	-1.49898***	-1.34547***	-1.46340***	-1.61366***
ϕ_2	1.13605***	1.27172***	1.17763***	1.18704***	1.17761***	1.27436***	1.13871***	1.07088***	1.20367***	1.30029***
ϕ_3	-0.00467	-0.10341***	0.00568	-0.01026***	-0.00716	0.01073***	0.02647***	-0.02185***	-0.00673*	0.00118
ϕ_4	0.92107***	0.93365***	0.92408***	0.93120***	0.91359***	0.93752***	0.92314***	0.93474***	0.92990***	0.91967***
t	340.8403***	340.8118***	340.8364***	340.8223***	340.8409***	340.8211***	179.9338	340.8312***	247.9139	340.8447***

Table 4.
Variance of the Vasicek's betas

The table reports the coefficients related to the estimation of the variance processes for the dynamic Vasicek's betas of the Fama-French industry portfolios. The estimation period is from July 1st, 1963 to April 30th, 2021. The significance levels at 1%, 5% and 10% are respectively represented in the following way: ***, **, *.

	DURBL	ENRGY	HITEC	HLTH	MANUF	NODUR	OTHER	SHOPS	TELCM	UTILS
γ_0	0.96368***	0.96417***	0.96281***	0.96411***	0.96422***	0.96159***	0.97272***	0.95914***	0.96315***	0.96314***
γ_1	-1.82132***	-1.19209***	-2.32421***	-1.26365***	-1.74894***	-2.03581***	-1.69631***	-1.90453***	-1.66039***	-2.30569***
γ_2	1.34216***	0.91870***	1.60522***	0.94322***	1.24443***	1.52714***	1.11035***	1.34570***	1.21088***	1.67124***
γ_3	0.03311***	-0.00724***	0.12166***	-0.00972***	0.01252**	-0.07509***	-0.02907***	0.00817***	-0.04797***	-0.15184***
γ_4	0.92129***	0.95761***	0.88811***	0.95112***	0.92531***	0.91590***	0.92068***	0.91590***	0.93098***	0.89350***
t	159.2822***	340.8088***	101.7028***	340.8411***	340.8051***	161.5891***	340.8412***	340.8305***	340.6182	67.7242***

Table 5.
The SHORT term predictions

The table reports the short term predictions for the Vasicek's betas related to the Fama-French industry portfolios. The predictions respectively consider 5 days ahead (Panel 5.1) and 22 days ahead (Panel 5.2). The significance levels at 1%, 5% and 10% are respectively represented in the following way: ***, **, *.

Panel 5.1: The SHORT term predictions (5 days ahead)

	DURBL	ENRGY	HITEC	HLTH	MANUF	NODUR	OTHER	SHOPS	TELCM	UTILS
c	0.00698***	0.00522***	0.02914***	0.00588***	0.00621***	0.00321***	0.00171	0.01029***	0.00638***	0.00879***
Beta_vas(-1)	0.99945***	0.95137***	0.96598***	0.97981***	0.99486***	1.00005***	0.96332***	0.99199***	0.96207***	1.01305***
Beta_vas(-5)	-0.02081***	0.05210***	-0.00711***	0.02232***	0.00329	-0.00369	0.01855***	0.00343	0.03339***	-0.02325***
Beta_vas(-22)	0.00148	-0.00556	0.00579***	-0.00342	-0.00716**	0.00388**	-0.00476*	-0.00272	-0.00927***	-0.00414***
Beta_vas(-66)	0.00551***	-0.00344	0.00200**	-0.00247	-0.00245	-0.00219*	0.01793***	-0.00337*	0.00454***	0.00291***
Beta_vas(-132)	0.00712***	-0.00001	0.00318***	-0.00236	0.00503***	-0.00162	0.00311*	-0.00001	0.00254	0.00156**
Rw-squared	**99.74%**	**99.87%**	**99.46%**	**99.83%**	**99.82%**	**99.89%**	**99.53%**	**99.76%**	**99.66%**	**99.83%**

Panel 5.2: The SHORT term predictions (22 days ahead)

	DURBL	ENRGY	HITEC	HLTH	MANUF	NODUR	OTHER	SHOPS	TELCM	UTILS
c	0.04412***	0.02123***	0.15015***	0.02456***	0.01650***	0.00518***	0.06696***	0.05534***	0.04024***	0.04949***
Beta_vas(-1)	0.76542***	0.85163***	0.77479***	0.85133***	0.94772***	0.84391***	0.76810***	0.93113***	0.86288***	0.89564***
Beta_vas(-5)	0.09100***	0.11591***	0.00895**	0.13689***	0.00847	0.08417***	0.06362***	0.01714	0.00281	-0.00543
Beta_vas(-22)	0.04103***	0.00467	0.03334***	-0.01876**	-0.00165	0.07406***	0.02177***	-0.00776	0.03327***	0.03609***
Beta_vas(-66)	0.02338***	0.00238	0.01968***	0.01301**	-0.00660	-0.01754***	0.06709***	-0.01156**	0.00407	0.00349*
Beta_vas(-132)	0.03266***	0.00243	0.00753***	-0.00821**	0.03428***	0.00898	0.01001***	0.01348***	0.05511***	0.01776***
Rw-squared	**98.36%**	**99.25%**	**96.30%**	**99.19%**	**98.76%**	**99.44%**	**97.03%**	**98.64%**	**98.22%**	**98.61%**

Table 6.
The LONG term predictions

The table reports the short term predictions for the Vasicek's betas related to the Fama-French industry portfolios. The predictions respectively consider 66 days ahead (Panel 6.1) and 132 days ahead (Panel 6.2). The significance levels at 1%, 5% and 10% are respectively represented in the following way: ***, **, *.

Panel 6.1: The LONG term predictions (66 days ahead)

	DURBL	ENRGY	HITEC	HLTH	MANUF	NODUR	OTHER	SHOPS	TELCM	UTILS
c	0.18314***	0.11572***	0.50431***	0.16328***	0.17681***	0.07782***	0.25937***	0.16970***	0.15135***	0.27579***
Beta_vas(-1)	0.39901***	0.70433***	0.20323***	0.64904***	0.64300***	0.75741***	0.33109***	0.62236***	0.46727***	0.42526***
Beta_vas(-5)	0.07592***	0.06138***	0.09300***	0.09006***	0.05615***	0.08587***	0.13744***	0.10128***	0.09441***	0.04924***
Beta_vas(-22)	0.09770***	0.04194***	0.07790***	0.06656***	-0.05015***	0.05706***	0.12374***	0.07624***	0.05873***	0.06823***
Beta_vas(-66)	0.06231***	0.02727***	0.07408***	-0.03203***	-0.05494***	-0.05095***	0.09988***	-0.06077***	0.07651***	0.05606***
Beta_vas(-132)	0.17162***	0.04346***	0.03043***	0.05694***	0.22330***	0.06746***	0.04032***	0.08553***	0.14588***	0.10635***
Rw-squared	**91.31%**	**93.27%**	**77.61%**	**93.58%**	**93.23%**	**96.33%**	**87.54%**	**94.49%**	**93.43%**	**85.29%**

Panel 6.2: The LONG term predictions (132 days ahead)

	DURBL	ENRGY	HITEC	HLTH	MANUF	NODUR	OTHER	SHOPS	TELCM	UTILS
c	0.30604***	0.35091***	0.65625***	0.37843***	0.29868***	0.26369***	0.36773***	0.33187***	0.24013***	0.49077***
Beta_vas(-1)	0.15552***	0.39682***	0.10390***	0.54038***	0.32030***	0.41657***	0.33814***	0.37056***	0.24404***	0.09848***
Beta_vas(-5)	0.07900***	0.05840**	0.04288***	0.04741	0.02829	0.06269***	0.10649***	0.00950	0.04493*	0.06540**
Beta_vas(-22)	0.09455***	0.02109	0.02105***	-0.05282***	-0.03464**	0.05849***	0.00306	-0.05959***	0.14207***	0.11307***
Beta_vas(-66)	0.19307***	0.16772***	0.08337***	0.04680***	0.16879***	0.05557***	0.14951***	0.13018***	0.12919***	0.11878***
Beta_vas(-132)	0.15712***	-0.00825	0.07363***	0.02751***	0.21032***	0.13092	0.02349***	0.20798***	0.18916***	0.07061***
Rw-squared	**84.67%**	**86.34%**	**58.68%**	**83.15%**	**86.70%**	**91.55%**	**82.61%**	**81.68%**	**88.11%**	**54.35%**

Figure 1.
The Dynamic Conditional Betas

The figure shows the dynamics of the conditional betas for the 10 Fama-French industry portfolios, from July 1st, 1963 to April 30th, 2021. The estimation of the dynamic conditional betas is based on the Diagonal BEKK(1,1) specification.

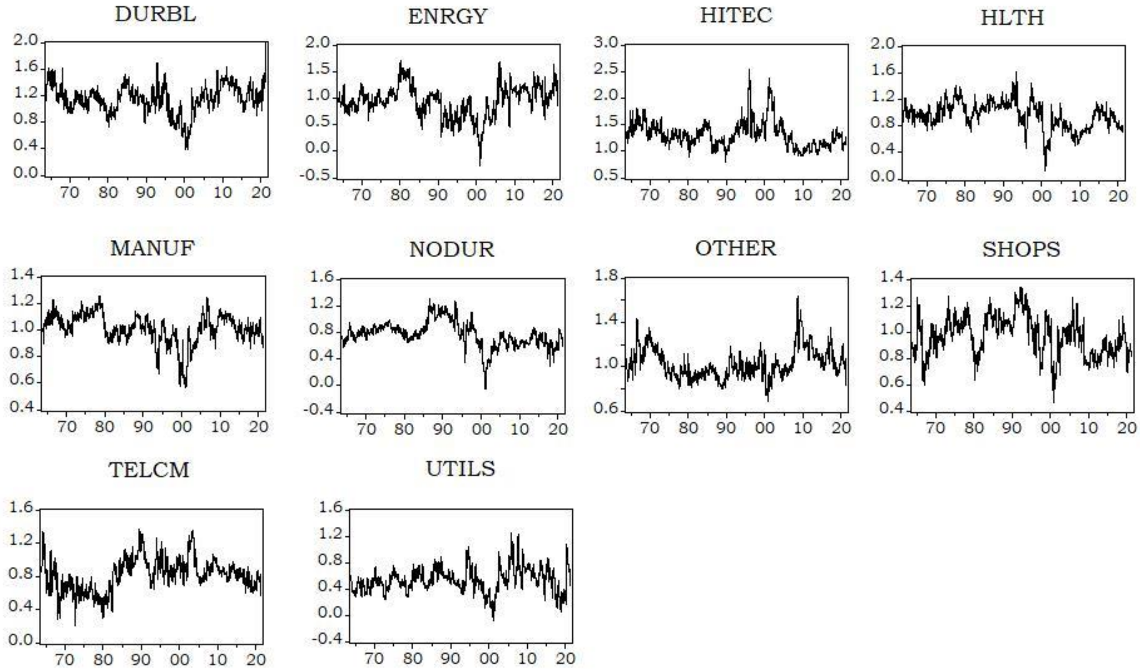

Figure 2.
Conditional Variance of the Betas

The figure shows the conditional variance of the dynamic conditional betas (Figure 2.1) as well as the conditional variance for all the betas (Figure 2.2), from July 1st, 1963 to April 30th, 2021. The conditional variance is based on an Exponential GARCH(1,1) specification.

Figure 2.1.
Conditional Variances of the Dynamic Conditional Betas

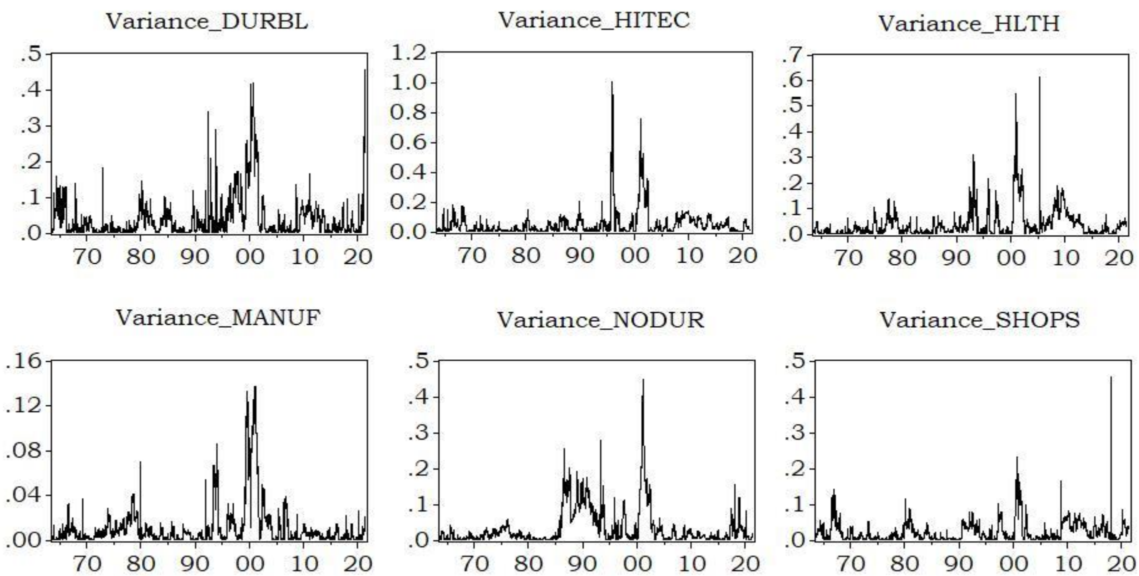

Figure 2.2
Conditional Variance for ALL Dynamic Conditional Betas

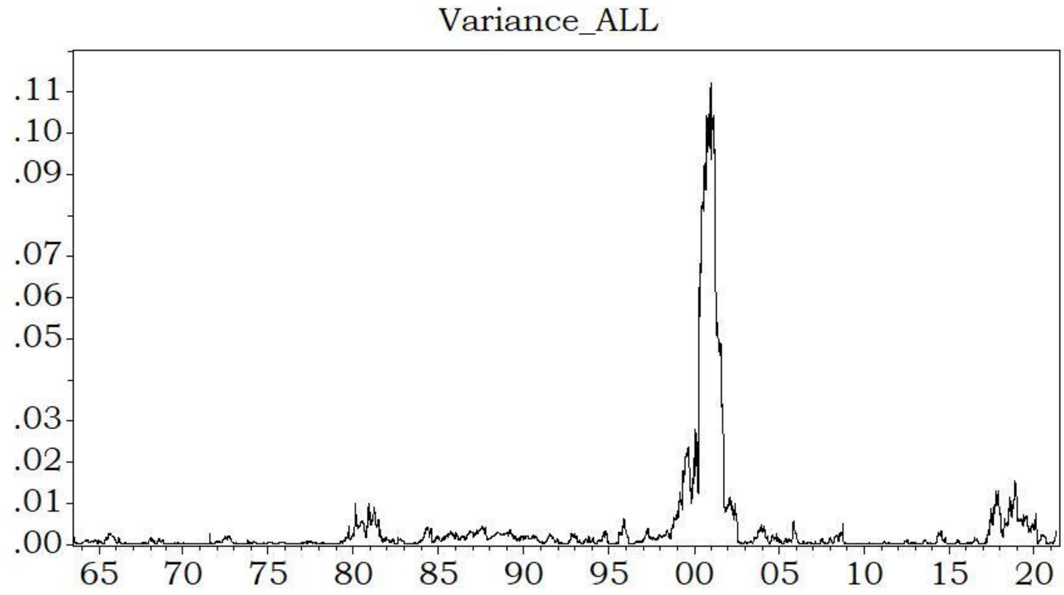

Figure 3.
The Average Dynamic Vasicek Betas vs. The Average Dynamic Conditional Betas
The figure shows the average dynamic Vasicek Betas vs. the average dynamic conditional Betas, from July 1st, 1963 to April 30th, 2021. The estimation procedure relies on the Diagonal BEKK(1,1) specification.