Collateralized Debt Networks with Lender Default *

Jin-Wook Chang †

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Abstract

The Lehman Brothers' 2008 bankruptcy spread losses to its counterparties even when Lehman was a lender of cash, because collateral for that lending was tied up in the bankruptcy process. I study the implications of such lender default using a general equilibrium network model featuring endogenous leverage, endogenous asset prices, and endogenous network formation. The multiplex graph model has two channels of contagion: a counterparty channel of contagion and a price channel of contagion through endogenous collateral price. Borrowers diversify their lenders because of the counterparty risk, but they have to deal with lenders who lend at a higher margin. This diversification generates positive externalities by reducing systemic risk, but any decentralized equilibrium is constrained inefficient due to under-diversification. The key externalities here, arising from the tradeoff between counterparty risk and leverage (margin), are absent in models with exogenous leverage or exogenous networks. I use this framework to analyze the introduction of a central counterparty (CCP). I show that the loss coverage by the CCP reduces diversification incentives and exacerbates the externality problem which can rather increase systemic risk.

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†Board of Governors of the Federal Reserve System, *Email:* jin-wook.chang@frb.gov .

1. Introduction

The failure of the collateralized debt market was one of the major contributors of the financial crisis in 2008 (Gorton and Metrick, 2012; Copeland et al., 2014).[1] This paper studies such a market in a hybrid model that combines general equilibrium and network frameworks. A typical form of collateralized debt takes the form of one-on-one interaction between two counterparties – a borrower and a lender – because of customization (bespoke) of contract terms. A collateralized debt network, the collection of such one-on-one relationships, has two transmission channels of shocks – the price channel and the counterparty channel. The collapse in the prices of subprime mortgages in 2007 had a direct effect on many financial institutions that held related assets. But the initial shock was exacerbated by the resulting bankruptcy of the Lehman Brothers, which spread the losses to Lehman's counterparties (Copeland et al., 2014; De Haas and Van Horen, 2012; Singh, 2017). This counterparty loss triggered fire sales of assets which made prices to decline even further (Demange, 2016; Duarte and Eisenbach, 2018; Duarte and Jones, 2017). Therefore, a model that incorporates the interaction of both channels is necessary to capture the full picture of collateralized debt markets (Glasserman and Young, 2016).

Lender default can also be a source of a counterparty channel of shocks (Eren, 2014; Infante et al., 2018; Scott, 2014). Lehman's defaults on its *lender* obligations to return collateral to its borrowers caused a significant loss in 2008. All of Lehman Brothers' assets including borrowers' collateral were frozen under the bankruptcy procedure. Many borrowers had to over-collateralize[2] their positions to protect the lender (Lehman) in case of borrower default. While over-collateralization secured lender's position, it exposed the borrowers to losses when they could not recover their collateral. The borrowers did not know when their collateral would be returned to them, nor did they know how much they would recover from the bankruptcy process (Fleming and Sarkar, 2014) and paid a sizable cost throughout the bankruptcy and collateral recovery process.[3]

[1] Repurchase agreements (repo), asset-backed commercial papers (ABCP), and derivatives are typical examples of collateralized debt and the most common form of short-term financing among financial institutions.

[2] The financial market has evolved to insulate lenders from borrower counterparty risk over the past few decades. The evolution of securitization has made the cash flows of contracts remote from borrower bankruptcies and significantly eliminated tangible losses that could follow from borrower default (Gorton et al., 2010). However, borrowers are now exposed to the risk of lender default.

[3] Regardless of whether the borrowers stood to recover their assets over the long term, the inability to

The goal of this paper is to analyze such a collateralized debt market with both price channel and counterparty channel of contagion from lender default. The main research questions are the following. First, how do agents spread losses to each other through a given network of counterparties in a collateralized debt market? Second, how do counterparties form an endogenous debt network (borrow and lend to each other) when they account for this contagion channel? Third, how does regulation change the systemic risk when accounting for endogenous networks responses of the market?

I propose a general equilibrium model with multiplex network interaction featuring endogenous leverage (margin), endogenous price, and endogenous network formation to study this problem. The model has interaction between the price channel (fire sales) and the counterparty channel (both borrower and lender defaults) that affect endogenous network formation. This paper is the first attempt to endogenize leverage, asset prices, and network formation simultaneously, to the best of my knowledge.

Model. The model has six main features. First, agents trade an asset that can be used as collateral in a competitive market. Price changes in the asset market affect agents' nominal wealth as a *price channel*. Second, there is a multiplex network of collateralized borrowing and lending. Agents enter bilateral customized contracts specifying the face value of the debt and the amount of collateral. Third, agents disagree on the fair value of the asset ex ante. Agents trade the asset and use it as collateral to borrow and lend because of the *belief disagreement*. Fourth, the lender of a debt contract holds the collateral and can *reuse (rehypothecate)* it to borrow money from someone else. An agent can be a lender as well as a borrower at the same time. Fifth, agents are subject to *liquidity shocks* before paying back their debt. Because of this liquidity shock, agents may have negative nominal wealth and go bankrupt. Sixth, both borrower and lender defaults are considered. Borrowers must put up collateral, and failure to pay results in a costless transfer[4] of collateral to the lender. When the lender fails to return the collateral, the borrower has to go through a costly process to recover the collateral from the lender. This lender default cost[5] generates propagation

recover funds in the short term caused disruption for certain firms. MKM Longboat Capital Advisors closed its \$1.5 billion fund partly because of frozen assets, and the chief operating officer of Olivant Ltd. committed suicide, because the fund had \$1.4 billion value of assets, which was believed to be unlikely to recover from Lehman Brothers (Scott, 2014). Another good example is the case of MF Global, a prominent broker-dealer that went bankrupt in 2011. The bankruptcy procedure took nearly five years to resolve all the claims for customers and creditors of MF Global including their borrowers such as hedge funds. By the end of 2015, more than half of the borrowers recovered their collateral in full, but they had to go through the lengthy process of a bankruptcy procedure with considerable costs to stay involved and also could not access the assets that were used as collateral (SIPC, 2016).

[4]For example, typical repo contracts are exempt from automatic stay of bankruptcy provisions.

[5]The lender default cost is similar to the borrower default cost, which is prevalent in the literature. If there is a bankruptcy of a counterparty, then that will incur additional cost in terms of time, effort, and litigation costs, which are a deadweight loss to the economy. For example, there were over 100 hedge funds

through the *counterparty channel*.

Main results. The first implication of the model is that there is a tradeoff between counterparty risk and (contract-level) leverage which affects network formation. If there is no lender default cost, then a single intermediation chain is formed endogenously. Borrowers prefer to maximize their contract leverage (or minimize margin) by borrowing from the most favorable lender. The most optimistic agent borrows from the second-most optimistic agent, who borrows from the third-most optimistic agent, and so on. However, if there is a lender default cost, borrowers diversify their lenders because of the possibility of counterparty default losses[6]. The tradeoff between counterparty risk and leverage (margin) exists because borrowers have to deal with more pessimistic lenders who lend less for the same collateral.

The second implication of the model is that there are positive externalities from diversification. Diversification of counterparties reduces not only individual counterparty risk but also systemic risk by limiting the propagation of shocks and price volatility. If an intermediary becomes safer, then its borrowers become safer as well, so the aggregate counterparty risk becomes smaller. In addition, a lower level of debt leads to lower price volatility, making each agent's balance sheet more stable. Because agents do not fully internalize these externalities, any decentralized equilibrium is constrained inefficient because of under-diversification.

The third implication of the model is that the loss coverage by a central counterparty (CCP)[7] exacerbates the externality problems by eliminating individual agent's incentive to diversify. A CCP *novates* a contract between two counterparties—that is, replaces a contract between a borrower and a lender with two different contracts: a contract between the borrower and the CCP and a contract between the lender and the CCP. Novation procedure acts as a pooling of individual counterparty risks as the CCP handles and absorbs any losses from default costs. However, the tradeoff between counterparty risk and leverage disappears as individual counterparty risk is covered, and each agent will concentrate all of her borrowing with the single most favorable lender. The endogenous response to the introduction of CCP will transform the implicit network structure into a single-chain network, which arises in a decentralized equilibrium only if there is no default cost. Such reckless borrowing behavior increases systemic risk in the economy by increasing the riskiness of each agent's balance

that had prime brokerage accounts or debt obligations under Lehman Brothers, and these accounts were frozen during the bankruptcy of Lehman Brothers. These positions, valued at more than $400 billion, were frozen, which further exacerbated the liquidity shortage of the market (Lleo and Ziemba, 2014). The lender default occurred not only because of rehypothecation but also because of Lehman not holding the collateral in a segregated account (Fleming and Sarkar, 2014).

[6]This diversification of lender behavior is similar to firms hedging against bank lending channels by having multiple banks as lenders as in Khwaja and Mian (2008). The model in this paper has borrowers replacing firms and lender default as the risk the borrowers are hedging against.

[7] Introducing CCP is one of the key elements of the financial system reforms addressed by central banks and financial authorities after the financial crisis in 2008 (Singh, 2010).

sheet and price volatility. However, netting[8] conducted by the CCP decreases systemic risk so the overall effect to systemic risk is ambiguous.

The fourth implication of the model is that all of the agents hold positive amounts of cash in any equilibrium. If there is a crash in the asset price, the marginal utility of cash will become very high and a surviving agent can enjoy huge return from cash by buying up all the remaining assets at a cheap price. Thus, every agent holds a positive amount of cash. This competitive cash holding, due to general equilibrium effect from the asset market which is absent in typical network models, counteracts the risk-stacking behavior (increasing correlation with others) of agents that is common in financial networks literature.

Empirical facts. The model captures a few empirical facts. First, an increase in counterparty risk leads to an increase in the number of counterparties and a decrease in reuse of collateral and average leverage. Because agents want to diversify more when the counterparty risk increases, the number of linkages increase and the reuse of collateral decreases since the optimists borrow directly from the pessimists rather than indirectly through intermediation. As Singh (2011) documents, the velocity (reuse) of collateral decreased from 3 to 2.4 after the bankruptcy of Lehman Brothers, and the average leverage in the over-the-counter (OTC) market also went down. Also Craig and Von Peter (2014) show that the average number of linkages between financial institutions have increased about 30 percent over the four years after Lehman Bankruptcy. The opposite result happened in unsecured debt markets in which the banks reduced their number of counterparties (Afonso et al., 2011; Beltran et al., 2015). This stark comparison shows the role of collateral in network formation. Finally, Eren (2015) shows that hedge funds in such markets preferred to deal with more risky counterparties before the crisis, and they switched their counterparties to less risky ones after the crisis with a more diversified portfolio of counterparties. All of these empirical observations align well with the main dynamics of this paper.

Second, all of the agents hold positive amounts of cash in any equilibrium. In reality, even the most aggressive investors, such as hedge funds, tend to hold large amounts of daily liquidity that are almost equivalent to cash. In the model, even the most optimistic agents would like to hold some cash to prepare for the case of severe liquidity shocks, which may push down the market price of the asset below the fair price. Furthermore, the amount of cash held by the borrowers can exceed the amount of cash held by lenders. This somewhat counterintuitive result comes from the fact that the potential degree of underpricing is higher for the optimistic borrowers than that of the pessimistic lenders. This property of the

[8] A CCP can also perform *netting* of counterparty exposures. If agent A owes \$100 to agent B who owes \$100 to agent C, then the CCP can net out the obligations between the two contracts. As a result, agent A owes \$100 to agent C and agent B has no obligation at all.

equilibrium also matches the empirical facts well. As Aragon et al. (2017) documented, hedge funds, which are the ultimate borrowers in collateralized lending markets, hold 34 percent of their assets as daily liquid assets, whereas money market mutual funds, which are the ultimate lenders in collateralized lending markets, hold less than 20 percent of their assets as daily liquid assets, as documented by Aftab and Varotto (2017).

Related literature. This paper is closely related to two strands of literature – the literature on general equilibrium with collateral and the literature on financial networks. Glasserman and Young (2016) suggest three typical shock transmission channels – default cascades, price-mediated losses, and withdrawal of funds. This paper strives to incorporate the first two channels (counterparty and price channels) in a general equilibrium model and see how they interact.

Accounting for both price and counterparty channels in this paper is important, as they lead to very different incentives for network formation as well as different probabilities of cascades. Many financial network models have an equilibrium in which agents have overlapping asset/counterparty portfolios or common correlation structure (Allen et al., 2012; Cabrales et al., 2017; Elliott et al., 2018; Erol, 2018; Jackson and Pernoud, 2019). Agents have strong incentives to correlate their investments with those of their counterparties, because they can enjoy better payments from their counterparties when they are solvent while being insolvent when they expect lower potential payments from their counterparties. The general equilibrium model in this paper introduces an opposing force to such incentives which is marginal utility of cash. Agents do not hold perfectly correlated portfolio because, if everyone in the economy collapses, then the one who does not can make a huge return in such a state where the marginal utility of cash is enormous.

The network contagion part of this paper is based on the insights from exogenous network contagion literature. Eisenberg and Noe (2001) introduced an exogenous network as debt-like interdependencies with propagation through the payments, which is extended by Acemoglu et al. (2015). The payment equilibrium concept employed in such literature is used in this paper as well. Cifuentes et al. (2005), Gai et al. (2011), and Rochet and Tirole (1996) have analyzed networks with both counterparty channel and price channel for asset holdings as in this paper, but this paper also incorporates the price channel for the underlying collateral and endogenous network formation. Elliott et al. (2014) introduce equity-based interdependencies with discontinuous jumps in the payoffs of agents in the case of bankruptcy. This paper does not have equity-based interdependency but incorporate the idea of discontinuous jump in costs of bankruptcy. Di Maggio and Tahbaz-Salehi (2015) incorporates collateral in the network model in the context of the moral hazard problem. This paper is different by having an endogenous market price for the collateral and free network formation, while not incorporating the moral hazard problem.

Allen and Gale (2000), Babus and Kondor (2018), Babus (2016), Brusco and Castiglionesi (2007), Chang and Zhang (2019), Elliott et al. (2018), Erol and Vohra (2018), Farboodi (2017), and Freixas et al. (2000) studied endogenous network formation in financial networks; they consider the endogenous network structure—for example, core-periphery structure of intermediation—and possible inefficiencies and systemic risks. Unlike the models in these papers, this paper allows for endogenous contracts in addition to an endogenous asset market and collateral price for both before and after the liquidity shocks. Although the bargaining problem in this paper is based on treating agents as price-takers, the degree of bargaining power depends on the position in the network as in the model with more sophisticated bargaining protocols in Duffie and Wang (2017). Babus and Hu (2017) show endogenous intermediation and core-periphery structure due to better management of inventory and matching efficiency. The collateral in their model has fixed values and mainly another option of contracting. Whereas, the model in this paper endogenizes the amount of collateral, and collateral is the main focus of the market.

This paper also follows the literature regarding general equilibrium with collateralized debt. The literature started from the seminal paper of Geanakoplos (1997) and developed through Geanakoplos (2003), Geanakoplos (2010), and Simsek (2013) which introduce models with collateral and how heterogeneous beliefs about the payoff of the asset can generate collateralized debt and trade. Fostel and Geanakoplos (2015) and Fostel and Geanakoplos (2016) show how endogenous leverage is determined, and the distribution of payoffs affect the dynamics of leverage. Geerolf (2018) introduces pyramiding, using a contract backed by collateral as collateral, and analyzes its effect on equilibrium. This paper extends the number of possible reuse of collateral to any arbitrary number. The literature analyzes how asset prices and contract-level leverage are determined in equilibrium but does not answer how the network structure alters the result.

Also this paper incorporates lender default channel and how collateral, which is supposed to insulate counterparty risk, can still remain as a counterparty contagion channel. Eren (2014), Gottardi et al. (2017), Infante and Vardoulakis (2018), Infante (2019), Infante et al. (2018), and Park and Kahn (2019) investigated the lender default problem in collateralized lending and relevant deadweight loss, in addition to contract and intermediation dynamics. This paper incorporates the lender default feature into the endogenous network structure while also having endogenous contract terms and asset prices.

Finally, this paper is also related to the literature about central clearing and repo markets. Duffie and Zhu (2011) started the formal discussion about central clearing, which is extended by Biais et al. (2012), Duffie et al. (2015), Arnold (2017), and Frei et al. (2017), analyzing the effect and margin dynamics under a CCP. Biais et al. (2012) uses the search effort cost as the moral hazard problem of clearing members, which is similar to the result of this paper. This paper, however, has endogenous network changes with a CCP, and the externality arises from the leverage and counterparty decisions rather than search effort decisions. Paddrik and Young (2017) and Paddrik et al. (2019) perform a systemic risk analysis on the CCP with a network shock transmission model and empirical analysis with data. This paper focuses more on analysis with endogenous network change after the introduction of CCP.

2. Model

Timeline. There are three periods $t = 0, 1, 2$. There are two goods – cash and an asset denoted as e and h, respectively. Cash is the only consumption good, and it is storable — one unit of cash at t becomes one unit of cash at $t + 1$. The asset yields s amount of cash at $t = 2$, and agents gain no utility from just holding the asset. Each agent has subjective belief on s at $t = 0$. The true s is publicly revealed to everyone at the beginning of $t = 1$, and everyone agrees upon s at $t = 1$. Each agent's preference is risk-neutral and determined by how much cash she consumes at $t = 2$. Therefore, agents are facing an investment problem. Each agent is endowed with e_0 amount of cash and h_0 amount of asset.

Agents. There are n types of agents, and the set of all agents is $N = \{1, 2, \ldots, n\}$. From now on, agent j means agent of type j. Agent j believes $s = s^j$ with probability one.[9] Agents are ordered by subjective beliefs on the payoff of the asset as $s^1 > s^2 > \cdots > s^n > 0$. This belief disagreement is the reason why agents trade, borrow, or lend in $t = 0$. All agents agree upon the true value of asset payoff $s \in S \equiv \{s_1, \ldots, s_n\}$ after this information is publicly revealed at the beginning of $t = 1$. However, the asset payoff is realized at $t = 2$, so there is a time gap between uncertainty resolution and payoff realization. Also assume that $ne_0 > nh_0 s^1$, so the cash in the market is greater than the equivalent cash value of the supply of the asset even in agent 1's perspective, which is the most optimistic view.

Shocks. For each agent $j \in N$, there can be a negative liquidity shock[10] ϵ_j at $t = 1$. The size of the shock ϵ_j is independent and identically distributed across $j \in N$, and the common distribution function is denoted as G where the support of G is $[0, \bar{\epsilon}]$ and differentiable in the support for $j \in N$ with g as its density function. Denote agent i's density function as g_i (just for index purpose) for each $i \in N$ and define the convolution of the density functions

[9] This concentrated belief assumption is used in Geerolf (2018), and the assumption is for tractability.

[10] This ϵ_j can be interpreted as senior debt that precedes debt obligations among the agents in the current economy in the flavor of Diamond and Dybvig (1983). Liquidity shocks are very commonly used in the financial network literature as in Acemoglu et al. (2015) and Elliott et al. (2018), to see how such external shocks propagate through the network.

as $g_\Sigma = g_1 * g_2 * \cdots * g_n$ and its distribution function as G_Σ. Suppose that the upper bound of liquidity shock is large enough that $\bar{\epsilon} > e_0 + h_0 s^1$.

The probability of arrival of liquidity shock is $0 \leq \theta_j < 1$ for any $j \in N$. Assume that $\theta_j = \theta$ for all $j \in N$ unless otherwise noted for now. Even though the distribution of liquidity shocks are the same, the arrival rate of the liquidity shock may differ across agents. Denote $\epsilon_j = 0$ if j did not receive liquidity shock at $t = 1$ (which is a measure zero event if j received the shock). There are no additional asset endowments at $t = 1$. Without loss of generality, there are no additional endowments of goods at $t = 2$.

Markets. Agents are fully competitive and know each other's type. Since agents are competitive, every agent believes that she is a price-taker. This assumption is following the tradition of general equilibrium literature and abstracting out from market power and bargaining problem.[11] Also, agents agree to disagree over the payoff s of the asset. The markets for both goods are competitive Walrasian markets. The price of cash is normalized to 1 at any period, and the price of the asset is p_t for $t = 0, 1, 2$.

Contracts. At $t = 0$, agents can buy or sell the asset in the competitive market. Also at $t = 0$, agents can borrow cash using an asset as collateral or lend money taking an asset as collateral. All borrowing contracts are 1-period contract between $t = 0$ and $t = 1$.[12] A borrowing contract consists of:

(1) the amount of collateral posted c_{ij},

(2) the amount of promised cash per 1 unit of collateral y_{ij}, and

(3) the identities of the lender and the borrower i, j.

Denote y_{ij} as promised cash amount in $t = 1$ from j to i per unit of collateral. All borrowing contracts are non-recourse, so the actual payment from j to i is $x_{ij} = \min\{y_{ij}, \tilde{p}_1\}$ per unit of collateral. Denote $q_{ij}(y_{ij})$ as the amount of cash i lends to j in $t = 0$ per unit of collateral. The second index of the subscript of q_{ij} will be omitted from now on, since the identity of the borrower becomes irrelevant because of competition (and non-recourseness). This borrowing amount can be considered as the price of the contract and q_i is a function of the promise. The gross interest rate is $1 + r_i(y_{ij}) \equiv \frac{y_{ij}}{q_i(y_{ij})}$.

Collateral Exposures. Denote c_{ij} as the amount of collateral posted by the borrower j to the lender i. This c_{ij} amount of asset is held by the lender until $t = 1$. If the borrower j

[11] One way to interpret this assumption is to consider that each agent j consists of a continuum (or hundreds) of homogeneous agents within the same type of j with perfectly correlated uncertainties. Since there is no asymmetric information, the model abstracts out from any adverse selection problem.

[12] Even if 1-period contracts between $t = 1$ and $t = 2$ are allowed, agents will only trade borrowing contracts between $t = 0$ and $t = 1$ endogenously. This is because there is no belief disagreement at $t = 1$.

pays back the full amount of promise $c_{ij}y_{ij}$, then the lender returns the collateral. Otherwise, the lender keeps the collateral and the cash value of the collateral is $c_{ij}p_1$. The lender who is holding the collateral can reuse the collateral to borrow cash from someone else which will be clear later in the collateral constraint in subsection 2.2. Let $h_{i,1}$ denote the amount of asset agent i holds that has not been used as collateral at $t = 0$.

Debt Network. A (collateralized) debt network is a weighted directed multiplex (multilayer) graph formed by nodes N and links with 2 layers $\alpha = 1, 2$ defined as $\vec{\mathcal{G}} = \left(\mathcal{G}^{[1]}, \mathcal{G}^{[2]}\right)$, where $\mathcal{G}^{[\alpha]} = \left(N, L^{[\alpha]}\right)$, $L^{[1]}_{ij} = c_{ij}$, and $L^{[2]}_{ij} = y_{ij}$. Define the adjacency matrices $C = [c_{ij}]$ and $Y = [y_{ij}]$ as collateral matrix and contract (promise) matrix, respectively. A debt network can be represented by a double of (C, Y) and describes how much each agent borrows from or lends to other agents. Following the convention, set $c_{ii} = 0$.

Lender Default. The lenders are obliged to return the collateral when the borrower pays the promise in full. However, if a lender has negative wealth at $t = 1$, then the lender goes bankrupt and defaults on the contract. The deadweight loss from the lender default [13] is the *cash* cost $\zeta(c)[p_1 - y]^+$, where $\zeta(c)$ is a function of the amount of collateral posted c, and y is the promised cash amount. If agent j is borrowing from agent i and the lender i goes bankrupt, the borrower j has to pay $\zeta(c_{ij})[p_1 - y_{ij}]^+$ amount of cash as the lender default cost. The lender default cost coefficient function is twice-continuously differentiable and $\zeta(0) = 0,\ \zeta'(0) = 0,\ \zeta'(c) > 0,\ \zeta''(c) > 0,\ \zeta(c) \leq c,\quad \forall 0 < c \leq nh_0$. Hence, the cost increases convexly as the total exposure to bankrupt lender increases and it is multiplied by the amount of liquidity shortage from lender default $[p_1 - y_{ij}]^+$, that is the excess payoff for borrower j is supposed make.

2.1. Discussion and Examples

Timeline. The timeline of the model, which is depicted in figure 1, can be summarized as the following. Agents are endowed with cash and asset at the beginning of $t = 0$. Agents buy or sell the asset and also form a collateralized debt network at $t = 0$. At the beginning of $t = 1$, asset payoff s becomes publicly known and liquidity shocks $\epsilon \equiv (\epsilon_1, \ldots, \epsilon_n)$ are realized. Because of the liquidity shocks, some agents may have more obligation to outside senior debt than their cash inflow (that is, ϵ_j is greater than the total cash value of her wealth) and go bankrupt. All the debt is paid back during $t = 1$, either by the promise amount or by giving up the collateral. The collateral is returned to the borrower (if not

[13] I assume that there is no collateral lost during the bankruptcy process, so all of the collateral will be eventually returned to the borrower. This assumption resembles the Lehman bankruptcy case in which all the collateral returned to the original borrowers. In the case of MF Global, the minority of the borrowers lost a fraction of their collateral. We abstract from the loss of collateral under the bankruptcy procedure. This assumption is justified by the endogenously arising rehypothecation constraints in the next section.

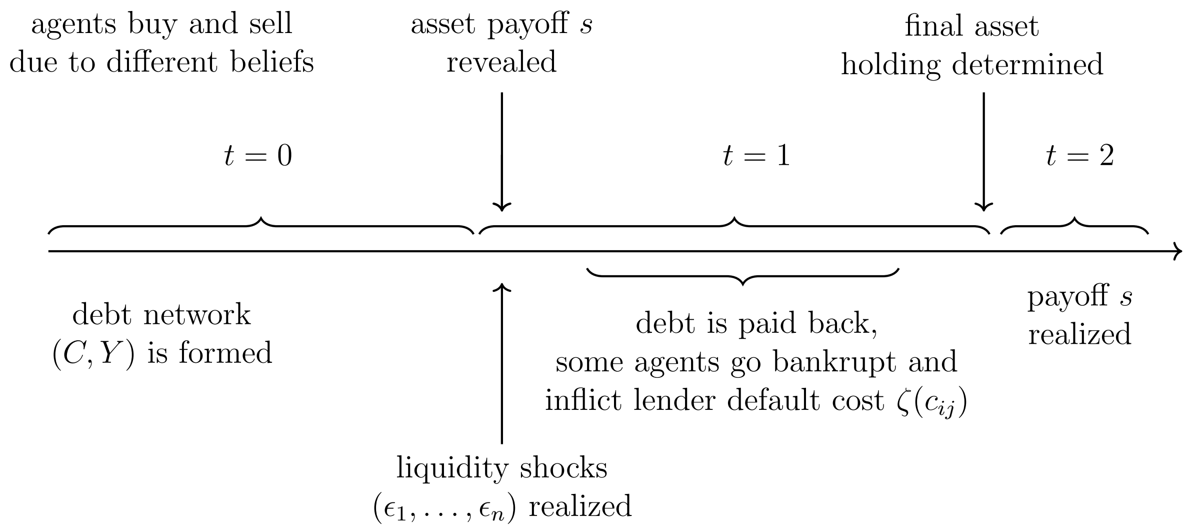

Figure 1: Timeline of the Model

defaulted) by the lender, but some borrowers may have to pay additional lender default costs if their counterparties went bankrupt. At the end of $t = 1$, all agents' final asset holdings are determined. At $t = 2$, the payoff of the asset is realized, and agents consume all the cash they have and enjoy utility.

Uncertainties. The model has two sources of uncertainty, the revelation of the payoff of the asset $\tilde{s}$ and the realization of negative liquidity shocks for each agent. At the beginning of $t = 1$, both of the uncertainties are resolved. Therefore, there will be no heterogeneous beliefs on the actual asset payoff in $t = 1$, and everyone agrees upon the asset return. However, the actual payoff realization of the asset occurs in $t = 2$, while they still have to pay back the debt they promised for $t = 1$ and toward the liquidity shocks. If the market is not under distress, then there will be no reason that p_1 is different from the commonly known cash payoff of s. However, because of the liquidity (cash) shortage in the market, there may not be enough cash in the market to buy all the assets at the fair price of s. Figure 2 is an example tree that depicts the underlying states and price realizations. There is a finite number of different s realization and continuum of different ϵ shocks. Agent 1 believes that only the first set of states in $t = 1$ occurs with positive probability. Agents 2 and 3 believe that only the second set of states and the third set of states in $t = 1$ occur with positive probability, respectively. The asset price in $t = 1$, p_1 depends on the state realization s and liquidity shock realization ϵ. Thus, each agent has their own distribution of prices as depicted in figure 2. Given the subjective distributions, each agent buys or sells, borrows or lends for different promises, and the equilibrium prices at $t = 0$ for the asset and for all the promises will be determined. Note that agents agree upon the distribution of liquidity

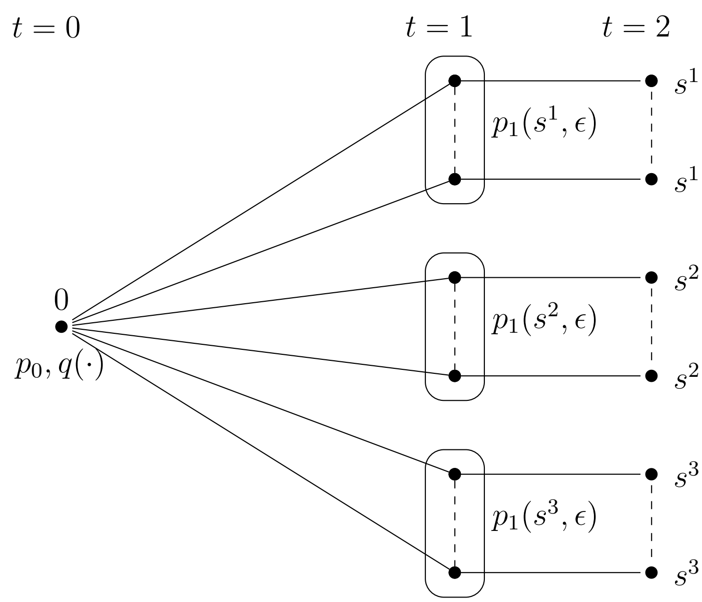

Figure 2: Tree of States and Price Realizations

shocks. Each agent's subjective belief simply puts different upper bounds on price which is s^j for agent $j \in N$.

Example of a Collateralized Debt Contract. Figure 3 visualizes the flow of cash and collateral for a collateralized debt contract. The red solid lines depict collateral flows, and the blue dotted lines depict cash flows between the agents. The top-left side of the figure visualizes the transaction at $t = 0$, where agent j posts collateral to the lender i in the amount of c_{ij} and i lends cash in the amount of $c_{ij}q_i(y_{ij})$ to agent j. If the price of the asset p_1 is greater than the promise y_{ij} at $t = 1$, then the borrower j pays the promise and the lender i returns the collateral as seen in the top right side of the figure. The bottom two sides visualize the other case. The bottom-left side of the figure has the same transaction at $t = 0$ as in the top case. However, the price of the asset p_1 is now lower than the original promise y_{ij}, and the borrower does not pay the promise at $t = 1$ as seen in the bottom right side of the figure. The lender i just keeps the collateral for herself when the borrower defaults.

Borrower Default. Because contracts are nonrecourse debt secured by collateral, every borrower with the same promise and collateral makes the same delivery. Shocks to the borrower's wealth do not get to transfer to deliveries toward lenders. Therefore, the lenders are insulated from the borrower's bankruptcy risk. In reality, this is precisely the reason why lenders require collateral from the borrowers. The lenders are protected by the exemption from automatic stay for repos under borrower bankruptcy (Antinolfi et al., 2015).

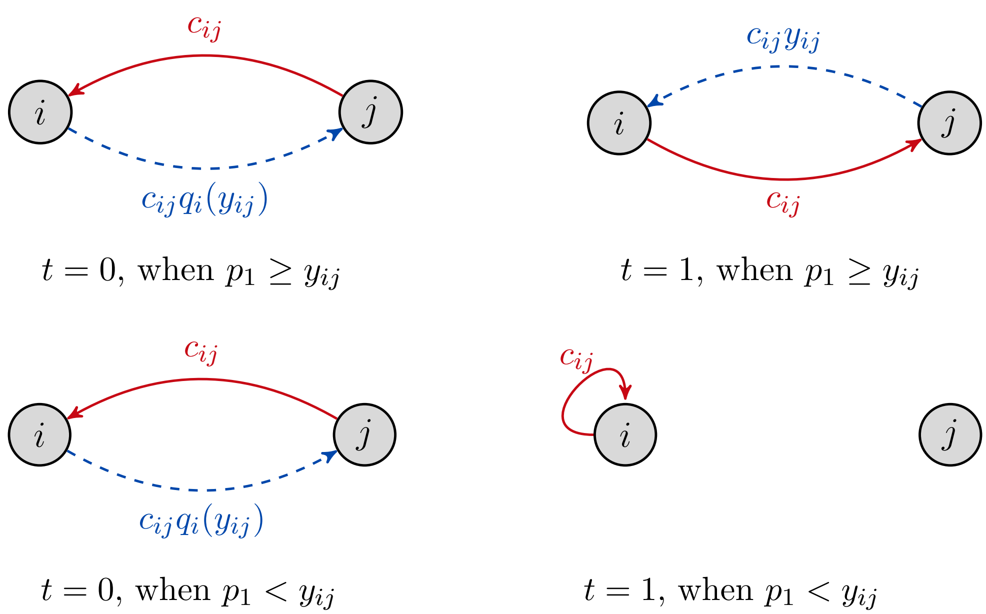

Figure 3: Flows of Cash and Collateral for Two Cases

Note: Blue dashed arrows represent flows of cash and red arrows represent flows of collateral. The top two figures represent the case without borrower default, and the bottom two figures represent the case with borrower default.

Lender Default. The lender default cost includes the opportunity cost of time and effort caused by involvement into a costly and lengthy bankruptcy procedure[14], immediate liquidity needs caused by a depositor run on an agent (bank) with large exposure to the bankrupt agent, legal costs for hiring lawyers, opportunity cost of investment, reputation cost from the clients of a financial institution and so on. Because of this lender default cost, borrowers face counterparty risk, and they may want to diversify their counterparties.

The convexly increasing cost structure in the model is not only a tractable alternative to assuming risk-aversion of the agents, which induces diversification behavior, but also a representation of realistic implications. If a hedge fund posted one Treasury bond as collateral to the Lehman Brothers, then they might find out where the original collateral went and retrieve it easily. However, if the hedge fund posted one thousand different bonds as collateral to the Lehman Brothers, then this may take much more time and cost to identify and retrieve all of the collateral of the hedge fund.[15] As a lot of investment opportunities

[14]These costs are similar to bankruptcy and liquidation costs in Elliott et al. (2014) and Acemoglu et al. (2015).

[15]For example, the Lehman Brothers' Europe branch had \$2.16 billion value of collateral in segregated accounts which are much easier to recover. In December 2009, a U.K. High Court judge held that clients whose assets should have been segregated but were instead commingled would not receive the same pro-

require a large lump-sum of cash, this hold of liquidity caused by a slowdown due to a larger pool of assets to the process would increase the opportunity costs exponentially.

The slope of ζ can proxy for how risk-averse agents are. For example, risk-averse agents would worry more about lender bankruptcy when their risk-aversion goes up, and they would diversify their lenders, or even reduce the amount of the total debt. Similarly, as ζ' increases faster, the agents are more willing to diversify lenders or even reduce the amount of the total collateral exposure. Therefore, this convexly increasing cost assumption represents the risk-aversion and aligns with the institutional facts of the bankruptcy related costs.[16] Other specifications, such as concave or constant cost structure, miss the key mechanism of network formation which is the tradeoff between counterparty risk and leverage. This property will be examined in Section 3.

Rehypothecation. The model allows reuse of the collateral held by the lender. Such reuse of collateral is called rehypothecation, in the financial market and rehypothecation is prevalent in a wide variety of collateralizable assets including repo contracts of Treasury bonds (Singh, 2017). In reality, borrowers prefer to allow rehypothecation of their collateral. Even after the fall of the Lehman Brothers, most borrowers continued to allow rehypothecation of their collateral (Singh, 2017). The reason for the prevalent use of rehypothecation is that reuse of collateral generates more funding and market liquidity for the borrowers themselves. Since the lender can reuse the collateral to borrow money from someone else, the lender can provide even more cash to the borrower for the same collateral, and this increases funding liquidity. Furthermore, since the collateral can be used multiple times, the price of the collateral also goes up. This price effect can be thought of as the velocity of capital (Singh, 2010) or the collateral multiplier (Gottardi et al., 2017), which contributes to higher market liquidity of the asset that can be used as collateral.

Figure 4 depicts how rehypothecation works. Agent k borrows cash from agent j and posts c_{jk} amount of collateral. Agent j in return lends $c_{jk}q_j(y_{jk})$ amount of cash to k. Now c_{jk} amount of collateral is sitting on j's balance sheet, and she can reuse the collateral to borrow cash from agent i. In this contract, j posts c_{ij} amount of collateral, i buys the contract with price $q_i(y_{ij})$ per collateral, and $c_{ij}q_i(y_{ij})$ is the total amount of cash lent to j. At $t = 1$, the opposite flows of cash and collateral occur. Agent k pays his promised cash amount $c_{jk}y_{jk}$ to j, and j pays $c_{ij}y_{ij}$ to i. At the other side, the lenders return their

tections as those entities whose asset had actually been segregated. But, in August 2010, an appeals court reversed the decision and ruled that clients whose money should have been segregated would be treated as if their funds had been. The decision slowed the return of assets to clients as it required a longer time of sorting through the bankruptcy procedure (Scott, 2014).

[16]Note that this cost could have been symmetrically applied to the borrower default as well. The results in this paper mostly hold for the case with the borrower default cost with a similar structure of convexly increasing cost. The only difference it makes is the difference in leverage (contract price) determination.

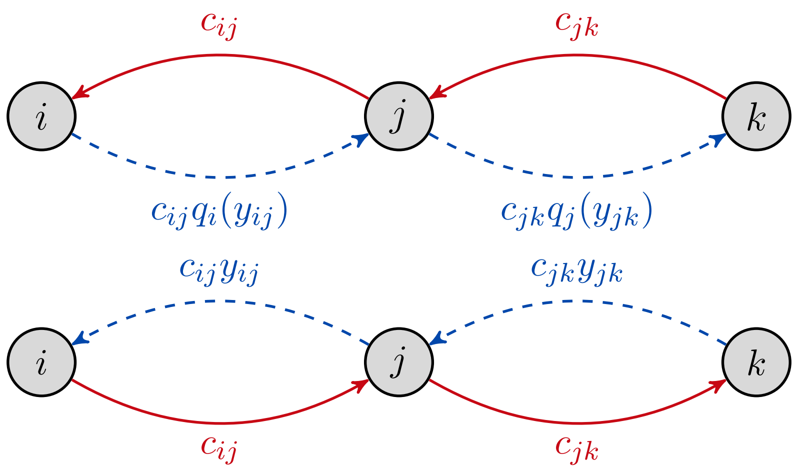

Figure 4: Flows of Cash and Collateral of Rehypothecation

Note: Blue dashed arrows represent flows of cash and red arrows represent flows of collateral. The top two figures represent the flows at $t = 0$, and the bottom two figures represent the flows at $t = 1$.

collateral to the borrowers. Agent i returns c_{ij} amount of collateral to j, and agent j returns c_{jk} amount of collateral back to k. The same collateral can be reused for an arbitrary number of times in contrast to other models of rehypothecation as in Geerolf (2018), Gottardi et al. (2017), Infante and Vardoulakis (2018), and Park and Kahn (2019).

Figure 5 shows an example of borrowing without rehypothecation and borrowing with rehypothecation. Suppose agents i, j, and k all have the same cash endowment of 50, and they have different beliefs as $s^i = 40,\ s^j = 80,\ s^k = 100$. Also suppose that there is no risk in $t = 1$, the asset price in $t = 0$ is $p_0 = 100$, and the interest rate is zero. Agent k is the most optimistic agent and would like to buy as much of the asset as possible. Agent k can increase the amount of asset purchase by leveraging more. If agent k wants to borrow from agent i, any promise above 40 will not be made by k. This is because agent i believes the payoff of the asset is 40, and any promise above 40 will just be the same as 40 because of borrower default under agent i's perspective. Then, the maximum amount of cash that k can borrow from i is 40. If agent k wants to borrow from agent j, then k will promise up to 80, which provides k a higher leverage than the leverage of borrowing from i. However, since agent j's endowment of cash is only 50, k cannot borrow more than 50 from i if there is no rehypothecation allowed. In contrast, if j is allowed to reuse the collateral, then j can borrow 40 from i. Now the effective cash available for j becomes $50 + 40 = 90$, and k can borrow 80 from j which is greater than the borrowing amount of 50 under no rehypothecation. The leverage of k with no rehypothecation is $100/(100 - 50) = 2$, while the leverage of k with rehypothecation is $100/(100 - 80) = 5$. Therefore, agent k can increase leverage by 150

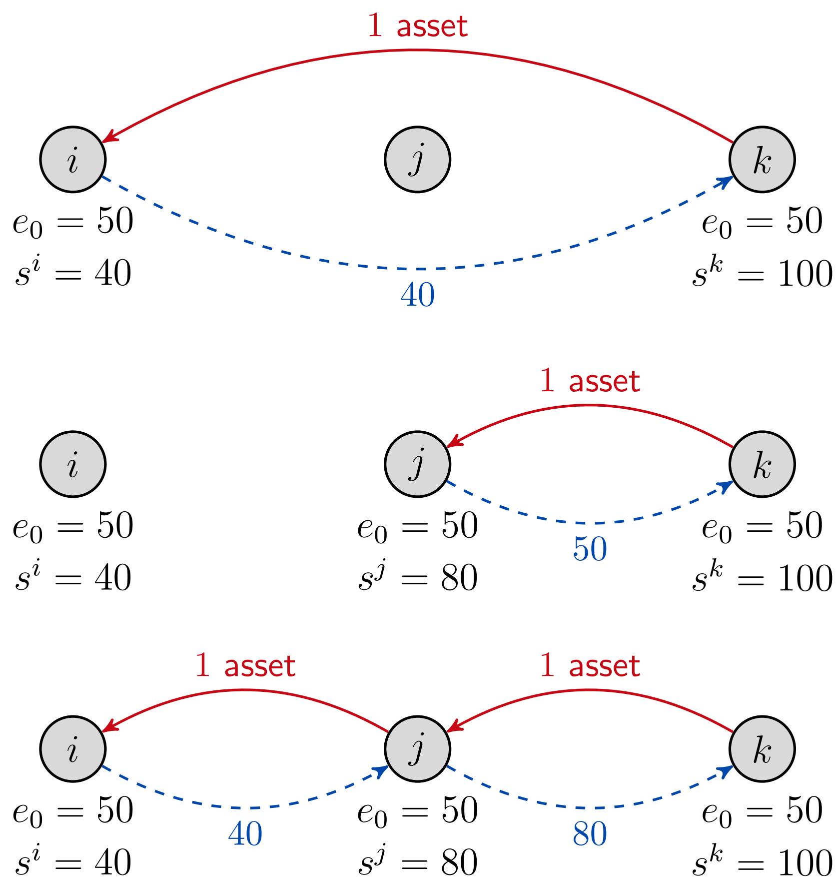

Figure 5: Example of Rehypothecation Effect

Note: Blue dashed arrows represent flows of cash and red arrows represent flows of collateral. The top figure represents the case of borrowing 40 directly from i, the middle figure represents the case of borrowing 50 from j, and the bottom figure represents the case of borrowing 80 from j who rehypothecates and borrows 40 from i again.

percent by allowing rehypothecation and would prefer to do so to increase her return.

2.2. Optimization Problem and Equilibrium Concept

Now that all the model structure is defined, an agent's optimization problem can be defined. Each agent maximizes their expected payoff in $t = 2$ at the beginning of $t = 0$ by choosing her investment portfolio. Each agent $j \in N$ can

(1) hold cash, amount denoted as e_1^j,

(2) can purchase the asset directly and carry it to the next period, in the amount denoted as $h_{j,1}$,

(3) borrow money from agent $i \in N$, posting collateral in the amount of c_{ij} and promise per collateral as y_{ij}, or

(4) lend money to agent $k \in N$, holding collateral in the amount of c_{jk} and promise per collateral as y_{jk}.

Note that the portfolio decision does not affect the macro variables, such as contract prices $q_{\cdot}(\cdot)$ and asset price p_0, under agent j's perspective, because each agent is a price-taker. For a given portfolio, the agent's expected wealth (cash equivalent of total cash and asset holding of the agent) in $t = 1$ is determined. However, these wealth values should be evaluated by the marginal value (utility) of cash for each state, which is s/p_1. The marginal value of cash could be greater than 1 if the asset price p_1 is under the fundamental value of the asset s. This underpricing can happen if the economy does not have enough aggregate cash in $t = 1$ due to liquidity shocks and bankruptcy-induced lender default costs. The market is liquidity (cash) constrained in such states. Thus, for each realization of liquidity shocks ϵ, agent j's nominal wealth changes, but the marginal value of cash also changes as well. Agent j's maximization problem becomes

$$\max_{\substack{e_1^j, \{c_{ij}, y_{ij}\}_{i \in N}, \\ h_{j,1}, \{c_{jk}, y_{jk}\}_{k \in N}}} E_j \left[\left(e_1^j - \epsilon_j + h_{j,1} p_1 + \sum_{k \in N \setminus \{j\}} c_{jk} \min\{y_{jk}, p_1\} \right. \right.$$
$$\left. \left. - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{y_{ij}, p_1\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij}) [p_1 - y_{ij}]^+ \right) \frac{s}{p_1} \right]^+ \tag{1}$$

s.t.

$$h_{j,1} + \sum_{k \in N \setminus \{j\}} c_{jk} \geq \sum_{i \in N \setminus \{j\}} c_{ij},$$
$$e_0 + h_0 p_0 = e_1^j - \sum_{i \in N \setminus \{j\}} c_{ij} q_i(y_{ij}) + \sum_{k \in N \setminus \{j\}} c_{jk} q_j(y_{jk}) + h_{j,1} p_0,$$

where $B(\epsilon)$ is the set of bankrupt agents for given liquidity shock realization ϵ, $[\cdot]^+ = \max\{\cdot, 0\}$, and $\mathbb{1}[\cdot]$ is an indicator function. The first constraint is the collateral constraint, and the second constraint is the budget constraint. The collateral constraint implies that agent j should have enough assets, either from direct purchase or from collateral posted by k who is borrowing from j to post collateral. The underlying implication of the collateral constraint is the same as in Geanakoplos (1997), but this model keeps track of the identity of borrowers and lenders to analyze the network effect and rehypothecation structure.

The equilibrium concept that will be used throughout the paper is a hybrid version of general equilibrium with price functions that are affected by the network structure as follows.

Definition 1. *For a given economy* $(N, (s^j, \theta_j, e_0, h_0)_{j \in N}, \zeta, G)$*, a septuple* $(C, Y, e_1, h_1, p_0, \tilde{p}_1, q)$ *where* $C, Y \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$*,* $e_1, h_1 \in \mathbb{R}^n_+$*,* $p_0 \in \mathbb{R}_+$*, and functions* $p_1 : \mathbb{R}^n_+ \to \mathbb{R}_+$ *and* $q_j : \mathbb{R}_+ \to \mathbb{R}_+$ *where* $q \equiv (q_1, \ldots, q_n)$ *is a* ***network equilibrium*** *if* (C, Y, e_1) *solves the agent maximization problem while satisfying budget and collateral constraints, markets are cleared as* c_{ij} *for the solution of agent* j *is the same as* c_{ij} *for the solution of agent* i *for all* $i, j \in N$*, asset market clears as* $\sum_{j \in N} h_{j,1} = H \equiv \sum_{k \in N} h_0$*, and* p_0*,* $\tilde{p}_1$ *realized at* $t = 1$ *and* q *are determined by no arbitrage conditions for the given network structure in* $t = 1$*.*

The network dynamic is essentially occurring in $t = 1$ through repayment and default costs from bankruptcy. This $t = 1$ network effect also feeds back into $t = 0$ optimization decisions which lead to network formation.

3. Network Equilibrium

This section characterizes the network equilibrium, the general equilibrium with collateralized debt network formation, and payment realization after network propagation. The payment realization in $t = 1$ shows how the network structure and shocks affect the market price and the final wealth (and equivalently payoffs) of the agents. The endogenous network of collateralized debt contracts in $t = 0$ is formed based on the consideration of the properties of a network and how agents clear markets of the asset and contracts. This section will solve for the equilibrium backwards: First, analyze the network contagion, fire sales, and price determination properties in $t = 1$ and then derive the optimal contract decisions and network formation in $t = 0$ for the given expected price distribution.

3.1. Payment Equilibrium in Period 1

Since $t = 2$ is merely the realization of the payoff of the asset and utility, we move to $t = 1$ and solve for the equilibrium prices and wealth for a given debt network (C, Y), cash holdings e_1, shock realization ϵ, and payoff revelation of the asset s. Each agent $j \in N$ pays back their promised amount of cash to her lender i in the amount denoted as x_{ij}, which follows the payment rule, $x_{ij} = \min\{y_{ij}, p_1\}$. Each agent's total nominal wealth (evaluated by cash), denoted as m_j, could be negative after the payments subtracted by liquidity shock ϵ_j for all $j \in N$. An agent with negative wealth goes bankrupt, and their wealth does not enter into the demand side of the market. Only agents with positive post-payment wealth can enter the asset market at $t = 1$ and affect the market price. If the asset is underpriced ($p_1 < s$), then all the agents will spend all of their wealth to buy the asset, because the asset return is greater than the cash return. The price that makes the aggregate wealth equal to nh_0p_1 will be the market clearing price. Thus, for given debt network (C, Y), cash holdings vector $e_1 \equiv (e_1^1, e_1^2, \dots, e_1^n)'$, asset holdings vector $h_1 \equiv (h_{1,1}, h_{2,1}, \dots, h_{n,1})'$, uncertainty realizations of liquidity shocks $\epsilon \equiv (\epsilon_1, \dots, \epsilon_n)'$ and asset payoff s, and given lender default cost function ζ, we can obtain the vector $M \equiv (m_1, \dots, m_n)$ of nominal wealth of each agent and the resulting market price of the asset as well as asset holdings. This market clearing price and allocation can be defined as payment equilibrium[17], which is an intermediate equilibrium of $t = 1$ as follows.

Definition 2. *For a given period-1 economy of $(N, C, Y, e_1, h_1, \epsilon, s, \zeta)$, a **payment equilibrium** is (M, h_2, p_1), where M is the wealth vector, h_2 is the asset holding vector, and p_1 is the price of the asset such that M satisfies the payment rule, h_2 is determined after the bankruptcy and default costs, and p_1 makes the asset market clear.*

From the payment rule $x_{ij} = \min\{y_{ij}, p_1\}$, contracts with promise of $y_{ij} > p_1$ will be paid less than the face value—that is, just the price of the asset—and the contracts with promise of $y_{kl} \leq p_1$ will be paid in full for any $i, j, k, l \in N$. If an agent j's total wealth m_j is negative, then the agent cannot even fulfill its obligations to the senior outside debtors (that is, the liquidity shock of ϵ_j), and the agent will go bankrupt. The model considers any event or cost related to the bankruptcy as outside of the collateral debt network, other than the counterparty (lender) default cost.[18] As defined before, $B(\epsilon)$ is the set of agents who go bankrupt under the shock vector ϵ. The market clearing price will indirectly determine this set because, in some cases, an agent could have survived in high p_1 but would go bankrupt in low p_1. Thus, this set might not be well defined as there could be multiple sets that constitute payment equilibria. Among multiple $B(\epsilon)$'s, selecting the smallest set of $B(\epsilon)$ that holds as payment equilibrium implies selecting the maximum price payment equilibrium. This equilibrium selection rule is well defined, which will be shown later in

[17] In the exogenous debt network literature stemming from Eisenberg and Noe (2001) and to papers such as Acemoglu et al. (2015), the main equilibrium concept is almost the same as the payment equilibrium (the name which I coined from this literature) in this paper. This intermediate step also provides a comparison between the model in this paper and the literature of exogenous financial networks and propagation dynamics. The crucial difference of the model in this paper is that the model here has an additional market for the asset used as collateral which induces the network propagation and the asset price feedback to each other.

[18] In a similar logic, suppose that the agents will fulfill their promises to each other unless they go bankrupt. This structure means the collateralized debt is a contingent contract (ultimately) by the use of collateral and the lenders will try to fulfill their obligations of returning the asset even under the situation when they have to pay the cost of retrieving the collateral from a bankrupt counterparty.

this subsection. Omit the subscript of p_1 from now on throughout this subsection since this subsection only focuses on $t = 1$.

The total nominal wealth of agent j after all the payment is

$$m_j(p) = e_1^j - \epsilon_j + h_{j,1}p + \sum_{k \in N \setminus \{j\}} c_{jk} \min\{p, y_{jk}\} \\ - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij})[p - y_{ij}]^+,$$

where $e_1^j - \epsilon_j$ is the remaining cash you have from $t = 0$ subtracted by (possibly zero) liquidity shock ϵ_j. To consider the wealth that is actually effective in demand when we compute the equilibrium, define the *effective nominal wealth* of each agent as $[m_j(p)]^+$. If $m_j(p) < 0$, then agent j goes bankrupt, that is $j \in B(\epsilon)$, and agent j will liquidate all of their holdings to pay ϵ_j. Thus, the equilibrium asset holding $h_{j,2}$ is determined by

$$h_{j,2} = \frac{[m_j(p)]^+}{p},$$

when $p < s$. If $p = s$, $h_{j,2} \leq \dfrac{[m_j(p)]^+}{p}$ but the asset holding cannot be pinned down and also is irrelevant to pin down due to invariance in final utility at $t = 2$ between holding the asset by paying the fair price and holding the equivalent amount of cash.

The aggregate cash value of the supply of the asset should equal to the aggregate cash value of the demand of the asset. As long as $p \leq s$, there will be an agent who would spend all the excess cash they have to buy the asset. The cash value of the aggregate supply is

$$\sum_{j \in N} h_{j,1}p = nh_0 p.$$

The equality is coming from the market clearing (with budget and collateral constraints) from $t = 0$. The cash value of the aggregate demand is a function of the asset price as well. If the price reaches s and there is enough money to buy up all the supply, then that is an equilibrium. Therefore, the aggregate effective cash value of demand in the market becomes

$$\sum_{j \in N}[m_j(p)]^+ = \sum_{j \in N} \left[e_1^j - \epsilon_j + h_{j,1}p + \sum_{k \in N \setminus \{j\}} c_{jk} \min\{p, y_{jk}\} \right. \\ \left. - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij})[p - y_{ij}]^+ \right]^+ .$$

Therefore, the market clearing condition that determines the price becomes

$$\sum_{i\in N}[m_i(p)]^+ = \sum_{j\in N} h_{j,1}p \quad \text{if } 0 \le p < s \tag{2}$$

$$\sum_{i\in N}[m_i(p)]^+ \ge \sum_{j\in N} h_{j,1}p \quad \text{if } p = s. \tag{3}$$

The aggregate effective nominal wealth increases as the price increases (see Lemma 5 in the appendix) and the lender default cost decreases. But, then again there is a feedback from the nominal wealth to the price. Note that the equality should hold unless $p = s$. We can interpret this as the price is going to be the level that makes the aggregate amount of liquidity that can cover both all available assets and the costs from defaults, cash-in-the-market pricing. Another case to consider is when $p = 0$. If there is any extra cash left in the economy, then $p = 0$ cannot be true. However, if there is no cash left in the economy after paying out the liquidity shocks and default costs, then $p = 0$ can occur, the market is broken down, and all the asset holdings become indeterminate as in the case of $p = s$.

The class of possible debt networks for the double (C, Y) is very large. In order to make the analysis plausible, I restrict attention to the networks in which collateral from the borrower is enough for the lender to pay her own promises. I call the class of such networks as the networks under *intermediation order*. In fact, the endogenous network formation in $t = 0$ will show that the equilibrium networks should be under intermediation order which will be showed in the next subsection. For a given level of payment $\hat{y}$, agent j should hold enough collateral (either held directly as $h_{j,1}$ or indirectly by lending as c_{jk}) that promises greater than or equal to $\hat{y}$ to cover all the debt promised to pay $\hat{y}$ or greater value. Thus, a network is under *intermediation order* if

$$\sum_{\substack{i\in N\setminus\{j\}\\ y_{ij}\ge\hat{y}}} c_{ij} \le h_{j,1} + \sum_{\substack{k\in N\setminus\{j\}\\ y_{jk}\ge\hat{y}}} c_{jk} \quad \text{for any } \hat{y}\in\mathbb{R}^+ \text{ and } j\in N. \tag{4}$$

This intermediation order is equivalent to only allowing pyramiding of contracts – promising a delivery using another contract as a collateral introduced by Geanakoplos (1997). If agent j uses the contract by agent k with promise of y_{jk} as collateral to promise y_{ij} to agent i, the actual delivery becomes $\min\{y_{ij}, \min\{p, y_{jk}\}\} = \min\{p, \min\{y_{ij}, y_{jk}\}\}$, so $y_{ij} \le y_{jk}$ to be a non-trivial pyramiding of the contract.

For example, if agent k promises 20 to j but j reuses the collateral and promises 30 to i, then, agent j violates the intermediation order. In this case, agent j might not have enough cash to repay i, if the price is between 20 and 30, so he cannot receive the collateral from i

and then return it to k. The intermediation order guarantees that if the ultimate borrower (collateral provider) fulfills her promise, then the intermediary (who reuses the collateral) also has enough cash to fulfill his promise to the ultimate lender (cash provider). Note that intermediation order implies collateral constraints.

Under the intermediation order, we can interpret the market clearing condition in a more intuitive way. The negative liquidity shocks ϵ destroys the aggregate available cash. The destruction of cash for the demand can be decomposed into three factors:

(1) each agent's liquidity shock ϵ_j,

(2) lender default costs from bankrupt lenders $\sum_{i\in B(\epsilon)} \zeta(c_{ij})[p - y_{ij}]^+$, and

(3) second-order bankruptcy from the first two effects which amplifies (2).

The second and third factors create a feedback loop in the market through the price channel and the counterparty channel similar to debt network models without collateral. Note that the first factor is bounded above by the nominal wealth of the agent before the negative liquidity shock, $\mu_j(p) \equiv m_j(p) + \epsilon_j$, because any excess liquidity shock still makes the same effective nominal wealth of zero. For a given price p, the actual destruction of cash from liquidity shock to j relevant to the demand of the asset market is $\min\{\epsilon_j, \mu_j(p)\}$.

For a given price p, the excess cash of the network can be computed as the *cash savings from* $t = 0$ subtracted by the *destruction of cash*. Hence, the *remaining cash* becomes

$$RM \equiv \sum_{j\in N} e_1^j - \sum_{j\in N} \min\{\epsilon_j, \mu_j(p)\} - \sum_{j\in N} \sum_{i\in B(\epsilon)} \zeta(c_{ij})[p - y_{ij}]^+,$$

which is the cash saved from $t = 0$ subtracted by the liquidity shocks and lender default costs. The remaining cash can be considered as the demand side. For the supply side, the amount of collateral that are sold in the market is the amount of collateral from bankrupt agents' balance sheets, that is the *total fire sales of the assets* denoted as

$$FS \equiv \sum_{j\in B(\epsilon)} \sum_{\substack{k\in N \\ p<y_{jk}}} (c_{jk} + h_{j,1}) - \sum_{j\in B(\epsilon)} \sum_{\substack{i\neq j \\ p<y_{ij}}} c_{ij}.$$

Suppose that the price p is neither 0 or s. Then, the market clearing condition, equation (2) becomes

$$\pi(p) = \frac{RM}{FS}, \tag{5}$$

which is the *remaining cash* divided by the *total fire sales of the assets* that are under bankrupt agents' balance sheets.

By the intermediation order, the denominator is always nonnegative. However, if there are no assets to be bought (that is, denominator of $\pi(p)$ is zero), then the price of the asset will be trivially its fair value price s. If there is no asset to be sold by the bankrupt agent, then there is no reason to lower the price of the asset. If there are enough cash in the market to cover the extra supply (fire sales) with the fair price (i.e. $\pi(p) \geq s$) then the price is also set as fair value price s. If there are some leftover cash after the payments and costs that is not sufficient to buy all of the assets in fair price, then the market price will be $\pi(p) < s$ which we define as *liquidity constrained price* of the asset.

The post-shock market clearing condition, equations (2) and (3), can be rearranged to obtain the price equation as follows.

$$p = \begin{cases} 0 & \text{if } \sum\limits_{j \notin B(\epsilon)} e_1^j \leq \sum\limits_{i \notin B(\epsilon)} \epsilon_i \text{ for any } p \in [0, s] \\ s & \text{if } \pi(p) > s \text{ or } FS = 0 \\ \pi(p) & \text{otherwise.} \end{cases} \tag{6}$$

Under the intermediation order, I can show that a payment equilibrium always exists and the set of equilibrium prices is a complete lattice.

Proposition 1 (**Existence and Lattice Equilibrium Prices**). *For any given collateralized debt network $(N, C, Y, e_1, h_1, \epsilon, s, \zeta)$ with $C > 0$ that is under intermediation order, there exists a payment equilibrium (M, h_2, p_1). Furthermore, among the possible equilibria, there always exists a maximum equilibrium that is $(\overline{M}, \overline{h_2}, \overline{p_1})$, where $\overline{p_1}$ is the highest price among all possible equilibrium prices.*

All the proofs are relegated to the appendix. The intuition of the proof is the following. The delivery $x_{ij} = \min\{y_{ij}, p\}$ toward the lender increases as price p increases. By intermediation order, every borrower or intermediary payoff also increases as p increases. Thus, every individual nominal wealth increases in p. Since individual nominal wealth m_j is increasing in p, as shown by lemma 5 in the proof, the aggregate nominal wealth is increasing in asset price p and decreasing in lender default cost ζ. Since increase in wealth also means bankruptcy is less likely, the lender default cost also decreases when p increases. Therefore, every single variable that is included in the market clearing condition (weakly) increases in price p. Although the equilibrium is determined by the vector of all the wealth, we can summarize each equilibrium by price level p, and there exists a fixed point price that clears the market.

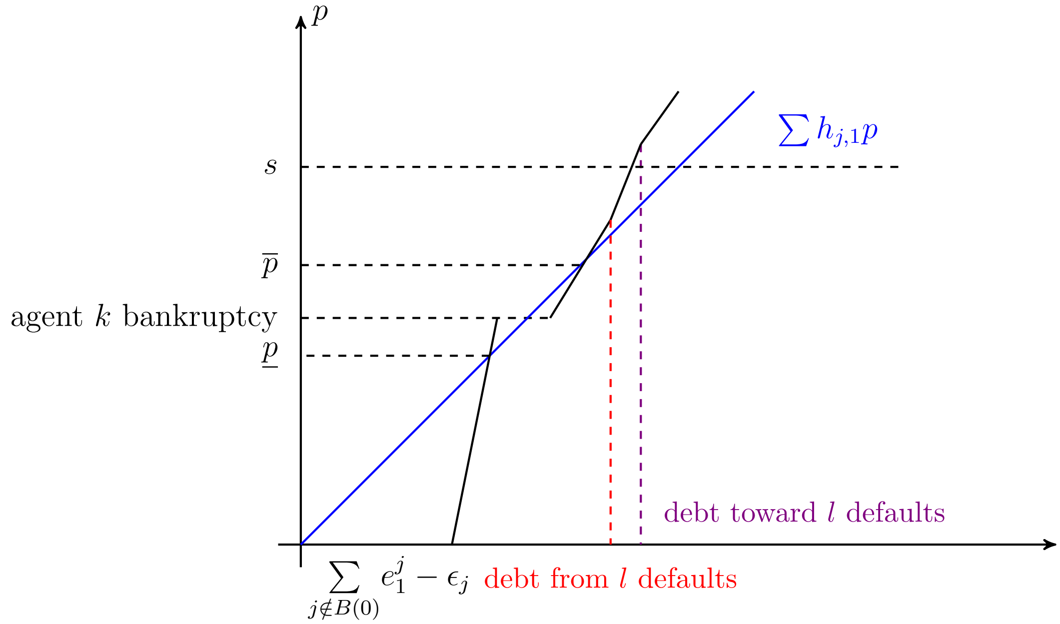

Figure 6: Multiple Payment Equilibria for Market Clearing

Although we can show that the payment equilibrium always exists, we cannot guarantee its uniqueness. This multiplicity is mainly due to the jumps in $m_j(p)$ at the point of bankruptcy of other agents. The actual bankruptcy set may also depend on the market clearing price as $B(\epsilon|p)$. An agent may have $m_j(p) > 0$ for given price p and bankruptcy set $B(\epsilon|p)$, but her wealth may be negative at p' and given bankruptcy set $B(\epsilon|p')$ so $m_j(p') < 0$. Her bankruptcy will generate even more second-order bankruptcy costs and make p' to be true. The following proposition summarizes this relation between multiplicity (and uniqueness) and bankruptcy.

Proposition 2 (Multiplicity and Bankruptcy). *For any given collateralized debt network $(N, C, Y, e_1, h_1, \epsilon, s, \zeta)$ with $C > 0$ that is under intermediation order, there may be multiple equilibria. If p and p' are two distinct prices from the two different payment equilibria, then $B(\epsilon|p) \neq B(\epsilon|p')$.*

Figure 6 depicts an example of multiple equilibria. Define the sum of lender default costs coming from agent l's bankruptcy for price p as $\beta_l(p) \equiv \sum_{j \in N} \zeta(c_{lj})[p - y_{lj}]^+$. Also denote the nominal wealth subtracted by negative liquidity shock as $\mu_l(p) \equiv m_l(p) + \epsilon_l$. There are kinks at prices in which each contract defaults and discontinuous jumps at prices in which each agent goes bankrupt. The first type of kink occurs for $p < y_{ij}$ which affects both $m_i(p)$ and $m_j(p)$, and the second type of jump occurs at the point where $m_j(p) = 0$. From the second statement of proposition 2 and equation (11) in the proof, the existence of lender default cost plays a significant role in generating multiplicity and also the counterparty

Figure 7: Counterparty Irrelevance Example

contagion effect through the second-order bankruptcy. Due to the multiplicity[19] and the lattice property, we assume $B(\epsilon)$ to be the bankruptcy set from the maximum equilibrium price—that is, $B(\epsilon) \equiv B(\epsilon, \overline{p})$, from now on. Also, a *maximum equilibrium selection rule* means choosing the equilibrium with the maximum equilibrium price. We will focus on the results on equilibrium with the maximum equilibrium selection rule from now on as in Elliott et al. (2014).

Trivially, if there is no default cost—that is $\zeta(c) = 0$ for any c—then the payment equilibrium is unique from the second statement of proposition 2. Also without a default cost, change in counterparty connections does not matter as long as the total borrowing and lending amount remain the same. The following proposition states this property.

Proposition 3 (**Counterparty Irrelevance**). *If there is no lender default cost—that is, $\zeta(c) = 0$ for all $c \geq 0$—then the payment equilibrium is unique for any given network. Furthermore, two networks (C, X) and $(\hat{C}, \hat{X})$ with the same indegrees and outdegrees—that is, $\mathbb{1}(C \circ X) = \mathbb{1}(\hat{C} \circ \hat{X})$ and $(C \circ X)\mathbb{1} = (\hat{C} \circ \hat{X})\mathbb{1}$—will have the same payment equilibrium.*

This proposition shows the necessity of assuming the existence of a lender default cost (or any counterparty risk) in order to generate interesting interaction among agents. Figure 7 shows an example of two different networks with the same equilibrium outcome. If all the links have the same weight—that is, $c_{ij} = c$ and $y_{ij} = y$ for all $i, j \in N = \{1, 2, 3, 4\}$—then the two networks have the same indegrees and outdegrees. Because of the absence of a default cost, agent's individual connection does not matter as long as the total borrowing and lending for each agent are the same. The two networks will have the same equilibrium price and allocation. Also the result holds for networks that are not under intermediation order.

The result is not so surprising since the main reason for using a collateral is to insulate the lender from the counterparty risk. A collateralized debt network has no counterparty risk as in the anonymous market. This irrelevance result can be extended to a model with

[19]The existence of multiple equilibria implies that there could be even more instability than looking at just the maximum result (Roukny et al., 2018).

default cost caused by borrower's default. For example, the actual payment when retrieving the collateral from the borrower in case of default can be less than the actual value of the collateral p, which is $(1-\phi)p$ with $\phi > 0$ due to fire sales cost or collateral seizure cost. The existence of default costs will only scale down the entire values of the collateral for the lender (that is, less lending in aggregate) but would not change the irrelevance result.

From now on, define *systemic risk* under the belief of agent j as the expected difference between the ex post fair value of the asset and the actual price of the asset, $\int (s^j - p) d\tilde{G}(\epsilon)$, where $\tilde{G}$ is the joint distribution of G_i's for all $i \in N$. This notion of systemic risk is following the definition of *systemic loss* in value defined in Glasserman and Young (2016). Even though the fundamental value of the asset is s, the underpricing of the asset, $s-p$, comes from the liquidity shocks and lender default costs which vary by the network connections. The systemic risk definition here is taking the ex ante expected value of the systemic losses for each subjective belief. The sum of all the systemic risks for each ex ante belief will be closely related to the social welfare computation, which we will talk later in the next subsection. The aggregate default costs after the revelation of s and realization of ϵ will determine the difference between the two values, and the difference represents how severe the mispricing is due to the total sum of deadweight losses.

Now we briefly describe how to solve the equilibrium in quantitative analysis under the maximum equilibrium selection rule.

Equilibrium Search Algorithm: Consider the following algorithm of finding the maximum payment equilibrium.

0. Set $B^{(0)}(\epsilon) = \varnothing$. Start with step 1.
1. For any step k, given $B^{(k-1)}$, compute $p^{(k)}$ that satisfies the market clearing condition 6.
2. For given $p^{(k)}$, compute $m_j(p^{(k)})$ with given $B^{(k-1)}$ and update $B^{(k)}$ with the new $m_j(p^{(k)})$.
3. If $B^{(k-1)} = B^{(k)}$, then we have the maximum equilibrium. Otherwise, move to the next step $k+1$ and repeat procedures 1 and 2.

This algorithm guarantees to find the maximum payment equilibrium price of the given network. Also, the algorithm finishes within n steps because the second-order bankruptcy (or cascades) could only occur at the maximum of $n-1$ times if it happens for one agent by one.

3.2. Network Formation in Period 0

Given the results from $t = 1$, agents form beliefs on the distribution of p_1 and $B(\epsilon)$ under shock realizations. As discussed in the model section, agent j solves the maximization problem:

$$\max_{\substack{e_1^j, \{c_{ij}, y_{ij}\}_{i \in N}, \\ h_{j,1}, \{c_{jk}, y_{jk}\}_{k \in N}}} E_j \left[\left(e_1^j - \epsilon_j + h_{j,1} p_1 + \sum_{k \in N \setminus \{j\}} c_{jk} \min \{y_{jk}, p_1\} \right. \right.$$
$$\left. \left. - \sum_{i \in N \setminus \{j\}} c_{ij} \min \{y_{ij}, p_1\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij}) [p_1 - y_{ij}]^+ \right) \frac{s}{p_1} \right]^+$$

s.t.

$$h_{j,1} + \sum_{k \in N \setminus \{j\}} c_{jk} \geq \sum_{i \in N \setminus \{j\}} c_{ij},$$
$$e_0 + h_0 p_0 = e_1^j + h_{j,1} p_0 - \sum_{i \in N \setminus \{j\}} c_{ij} q_i(y_{ij}) + \sum_{k \in N \setminus \{j\}} c_{jk} q_j(y_{jk})$$

From now on, substitute the probability measure for each individual's expected utility with F_j for ϵ and omit the + superscript and denote $E_j[\cdot]$ as nonnegative expected nominal wealth. Any negative wealth will be counted as zero in agent j's perspective. Note that each individual g_i is still relevant when considering the lender default costs and the correlated bankruptcy of agent j. For example, if agent i goes bankrupt, then agent j may also go bankrupt because of the cost of $\zeta(c_{ij})$ regardless of the size of the ϵ_j. In this case, agent j would not consider the lender default cost to be a problem greater than the size of her nominal wealth in $t = 1$ under that shock. Denote this implied *expected lender default amount of i under j's subjective belief* as $\omega_{ij}(y) \equiv E_j \left[[1 - y/p_1]^+ \mathbb{1}[i \in B(\epsilon)] \right]$. Then, the marginal increase in counterparty risk of borrowing from agent i for agent j becomes $\zeta'(c_{ij}) s^j \omega_{ij}(y)$.

An agent has five different ways to use her budget: holding cash, buying the asset, buying the asset with leverage, lending cash to others, and lending cash with leverage. For each additional unit of cash, an agent should compare the five options for marginal returns. Agent j's return on holding cash is $E_j \left[\frac{s}{p_1} \right]$.

The cash return goes up as j holds less cash because there would be even more underpricing if all other agents go bankrupt. From the market pricing equation (6) in $t = 1$, price of the asset in $t = 1$ can become $p_1 = 0$ if all the agents who are holding cash in the economy at $t = 0$ go bankrupt. Even if the probability of liquidity shock θ_j is small for everyone, if there is a positive probability of bankruptcy of the agent, then there is a positive probability of p_1 being zero, and the return on cash holding becomes infinity. Therefore, every agent in the equilibrium should hold a positive amount of cash. This pins down all the returns from borrowing and lending to the return of holding cash $E_j\left[\frac{s}{p_1}\right]$. The cash return becomes the benchmark return for any other investment decision the agent makes.

Lemma 1 (Positive Cash Holdings)**.** *If* $\bar{\epsilon} > ne_0 + h_0 s^1$*, then* $e_1^j > 0$ *for every* $j \in N$ *in any network equilibrium.*

This lemma implies that the model is distinctive from existing models in general equilibrium with collateral literature when agents have linear utility. Linear utility models in Geanakoplos (2010), Simsek (2013), and Geerolf (2018) all have borrowers holding only the assets and zero amount of cash. In reality, the ultimate borrowers such as hedge funds usually hold a significant proportion of their portfolio as cash equivalent assets. The network model here replicates this observed phenomenon by adding this liquidity shock in the intermediate period and the possibility of liquidity constrained-price, which is below the fair value of the asset.

The (marginal) return on lending depends on how much you lend but does not depend on which agent you lend to. This irrelevance comes from the fact that lenders do not have counterparty risk due to collateralization and no recourse contracts. Therefore, the contract price q_i does not depend on the identity of the borrower. Suppose j is lending a positive amount of cash without leverage—that is, j is a pure lender. The return equation of lending for j becomes

$$\frac{1}{q_j(y)} E_j\left[\min\left\{s, y\frac{s}{p_1}\right\}\right] = E_j\left[\frac{s}{p_1}\right].$$

The return of lending should equal the return of cash for no arbitrage (indifference). This equation also represents how the price of a contract (or interest rate) is determined if agent j does not leverage.

$$q_j(y) = \frac{E_j\left[\min\left\{s, y\frac{s}{p_1}\right\}\right]}{E_j\left[\frac{s}{p_1}\right]} = \frac{E_j\left[\min\left\{1, \frac{y}{p_1}\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}$$

As we have seen in the payment equilibrium part in $t = 1$, if the realization of the asset payoff increases, the asset is more likely to be underpriced than its fundamental value because of more exposure to liquidity shortage and lender default costs. Thus, if the nominal wealth of the agents are identical, the order of return of holding cash also follows the order of optimism over asset payoffs— that is, $E_j\left[\frac{s^j}{p_1}\right] > E_k\left[\frac{s^k}{p_1}\right]$ for any $j < k$. In fact, the inequality should always hold in an equilibrium as in lemma 2.

Lemma 2 (Cash Return Ordering)**.** *For any two agents in a network equilibrium, the cash return from the more optimistic agent is always greater than the cash return from the less optimistic agent—that is,* $E_j\left[\frac{s^j}{p_1}\right] > E_k\left[\frac{s^k}{p_1}\right]$ *for any* $j < k$ *and* $j, k \in N$.

The main intuition of the proof is as follows: If agent k, who is more pessimistic than agent j, has higher (subjective) return from cash holdings, then any other investment she is making should also have that same return by lemma 1. Suppose agent j mimics agent k's entire investment portfolio. Then the same investment cannot have a return greater than the return from cash holdings from agent j's original portfolio, because otherwise it violates the optimality of his own portfolio decision. But, if agent j and k face exactly the same cashflow and counterparty risks, then the return from that investment should always be higher from agent j's perspective because he is more optimistic about the asset return and the degree of underpricing (and marginal utility from cash) is higher under more optimistic belief. This implies that agent j can have higher return than agent k by mimicking a strategy that violates the original assumption of agent k having higher return from cash holding.

This cash return ordering from lemma 2 implies that interest rates of the same contract increases over an agent's optimism—that is, optimistic agents demand a higher interest rate than pessimistic agents do. This property will be verified again later by the contract pricing formula.

Return on buying the asset without leverage is $E_j\left[\frac{s}{p_0}\right]$, where p_0 is the asset price determined at $t = 0$. Since the return does not depend on p_1, this return is not (directly) influenced by counterparty risk. Hence, this return on asset is ordered directly by the agent's optimism—that is, $E_j\left[\frac{s}{p_0}\right] > E_k\left[\frac{s}{p_0}\right]$ for all $j < k$. Return on asset purchase with leverage is

$$\frac{s^j}{p_0 - q_i(y)} E_j\left[\left[1 - \frac{y}{p_1}\right]^+ - \zeta'(c_{ij})\left[1 - \frac{y}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\}\right],$$

where agent j is borrowing cash from agent i with c_{ij} amount and promises y. Similarly, return on lending with leverage is

$$\frac{s^j}{q_j(y') - q_i(y)} E_j\left[\min\left\{1, \frac{y'}{p_1}\right\} - \min\left\{1, \frac{y}{p_1}\right\} - \zeta'(c_{ij})\left[1 - \frac{y}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\}\right], \tag{7}$$

where j buys (lends money) a contract with promise y'. From the return comparisons and pure lender's no arbitrage condition, an agent's individual leverage decision could be derived, and the following lemma summarizes the result of leverage maximization.

Lemma 3 (Maximum Leverage)**.** *Suppose that agent j lends a positive amount of money to an agent (or buys the asset—that is, lends money to herself) and borrows a positive amount of money from agent i in a network equilibrium. Then, the following statements are true:*

1. *Agent j maximizes her leverage by borrowing the maximum amount of money she can borrow from agent i which is s^i.*
2. *If j is borrowing the same amount from agent i and k who have the same probability of bankruptcy with $i < k$, then j marginally prefers to borrow from i.*

The intuition of the proof is as follows: If borrower j and lender i agree on the distribution of prices below s^i, which only depends on liquidity shocks that both agents agree upon, then j and i agree upon the expected delivery. Since j has higher marginal utility of cash in $t = 0$ than i, agent j would like to increase borrowing at any point below s^i. At the point of s^i, agents disagree with the promised delivery above s^i. Agent j believes the price of the asset p_1 can be greater than s^i if the aggregate liquidity shock is not large enough, but i believes the price is bounded above by s^i even if there is zero liquidity shock. Therefore, the endogenous leverage is determined by the promise of $y = s^i$ and its price $q_i(s^i)$.[20] The logic can be considered as an extension of the three state case in Geanakoplos (2003).

With this lemma, we can focus only on networks following intermediation order along with the collateral constraints.

Corollary 1. *Any debt network from a network equilibrium is under intermediation order.*

Also by lemma 3 we can pin down $q_j(y)$ for agents who both borrow and lend. If agent j borrows y from agent i and lends y' to some other agent (or herself if she buys the asset directly) then her no-arbitrage contract price becomes

$$q_j(y') = q_i(y) + \frac{E_j\left[\min\left\{1, \frac{y'}{p_1}\right\} - \min\left\{1, \frac{y}{p_1}\right\} - \zeta'(c_{ij})\left[1 - \frac{y}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}$$

[20] This intuition also brings light to how complicated the model would be if concentrated beliefs are not assumed. For example, if agent's optimism is ordered by first-order stochastic dominance, then endogenous leverage depends not only on the relative hazard ratio, but also on the difference in marginal utilities of cash which also changes endogenously and is extremely intractable to pin down. Moreover, they differ with the distribution of liquidity shocks.

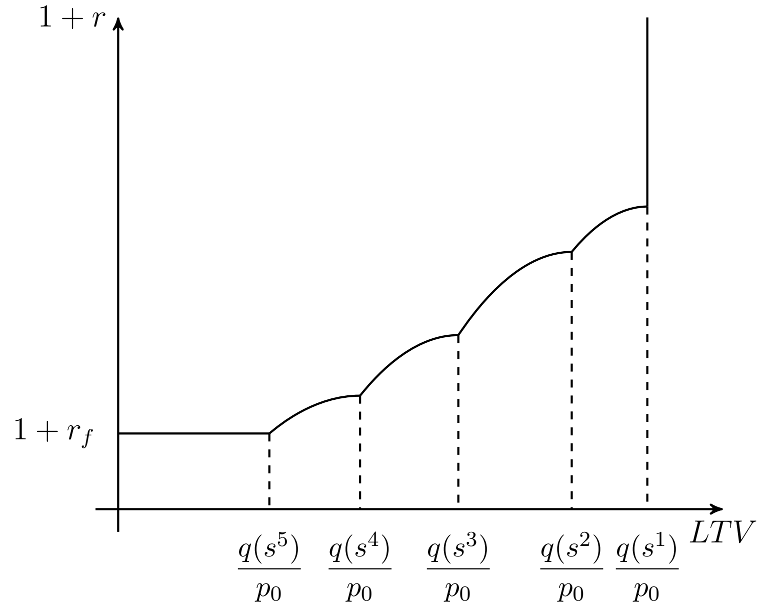

Figure 8: Credit Surface of Collateralized Debt

By lemma 3 we only need to focus on kink points for borrowing. Hence, any agent who is willing to borrow from agent j will face the willingness to pay as

$$q_j(y) = q_i(s^i) + \frac{E_j\left[\min\left\{1, \frac{y}{p_1}\right\} - \min\left\{1, \frac{s^i}{p_1}\right\} - \zeta'(c_{ij})\left[1 - \frac{s^i}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}. \tag{8}$$

The following proposition and figure 8 describe the relationship between interest rate and loan-to-value ratio.

Proposition 4 (Concave Credit Surface)**.** *In any network equilibrium, the contract price function $q(y)$ is piece-wise concave in the amount of promise y and has kinks and jumps at each payoff points $s^1, s^2, \ldots, s^{n-1}, s^n$. Furthermore, the credit surface of the equilibrium (the graph between leverage $q(y)/p_0$ and interest rate $y/q(y)$) is piece-wise concave and continuous in the amount of leverage $q(y)$ and has kinks at each corresponding payoff points $q(s^1), q(s^2), \ldots, q(s^{n-1}), q(s^n)$ and right derivative of each kink point is greater than the left derivative. Also, the interest rate goes to infinity at the point $q(s^1)$.*

Now the remaining parts of the equilibrium are the actual amount of cash holdings and the amount traded for each contract. From the return equation of leverage, equation (7), and the convexly increasing lender default cost ζ, borrowers would diversify their borrowings across different lenders. Even if agent j can borrow more from i in the amount of $q_i(s^i)$, higher

$\zeta'(c_{ij})$ would make j borrow from $i+1$ with lower leverage (or higher margin) of $q_{i+1}(s^{i+1})$ because of lower default cost $\zeta'(c_{(i+1)j})$. Hence, there is a tradeoff between leverage and counterparty risk. An agent wants to maximize leverage to maximize her return but she has to face higher counterparty risk due to increased concentration. If the agent wants to diversify her lenders, then she has to deal with more pessimistic agents who only provides low leverage (requires high margin) which implies low return for the borrower. Thus, the network becomes a multi-layered chain network instead of a single-chain network when ζ becomes large and θ_i is non-negligible. Thus, the equilibrium cash holdings depend on the gap between beliefs on asset payoffs and the slope of the contract price function.

3.3. Results on Decentralized OTC Market

Now with the given contract prices, the asset price can be analyzed. The first result is about who is buying the asset in the equilibrium. Not surprisingly, the agent with the most optimistic belief on the asset payoff, agent 1 who believes the asset payoff will be s^1, always buys the asset.

Lemma 4 (Natural Buyers)**.** *In a network equilibrium with maximum payment equilibrium selection rule, the most optimists, agent 1, buys the asset with positive quantity, thus* $c_{11} > 0$*. Similarly, agent* i *borrows from agent* $i+1$ *with positive amount,* $c_{i+1,i} > 0$ *for any* $i \in N, i \neq n$*.*

The intuition of the lemma is that if any agent $j > 1$ is buying the asset, then agent 1 will have even higher return than j by using the same leverage decisions as j unless agent 1's cash holding is huge enough to make her required return low. However, when agent 1's cash holding is large, then j's return of cash is enormous in case agent 1 goes bankrupt, and agent j should either increase his cash holding or increase the return on asset purchase—that is, p_0 goes down. But either of them should make agent 1's perceived return on asset purchase (with leverage) increase even faster because of $s^1/p_0 > s^j/p_0$. Thus, agent 1 should be a natural buyer of the asset (but not necessarily the only buyer). Similar logic can be applied to any subsequent contracts $\min\{p_1, s^i\}$ and by induction, we can show that a natural buyer of any contract with a promise of s^i is agent i. Note that agents other than agent 1 can also hold some amount of assets because it is possible to have

$$E_j\left[\frac{s^j}{p_1}\right] = \frac{E_j\left[s^j - \min\left\{s^i\frac{s^j}{p_1}, s^j\right\} - \zeta'(c_{ij})s^j\left[1-\frac{s^i}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{p_0 - q_i(s^i)}$$

for multiple $j \in N$. In this case, agent 1 holds more cash than agent j so that the possible underpricing coming from larger support for agent 1 is mitigated by being less vulnerable to liquidity shocks to others including agent j. Thus, $e_1^1 > e_1^j$ in such cases.

This property of optimists holding more cash than pessimists can be formalized for a certain parameter region. Belief disagreements are *harmonically dispersed* if $s^j s^{j+2} \leq (s^{j+1})^2$ for any $j < n - 2$. Harmonically dispersed belief disagreements imply that the belief of one agent among three consecutive agents are not too radically skewed. For example, belief disagreements are not harmonically dispersed if agent 2 and 3 believes s to be 20 and 10, respectively, but agent 1 believes s to be 100. Agent 1's belief should be less than or equal to 40 in order to be harmonically dispersed.

Proposition 5. *Suppose the network equilibrium is a single-chain network—that is, $c_{i+1,i} = c_{i+2,i+1} = c > 0$ for $i < n - 2$ and $c_{ij} = 0$ for any ij not in the path between agent 1 and n and $i \neq j$. Also suppose that the belief disagreements are harmonically dispersed. Then, agents hold cash as $e_1^1 > e_1^2 > \cdots > e_1^n$—that is, the order of amount of cash holdings is the same as the order of optimism on the asset payoff.*

Corollary 2. *If there is no lender default cost—that is, $\zeta(c) = 0$ for any $c \in \mathbb{R}^+$—and the belief disagreements are harmonically dispersed, then agents hold cash as $e_1^1 > e_1^2 > \cdots > e_1^n$—that is, the order of amount of cash holdings is the same as the order of optimism on the asset payoff.*

This result is in contrast to standard results in general equilibrium with collateral literature such as Geanakoplos (1997), Fostel and Geanakoplos (2015), Simsek (2013), and Geerolf (2018) in which optimists spend more, if not all, cash to purchase assets, and pessimists hold more cash and sell assets. Although agent 1 values the asset the most, they also have the highest marginal utility of cash in $t = 1$. Because the asset value is so high, the price of the asset is also vulnerable to liquidity shortage in the market. Under agent 1's perspective, the market should have nh_0s^1 amount of cash to clear the market with the asset's fundamental value. On the contrary, agent n believes the market can be cleared in fair value in $t = 1$ even with nh_0s^n amount of cash, and underpricing only happens when the economy is under severe liquidity shocks. Holding more cash is possible because of the possibility of leveraging through the lending chain. The down payment (cash paid for the levered purchase) for agent 1, $q(s^1) - q(s^2)$, can be less than the down payment for agent $n-1$, $q(s^{n-1}) - q(s^n)$. Also, the cash holding dispersion will be even more severe if leverage increases.

This cash holding result may seem unrealistic. However, the empirical facts support this result. On average, 34 percent of a hedge fund's assets can be liquidated within one day (without fire sale discounting) according to Aragon et al. (2017) using Form Periodical Filings over 2013–2015. This proportion is well above the proportion of money market mutual funds (MMMFs) in the SEC reformed regulation by 10 percentage points. Before the regulation, the daily liquid assets for MMMF were on average less than 20 percent, and even after the regulation, the daily liquidity in the portfolio is still below 31 percent (Aftab and Varotto, 2017). Because hedge funds are the ultimate asset buyers (as agent 1) in a collateralized debt market, and money market mutual funds are pure lenders in the market (as agent n), the empirical findings are consistent with the result of the proposition.

The next result and important step for the proof of existence is that individual agent's diversification behavior generates positive externalities through amplification and feedback effects in both asset price channel and counterparty channel in $t = 1$. If agent j diversifies more and lowers her own return because of counterparty risk concerns, then it will lower the leverage through $q(y)$ and also decrease price volatility in $t = 1$. Furthermore, this risk reducing behavior makes agent j's balance sheet m_j more stable and decreases the probability of j's bankruptcy. Thus, second-order bankruptcy contagion decreases even further.

Before stating the proposition, we have to define directions of lowering the aggregate debt level. Since any equilibrium network is under intermediation order, we can restrict our attention to directions that go across such a class of networks. First, for a given collateral matrix C, a collateral matrix C^* is *uniformly less indebted* if $c_{ij} \geq c^*_{ij}$ for any i, j and $c_{ij} > c^*_{ij}$ for at least one pair ij. The second direction comes from diversification. Define L_j as the largest holder of j's collateral, thus, $\max_{i \in N \setminus \{j\}} c_{ij} = c_{L_j j}$. For a given network equilibrium and its collateral matrix C, C^* is a *diversification of agent* j from C, if

1. $c_{L_j j} > c^*_{L_j j} \geq \max_{i \in N \setminus \{j\}} c^*_{ij}$, $c_{ij} \leq c^*_{ij}$ for all $i > L_j$,
2. $\zeta(c^*_{L_j j})\omega_{L_j j} \geq \zeta(c^*_{ij})\omega_{ij}$ for any $i > L_j$,
3. $\sum_{i \in N \setminus \{j\}} c_{ij} \geq \sum_{i \in N \setminus \{j\}} c^*_{ij}$,
4. $c_{ik} \geq c^*_{ik}$ for all $i, k \in N$ with $k \neq j$, and
5. (C^*, Y) is under intermediation order.

This diversification of agent j from an equilibrium collateral matrix implies that agent j has her counterparties more diversified than the original network in either intensive or extensive margins while still maintaining the perceived counterparty risk not exceeding the original largest holder of collateral.

Proposition 6 (Diversification Externality)**.** *Suppose that $(C, Y, e_1, h_1, p_0, \tilde{p}_1, q)$ is a network equilibrium. Suppose there is a collateral matrix C^* and either of the two conditions holds:*

1. *C^* is uniformly less indebted than C.*

2. *C^* is a diversification of agent j from C for $j < n$.*

Then, ex ante expected payment equilibrium price p_1 under $(N, C^*, Y, e_1, h_1, \cdot, \cdot, \zeta)$ is greater than that under $(N, C, Y, e_1, h_1, \cdot, \cdot, \zeta)$ and ex ante expected volatility of p_1 under $(N, C^*, Y, e_1, h_1, \cdot, \cdot, \zeta)$ is lower than that under $(N, C, Y, e_1, h_1, \cdot, \cdot, \zeta)$ for each subjective belief of $j \in N$.

This proposition implies that the higher the debt level is, either uniformly more indebted or under the direction of less diversification, the more the underpricing occurs both in terms of likelihood and intensity. The intuition[21] is that the increase in lender default cost as $\zeta(c)$ increases convexly in c and also the contagion intensifies through both second-order bankruptcy of counterparty channel and asset price channel. If a borrower is more indebted, the expected sum of lender default costs is higher. Also if a borrower is less diversified, the expected sum of lender default costs is higher because of convexity of ζ. The second-order bankruptcy contagion only makes it even worse in expected sense because that only increases the probability of bankruptcy even more. Thus, diversification generates benefits to all of the agents.

Given all of the tools from $t = 1$ payment equilibrium and $t = 0$ borrowing and lending behavior, we can prove existence of a network equilibrium as well as the properties of it.

Theorem 1 (Existence and Characterization of Network Equilibrium).
For a given economy $(N, (s^j, \theta_j, e_0, h_0)_{j \in N}, \zeta, G)$ and maximum equilibrium selection rule, there ***exists*** *a network equilibrium $(C, Y, e_1, h_1, p_0, \tilde{p}_1, q)$, and any network equilibrium is characterized as follows:*

1. *For any $y \in [s^{j+1}, s^j]$*

$$q(y) = q(s^{j+1}) + \frac{E_j\left[\min\left\{1, \frac{y}{p_1}\right\} - \min\left\{1, \frac{s^{j+1}}{p_1}\right\} - \zeta'(c_{(j+1)j})\left[1 - \frac{s^{j+1}}{p_1}\right]\mathbb{1}\left\{j+1 \in B(\epsilon)\right\}\right]}{E_j\left[\frac{1}{p_1}\right]},$$

   where we set $q(s^{n+1}) = s^{n+1} = 0$ and $\max_j E_j\left[\mathbb{1}\{n+1 \in B(\epsilon)\}\right] = 0$.

2. *For any $i, j \in N, i \neq j$, $y_{ij} = s^i$ and (C, Y) is under intermediation order.*

[21]Ibragimov et al. (2011) suggests a model with diversification of risk classes leading to systemic risk through commonality. This force is restricted by the competition in the asset market and high marginal utility of cash under crisis states in my model.

3. For any counterparties i, k of j with $c_{ij} > 0, c_{kj} > 0$,

$$\frac{s^j}{q(s^j) - q(s^i)} E_j \left[\min\left\{1, \frac{s^j}{p_1}\right\} - \min\left\{1, \frac{s^i}{p_1}\right\} - \zeta'(c_{ij}) \left[1 - \frac{s^i}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\} \right]$$
$$= \frac{s^j}{q(s^j) - q(s^k)} E_j \left[\min\left\{1, \frac{s^j}{p_1}\right\} - \min\left\{1, \frac{s^k}{p_1}\right\} - \zeta'(c_{kj}) \left[1 - \frac{s^k}{p_1}\right]^+ \mathbb{1}\left\{k \in B(\epsilon)\right\} \right].$$

4. For any $j, i \in N$ and $j \leq i$, $c_{ji} = 0$.

5. Cash holdings of each agent is determined by

$$e_1^j = e_0^j + h_0^j p_0 + \sum_{i \in N \setminus \{j\}} c_{ij} q(s^i) - \sum_{k \in N \setminus \{j\}} c_{jk} q(s^j) - h_{j,1} p_0.$$

6. The price of the asset at $t = 0$ is determined by

$$p_0 = q(s^1).$$

7. The price of the asset at $t = 1$, $\tilde{p}_1$ is determined by payment equilibrium for the given network (C, Y).

Also the set of network equilibria forms a complete lattice, and there exists a maximum leverage network equilibrium that is the equilibrium with the collateral matrix which has the highest aggregate debt among all other equilibria.

The theorem contains several implications. First of all, the theorem suggests the network structure change for the given economy, in particular the mechanism of network formation—tradeoff between leverage and counterparty risk. Any equilibrium collateral matrix should be an acyclical network as an agent only borrows from more pessimistic agents, and the contract matrix follows rank order due to lemma 3. Each agent can be both borrower and lender because of return differences under subject beliefs and intermediation rents. For negligible default cost (small ζ and θ_j), a single-chain network is formed that is agent j borrows from agent $j+1$ only for all $j < n-1$. This is because even if $c_{j+1,j} = \sum_{k \in N} c_{jk}$, the return from borrowing from $j+1$ is still greater than return from borrowing from $l > j+1$ as the counterparty risk increase is small. Figure 9 is an example of such a network. This resulting intermediation chain resembles the intermediation pattern in Glode and Opp (2016), because the agents with the closest beliefs trade with each other which maximizes the gains of trade. Agents are not concerned about diversifying their counterparty and

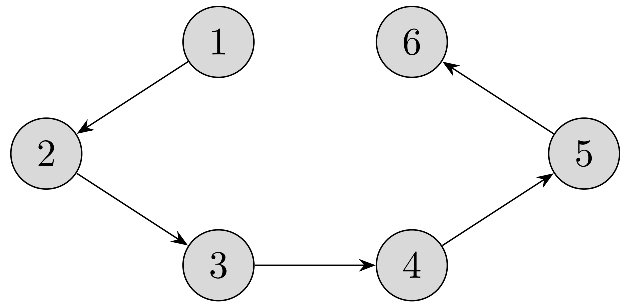

Figure 9: Single-Chain Network

choose the most profitable counterparty—that is, the most optimistic agent after herself—and concentrates all the borrowing. However, if the default cost ζ is non-negligible, then a multi-chain network is formed in equilibrium. Figure 10 is an example of such a multi-chain network. Agent j borrows not only from $j+1$ but also from $j+2$ as well. Agents would rather diversify their counterparties and would like to link with several levels down of optimism. However, this comes at the cost of lower leverage (higher haircut). This network formation mechanism, the tradeoff between leverage and counterparty risk, makes the intermediation pattern distinct from Glode and Opp (2016).

The second implication is the lack of diversification. As shown in proposition 6, is that such diversification of lenders create positive externalities to other agents by making the overall network safer. However, such positive externalities from diversification are not included in individual agent j's concern. Therefore, the degree of diversification is always less than the optimal degree in the economy, and the equilibrium is constrained inefficient. Define the social welfare of the economy as the sum of ex ante expected utilities of all the agents as

$$\sum_{j \in N} E_j \left[m_j(\epsilon) \frac{s^j}{p_1(\epsilon)} \right].$$

An equilibrium is constrained inefficient if a social planner can generate higher social welfare by adjusting the allocation while the resource constraints and collateral constraints are satisfied and leaving the $t = 1$ market interaction decentralized.

Theorem 2. *Any network equilibrium under OTC market is constrained inefficient due to under-diversification if ζ is non-negligible.*

The third implication of theorem 1 is leverage stacking through the lending chain. Increase in $q(s^n)$ increases all the subsequent contract prices, which imply that the lending amount increases. Therefore, lending or leverage at any point in the lending chain has a

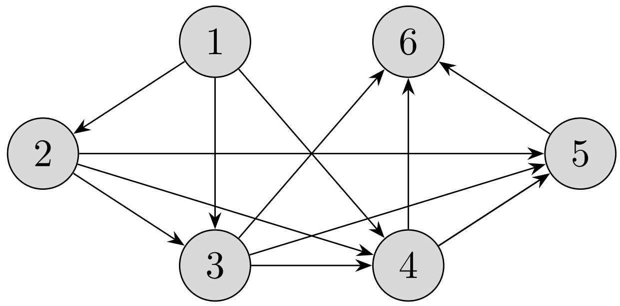

Figure 10: Multi-Chain Network

multiplier effect on the economy. This leverage multiplier effect due to reuse of collateral has been mentioned in Gottardi et al. (2017) as well. A distinct feature of the model in this paper is that different level in the lending chain has different multiplier effects. An increase in s^n will have a larger effect than an increase in s^2 as agent n's lending stacks $n-1$ times through the lending chain through equation 8. A real world implication could be that the increase in the confidence of the ultimate lender (agent n in the model or cash providers such as money market mutual funds in reality) can lead to huge increase in asset prices through this multiplier effect.

The fourth implication is the dispersion of gains of trade. Unlike the result in one link of borrowing and lending in Simsek (2013) and Geerolf (2018), where the gains of trade are fully concentrated to the borrower (agent 1), the gains of trade are dispersed across all agents through competition across different agents and also varying degree of liquidity shortage. The literature regarding the principal-agent problem in lending contract usually focuses on the special case in which borrowers have all the bargaining power (Gale and Hellwig, 1985; Holmstrom and Tirole, 1997), but the result of the network equilibrium shows that even if the market is competitive and each individual agent is a price-taker, there can be some surplus distributed to either side. In particular, even if we retract the model to a single contract case as $n = 2$, the dispersion of bargaining power still occurs. This is partly due to positive cash holdings for either side, which is coming from a liquidity shock and differential marginal utility of cash. Also in the network or lending chain literature context, this feature implies bargaining power between borrowers and lenders is determined endogenously in contrast with the papers such as Farboodi (2017) and Hugonnier et al. (2018), where they assume exogenous bargaining power as some constant. The more cash you are holding, the less power you have in terms of bargaining power as your outside option becomes less profitable and cannot charge a higher interest rate.

The fifth implication of theorem 1 is the endogenous market reaction to the change in counterparty risk. From theorem 1 and proposition 6, the connection between degree centrality and contract prices (interest rates) can be deduced. As the debt of the network increases, the equilibrium contract prices become lower. This is due to the second term of equation (8). The denominator increases while the numerator does not increase as much due to the boundedness of contract returns. The intuition for this result is the following. Since the network has a higher amount of debt, the market in $t = 1$ can suffer more from liquidity shocks and further propagation in case of bankruptcy. Agents prefer to hold cash in case of huge liquidity shocks and are also willing to lend less for the same promise as a lender. Similar comparative statics can be done for the equilibrium contract prices. For example, if all of the agent's liquidity shock arrival rate θ_j increases, then contract price for an agent who borrows cash would decrease as the return from the leverage decreases. Also, change in the asset payoff belief s^j would affect both the amount of debt as well as contract prices. The comparative statics results are summarized as the next proposition.

Before stating the proposition, define the velocity[22] of collateral in a network C as the volume of total collateral posted divided by the stock of source collateral[23]

$$Velocity(C) \equiv \frac{\sum_{i \in N} \sum_{j \neq i} c_{ij}}{\sum_{j \in N} h_{j,1}}.$$

This velocity of collateral represents volume of the reuse of collateral within the network. For example, if the network C is a single-chain network using all of the source collateral, which is all held by agent 1 repeatedly, then the velocity of C is $n - 1$ because $c_{21} = c_{32} = \cdots = c_{n,n-1} = c_{11}$ and $Velocity(C) = \dfrac{c_{21} + c_{32} + \cdots + c_{n,n-1}}{c_{11}} = n - 1$. The velocity of collateral is also an approximate measure of the average length of the lending chain in the network as argued in Singh (2017).

Proposition 7 (Comparative Statics on Borrowing Pattern)**.** *For a given network equilibrium with maximum equilibrium selection rule, the following statements are true.*

1. *If s^j increases (decreases) by the same amount for every $j \in N$, then the equilibrium contract prices and leverage for each agent increases (decreases). Also the number of links between agents weakly decreases (increases) and the velocity of collateral increases (decreases).*

2. *If θ_j increases (decreases) by the same amount for every $j \in N$, then the equilibrium contract prices and leverage for each agent decreases (increases). Also the number of*

[22]Since this model is not dynamic, the "velocity" here means how much a collateral moves around in the market.

[23]This definition is similar to the definition of the velocity of collateral in Singh (2017)—that is, the volume of secured transactions divided by the stock of source collateral.

links between agents weakly increases (decreases) and the velocity of collateral decreases (increases).

The results above can be summarized in the following theorem.

Theorem 3 (Network Change under Crisis)**.** *If the economy is under financial distress and the counterparty risks become greater as s^j decreases or θ_j increases, then agents diversify more, the asset price decreases, the average leverage decreases, the velocity of collateral decreases, and the average number of counterparties (weakly) increases.*

The results of theorem 3 are consistent with the empirical facts. As Singh (2017) documented, the velocity (reuse) of collateral decreased from 3 to 2.4 right after the bankruptcy of the Lehman Brothers[24] and the average leverage in the OTC market also went down. Also Craig and Von Peter (2014) shows that the average number of linkages between financial institutions have increased about 30 percent over the four years after the Lehman bankruptcy. The dynamics of theorem 3 has occurred even before the Lehman bankruptcy. In the wake of Bear Stearns' demise, hedge funds had increasingly used multiple prime brokers to mitigate counterparty risk. In fact, despite the traditionally concentrated structure of the prime brokerage business, as far back as 2006, about 75 percent of hedge funds with at least $1 billion in assets under management relied on the services of more than one prime broker (Scott, 2014). After the Lehman's bankruptcy, hedge funds increased the number of prime brokers they work with even further and the prime brokerage market became much more competitive (which translates into lower intermediation rents under theorem 3) after the crisis (Eren, 2015). On the contrary, the opposite result happened in unsecured debt markets. Afonso et al. (2011) and Beltran et al. (2015) find that the banks in the federal funds market reduced their number of counterparties after the Lehman bankruptcy. This stark comparison shows the importance of collateral in network formation.

Numerical examples in figure 11 and table 1 show the comparative statics in theorem 3. Figure 11 represents the collateral flow of promises of no risk, moderate risk, and significant risk cases respectively. Each numbered node represents the agent, and the arrowed link represents the direction and the size of the promise. As the risk increases, equilibrium network changes from a single-chain network with a large size of collateral flows to a multi-chain network with a smaller size of collateral flows. The no risk case has $\theta_j = \theta = 0$ for all $j \in N$, the moderate risk case has $\theta_j = \theta = 0.4$ for all $j \in N$, and significant risk case has $\theta_j = \theta = 0.8$ for all $j \in N$. As the liquidity shock and counterparty risk become more

[24]The velocity went further down to 1.8 as of 2015. Singh (2017) argues that the collateral landscape has changed further because of central banks' quantitative-easing policies and new regulations which are beyond the scope of this paper.

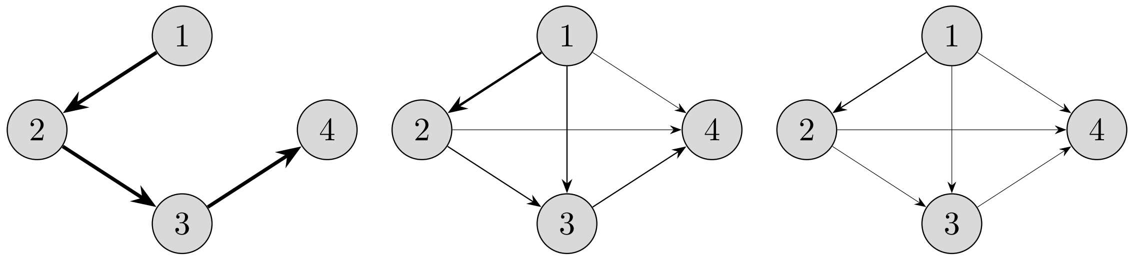

Figure 11: Network Comparative Statics – no risk, moderate risk, and significant risk

	No risk	Moderate risk	Significant risk
Pr(Bankruptcy)	0%	9.6%	25.4%
Leverage	10	2.0766	1.7411
Velocity	3	1.6870	1.4149
$ Volume	2400	756	431
# of links	3	6	6

Table 1: Network Comparative Statics

relevant, the probability of bankruptcy increases. The leverage of a natural buyer, agent 1, decreases by a huge margin, and the velocity of collateral decreases as agents diversify their counterparties which reduces the reuse of collateral. The number of links increases because of diversification and the total nominal cash volume of promises decreases.

3.4. Discussion

The sum of ex ante expected utilities of all agents is comprised of two major parts: the allocative efficiency and financial stability. The allocative efficiency is maximized under a single-chain network because each agent effectively buys (bets) the tranche of the asset that she believes in, however, a single-chain network also minimizes financial stability by the concentration of network and maximized leverage. The overall social welfare should depend on the balance between the two as in Gofman (2017). The sources of externalities are fire sales spillover or collateral externalities as in Duarte and Eisenbach (2018) and Dávila and Korinek (2017), and cascades through networks as in Eisenberg and Noe (2001) and Elliott et al. (2014).

The shape of ζ is important. We can consider many different cost specifications such as concave or constant costs. These cost functions will fail to replicate the risk-aversion behavior and fail to generate the main mechanism – the tradeoff between leverage and counterparty risk. One possible interesting cost structure can be a function that is concave (or constant) at the beginning and then later becomes convex—that is, $\zeta''(c) \leq 0$ for $c \in (0, \bar{c}]$ and $\zeta''(c) > 0$ for $c \in (\bar{c}, \infty)$. This shape will make each and every agent exposed to the agent who is the next most optimistic to her at least of $\bar{c}$ amount.[25] Even more degree of freedom is possible by allowing heterogenous costs for each and every pair as $\zeta_{ij}(c)$. Such heterogenous cost structure would be crucial in estimating the parameters empirically and replicating the core-periphery structure in OTC markets.

One of the important issues related to the lender default problem is the haircut differences. The haircut for a hedge fund's contract is typically greater than the haircut for a dealer bank's contract when they borrow from money market mutual funds. There are two ways to attain these haircut differences under the model of this paper. One way is interpreting the rehypothecated contracts as re-use of the contract cash flow and collateral flow as pyramiding in Geerolf (2018). Under the given pyramiding cash flows, the haircut becomes lower as the collateral goes down the lending chain because the belief discrepancy and possible underpricing (due to liquidity shortage) becomes smaller. Thus, this interpretation reconciles the haircut dispersion we observe from the data. Another way of attaining the haircut differences is by introducing size and cost heterogeneity. If a dealer bank is much larger than its counterparties—hedge funds and money market mutual funds—then the dealer may be able to trade with other agents under much lower haircut. The heterogeneity of size and cost can also help recover the commonly observed network structure – core-periphery networks. More formal analysis on the possible heterogeneity is left for future extensions.

Another important issue related to the lender default problem is allowing rehypothecation. Rehypothecation greatly enhances allocative efficiency but also generates the channel of contagion as we have seen in the bankruptcies of the Lehman Brothers and MF Global. If agents can only trade under standardized contracts or under CCP which does not allow rehypothecation, then this restriction of rehypothecation generates huge loss of gains from trade. Even the borrowers would prefer to allow the lenders to rehypothecate the collateral since that will increase their leverage even more by allowing better funding liquidity for the lender. An anecdotal evidence for this preference is that the borrowers kept using the master agreements that allowed rehypothecation over other contract agreements which prohibit rehypothecation even after experiencing the huge lender default of Lehman (Singh, 2017).

[25] This cost structure also makes sense in terms of institutional details since most of the Chapter 11 bankruptcy problems for small financial institutions are straightforward. This cost structure would make even more sense if other agents' exposure to the same lender also affects the borrower such as $\zeta\left(\frac{c_{ij}}{c_{i1} + c_{i2} + \cdots + c_{in}}\right)$.

4. Central Clearing

As discussed in the introduction, central clearing and the introduction of a central counterparty (CCP) is one of the major issues in market structure regulations. In this section, I define a theoretical way of introducing CCP and perform a counterfactual analysis on the impact of introducing CCP to a decentralized OTC market.

CCP novates one contract between a borrower and a lender into two contracts – a contract between the borrower and the CCP and a contract between the lender and the CCP. This implies the CCP can be considered as a new agent, defined as agent 0, and the CCP simply duplicates the already existing debt network C, Y into its balance sheet. This can be done by first adding all the columns of C, and each column sum will be c_{0i} for all $i \in N$. Then, add all the rows of C, and each row sum will be c_{i0} for all $i \in N$. The contract matrix Y can also be modified by adding the new row and column for 0 with all the relevant promises of s^j for each $j-1$ row and column. CCP also does pooling, which is buffering the counterparty risk with its own balance sheet. The CCP's cash holdings e_1^0 can be considered as a cash buffer, as CCP guarantee funds that are coming from n client agents with γ amount of contribution, so $e_1^0 = n\gamma$. This structure of participation fee is in fact, how the actual CCP manages its guarantee funds in the CCP's "default waterfall." Define the new debt network with CCP as $(C_{ccp},\ Y_{ccp})$.

CCP also nets out obligations between two counterparties. We can consider netting of borrower obligations as a transformation of the debt matrix $C \circ Y$ that is $\hat{C} \circ \hat{Y}$ s.t.

$$\hat{c}_{ij}\hat{y}_{ij} = [c_{ij}y_{ij} - c_{ji}y_{ji}]^+$$

for all $i,\ j \in N$. This can be considered by a transformation of matrix as $[C \circ Y - C' \circ Y']^+$. If this netting procedure is done for the original debt network, then this is a *bilateral netting procedure*. If we run the netting transformation procedure after the inclusion of CCP—that is, $[C_{ccp} \circ Y_{ccp} - C'_{ccp} \circ Y'_{ccp}]^+$—then it becomes the *multilateral netting*, $\hat{C}_{ccp} \circ \hat{Y}_{ccp}$, which is relatively straightforward operation equivalent to the double summation operation in Duffie and Zhu (2011).

The netting should be considered more carefully when it comes to lender obligations since the lender obligation may not be relevant under certain prices when the borrower defaults on their promises. The netting procedure works as follows.

1. For the given price p_1, compute the entry-by-entry indicator matrix $\Gamma \equiv \mathbb{1}(Y = X)$.

2. Compute the effective collateral matrix $C' \equiv C \circ \Gamma$.

3. Perform the CCP netting procedure above to derive $\hat{C}'_{ccp}$.

4. Redistribute the relevant collateral obligations from the updated $\hat{C}'_{ccp}$ to corresponding columns and rows for C.

This redistribution is done by whoever is the final holder of the asset. Under acyclical networks which arise endogenously in the model as seen in theorem 1, there is no indeterminacy of redistribution, so the new debt network is properly defined. Also any leftover wealth of the CCP is equally distributed to the surviving agents. Thus, the CCP's nominal wealth after payments becomes

$$m_0(\epsilon|p_1) = n\gamma - \sum_{j \in N} \sum_{k \in N} \zeta(c_{jk})[p_1 - y_{jk}]^+ \mathbb{1}\left\{j \in B(\epsilon)\right\},$$

and the CCP goes bankrupt when $m_0(\epsilon|p_1) = 0$. Note that an economy under CCP has the debt network that is still under intermediation order and there exists an equilibrium.

There are many important properties of a CCP in reality, such as enhanced transparency and collateral management, that are abstracted out from the model. Other than the pooling and netting of the contracts, I assume that the CCP is exactly the same as the other agents in the economy. The CCP still has to pay the same ζ cost for bankrupt lenders and does not have any additional benefits on collateral management or efficiency in margin settings. Also when the CCP goes bankrupt (if the nominal wealth of the CCP becomes negative), then all the agents suffer ζ cost as the OTC market case. Obviously, these assumptions are strong. For example, the vast majority of Lehman's clients who went through CCPs obtained access to their accounts within weeks of Lehman's bankruptcy (Fleming and Sarkar, 2014). This implies the ζ when the CCP is the borrower might be lower than ζ of the other agents. But, the cost of retrieving collateral after the CCP went bankrupt might be much higher than the lender default costs from the OTC markets. Also CCPs often do not allow rehypothecation or are sometimes themselves restricted in rehypothecating the assets. But, then this restriction comes with a cost of worse flow of collateral and liquidity as the velocity of collateral decreases (Singh, 2017). The main point of this analysis is rather to focus on the understudied property of endogenous reaction of the market, in terms of change in network formation. Any other properties are abstracted out from the model and are subject to further studies.

4.1. CCP without Netting

First, consider the effect of novation and pooling only. Since agents are protected from direct counterparty risk when the CCP survives, agent j's optimization problem becomes

$$\max_{\substack{e_1^j,\{c_{ij},y_{ij}\}_{i\in N},\\ \{c_{jk},y_{jk}\}_{k\in N}}} E_j\left[\left(e_1^j-\epsilon_j+h_{j,1}p_1+\sum_{k\in N\setminus\{j\}}c_{jk}\min\{y_{jk},p_1\}+\frac{m_0(\epsilon|p_1)}{\sum_{i\in N}\mathbb{1}\{i\notin B(\epsilon)\}}\right.\right.$$
$$\left.\left.-\sum_{i\in N\setminus\{j\}}c_{ij}\min\{y_{ij},p_1\}-\sum_{0\in B(\epsilon)}\zeta(c_{ij})[p_1-y_{ij}]^+\mathbb{1}\{i\in B(\epsilon)\}\right)\frac{s}{p_1}\right]^+ \tag{9}$$

s.t.

$$h_{j,1}+\sum_{k\in N\setminus\{j\}}c_{jk}\geq\sum_{i\in N\setminus\{j\}}c_{ij},$$
$$e_0+h_0p_0=e_1^j+h_{j,1}p_0+\gamma-\sum_{i\in N\setminus\{j\}}c_{ij}q(y_{ij})+\sum_{k\in N\setminus\{j\}}c_{jk}q(y_{jk}).$$

From proposition 6 and theorems 1 and 2, the following proposition holds.

Proposition 8. *For a given network equilibrium with maximum equilibrium selection rule under OTC market with collateral matrix C, suppose that a CCP without netting is introduced to the market.*

1. *If the CCP never goes bankrupt because of (implicit) guarantee by the government, then the new network with collateral matrix C_{ccp} has the highest systemic risk across all collateral matrix that will satisfy intermediation order and collateral constraints.*

2. *If agents have to contribute γ to the CCP and the size of the contribution is large enough to cover any lender default costs, then the new network with collateral matrix C_{ccp} has the highest the systemic risk across all collateral matrix that will satisfy intermediation order and collateral constraints.*

3. *If agents have to contribute γ to the CCP and the CCP can go bankrupt in some states, then the new network with collateral matrix C_{ccp} has higher systemic risk than the original network with collateral matrix C.*

The CCP's pooling feature eliminates direct counterparty risk concern from agents. They connect for the most favorable contracts with the most concentrated counterparties. This elimination of positive externalities from diversification amplifies the cost even more through an increase in debt. The intuition can be explained as a metaphor for fire insurance. If an

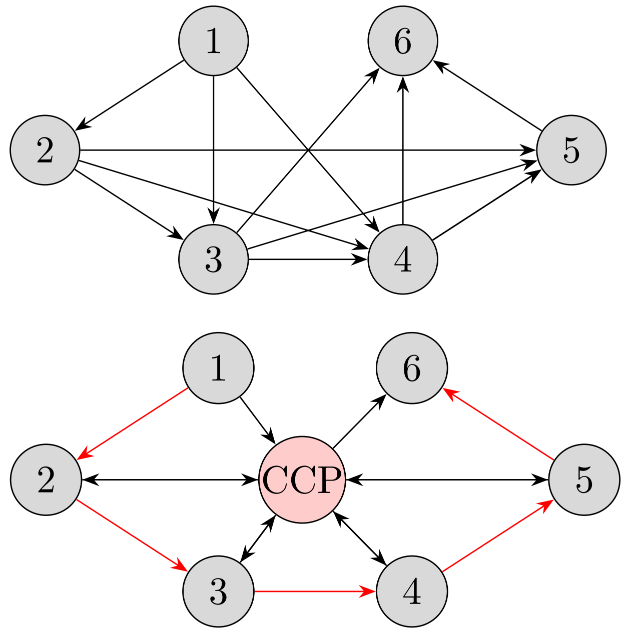

Figure 12: OTC Network and CCP Network

agent joins the fire insurance, her individual fire risk can be fully covered. However, since she does not care about her own fire risk anymore, the moral hazard problem occurs. She does not care for fire safety, which incurs individual effort cost and her probability of fire increases. Thus, the aggregate fire risk rather increases when the economy-wide fire insurance is introduced. In addition, her individual fire safety might have also prevented some spillover fire to other people. Thus, the amplification of aggregate fire risk occurs even further. If γ is high, then some agents, say $j+1, j+2, \ldots, n$, may not participate in the market, if they had the choice, since their return from borrowing or lending in the market does not justify paying γ. However, the individual incentives of the participants are still the same, since marginal incentives are the same. Even though the lending chain leverage may decrease, the network they have is going to maximize the systemic risk for the given component of the network.

The graphical dynamics of the above result is described in figure 12. The top graph is the decentralized OTC network where each agent diversifies their counterparties. The bottom graph is the new network after introducing a CCP in the middle. The notional link in the new network looks like the black links, which are only the contracts between the CCP and the other agents. However, the actual contract flows are the single-chain network in red links, which is different from the OTC network in the top graph. If the endogenous change in the network, from a multi-chain network to a single-chain network, is not taken into account, then the impact of introducing a CCP on systemic risk could be under-evaluated.

4.2. CCP with Netting

A CCP indeed provides positive benefits in reducing systemic risk through the netting. Bilateral netting does not reduce systemic risk at all, because there is no cycle in an endogenously formed network. However, multi-lateral netting does reduce counterparty exposure.

Proposition 9. *Bilateral netting does not affect systemic risk. Multi-lateral netting always decreases systemic risk.*

Multi-lateral netting can reduce risk even if there is no cycle. For example, if agent 1 is borrowing from 2 who is borrowing from 3 and agent 2 goes bankrupt, then agent 1 suffers from default cost. However, if CCP nets out the contracts, then agent 1 can pay 3 to retrieve her collateral and not suffer from default cost because of going through the additional chain of agent 2. Hence, the introduction of a CCP has the cost of systemic risk caused by the network structure (from higher leverage and concentrated counterparty risk) because of pooling and the benefit of reducing net counterparty exposure by multilateral netting.

Exogenous leverage models completely miss all these cost and benefit features. If there is an exogenously given leverage that is fixed as y and its market clearing price is fixed as $q(y)$, then agents will be divided into two groups, buyers (borrowers) and sellers (lenders) of the asset. Then, there is no tradeoff between leverage and counterparty risk since there is only one contract. Agents will fully diversify their counterparties, even for an infinitesimally small default cost. Thus, a complete bi-partite network as in figure 13 is the equilibrium network under exogenous leverage. Since agents are already diversifying fully, pooling has zero effect on network formation. On the other hand, since all the paths in the network have length of 1 and there is no cycle, netting has zero effect as well.

Proposition 10 (Irrelevance of CCP)**.** *If there is only one contract y that is available in the market, then the decentralized OTC equilibrium network is a complete bi-partite network. Furthermore, introduction of a CCP (with or without netting) to such market has no impact on leverage and endogenous network formation.*

4.3. Numerical Examples

In this subsection, I perform a quantitative analysis of the model to provide for numerical examples. There are four agents, each with endowments of 5000 cash and 25 assets, where $\zeta(c) = c^3$, and $S = \{10, 9, 8, 7\}$. The common shock distribution is a log-normal distribution with a mean of 5 and a standard deviation of 5. For 500 samples of this distribution and the

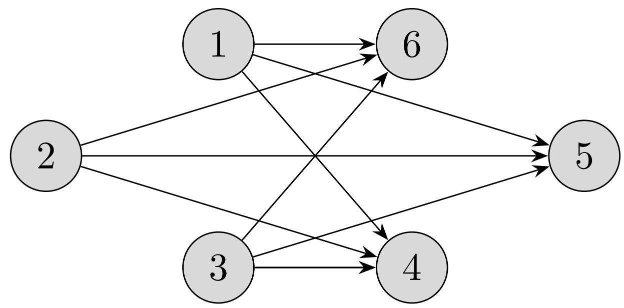

Figure 13: Single Leverage Complete Bi-partite Network

given seed of random number generation, the average shock size is 2406957 and the median shock size is 347.1644. The equilibrium selection rule is the maximum equilibrium selection rule. The algorithm is the following:

Quantitative Algorithm.

1. Guess the initial equilibrium collateral matrix C_0.

2. Compute the payment equilibrium prices $\tilde{p}_1$ and bankruptcy sets $B(\epsilon)$ for each simulated state ϵ out of k different states and for each subject beliefs s^j of agents. (total $n \times k$ matrix of prices and $n \times n \times k$ array of bankruptcy indicators)

3. Compute each agent's expected returns on each investment decision in $t = 0$.

4. Compute the market prices of the asset p_0 and contracts $q(y)$.

5. Derive agent's optimal portfolio decisions starting from agent 1 to agent n. By acyclicality and rehypothecation constraints and lemma 1, this procedure satisfies agents' optimality and market clearing conditions. Set the new collateral matrix as C_1

6. Compare C_0 and C_1. If the difference is above the tolerance level, then update $C_0 = C_1$ and go back to step 2. If the difference is smaller than the tolerance level, then set C_1 as the equilibrium network and compute the rest of the variables of the equilibrium.

First, suppose that the CCP never defaults as the government guarantees the solvency of CCP by tax payer's cash. Under this case, we compare three different cases of the market structure: decentralized OTC market, CCP without netting, and CCP with netting. For each market structure, we change the values of θ, which is the common arrival rate of liquidity shock, and compare the three cases for each θ value. In the graphs in figure 14 and 15, the blue solid lines represent the numbers from a decentralized OTC market, the red dashed lines represent the numbers from a market under a CCP without netting, and the black dotted lines represent the numbers from a market under a CCP with netting.

As in the top-left graph in figure 14, the leverage of the three cases starts with 10. In the OTC market, leverage drops around 2 and stays low as the increase in counterparty risk concern reduces the leverage. On the other hand, two cases with CCP have almost the maximum leverage because agents are not concerned with lender default costs, which is fully covered by the CCP. The top-right graph in figure 14 shows the sum of ex ante social welfare for each case. All of the cases have lower social welfare as the arrival rate of shock increases. However, the OTC market has the highest social welfare compared with the two CCP cases. This is due to agents' diversification in the OTC market, which is absent from the CCP markets. Also netting has an important impact as it limits the duplication of lender default costs from bankruptcies which makes a noticeable differences between the two CCP cases. However, the probability of bankruptcy is still the highest in the OTC market as can be seen in bottom-left of figure 14. The reason is that there exists a contagion channel in the OTC market which is nonexistent in CCP cases because the counterparty channel is insulated by the CCP. As predicted by the theory, the velocity of collateral in the network for the OTC market goes down as θ increases, while the velocity remains the same for two CCP cases.

Now, suppose that the CCP does not have the government guarantee and only covers its losses by the member contribution for the default guarantee fund γ. The size of γ is set as 1000. Under this case, the CCP can actually go bankrupt if the sum of the lender default costs is too large. The leverage graph in the top left of figure 15 shows an interesting shape. In the market with CCP without netting, the leverage rather increases almost to 30 and then start to revert back to 10, which is still much larger than the OTC market case. These dynamics come from the interaction between the counterparty channel and the price channel through the leverage. As $\theta = 0.2$ is still a small number, agents are willing to borrow and lend still very aggressively, however, when the CCP goes bankrupt with the low probability then it will make a huge crash in this case. Agents are gambling for the CCP to survive which is very costly for the agents. Also, since the CCP failure implies total market failure, agents are much less concerned about the event of market failure, because that implies the agents themselves are also out of the market as well. In the meantime, they can have large return from cash holdings if they survive. All of these features contributes to the enormous leverage. This colossal leverage also results in lower social welfare as can be seen in the top right of figure 15. The leverage for the case of CCP with netting is much lower than the case without netting. The first reason is, of course, the reduction of counterparty exposure due to netting and much lower likelihood of market breakdown. The agents do not expect the total market break down, but they do care about having more cash in case of a CCP

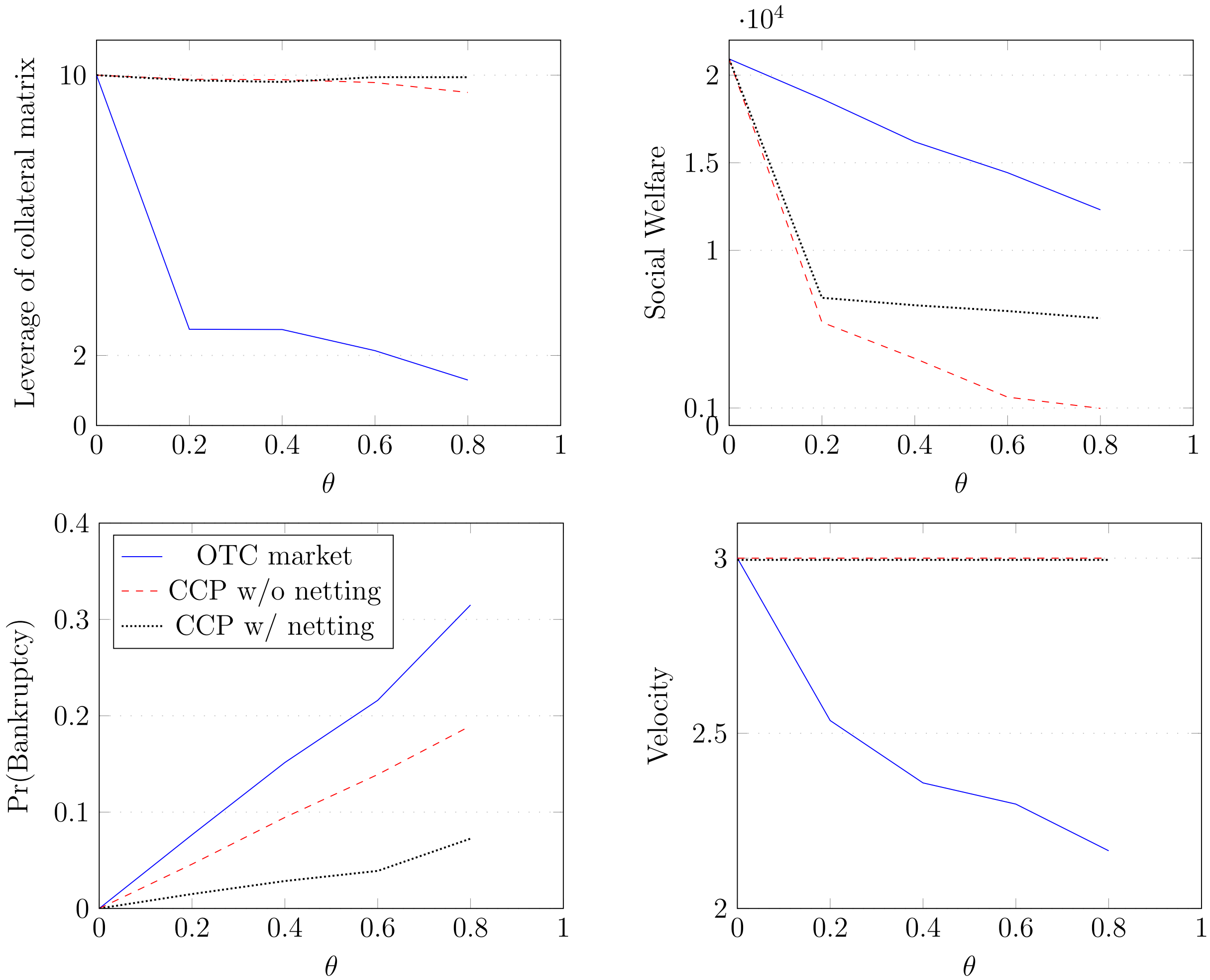

Figure 14: Numerical Results when CCP Never Defaults

failure, but they still survive. Another reason for the moderate leverage is the diversification behavior of agent 1. As the netting cancels out all the exposures between the intermediaries, agent 1 is still exposed to agent n's counterparty risk even after the netting. Therefore, agent 1 wants to diversify and reduces leverage. Since agents are internalizing some of the lender default costs and the netting reduces the total expected lender default costs for a given network, the social welfare under CCP with netting is greater than the social welfare under the OTC market. The bottom left of figure 15 also shows the similar pattern for bankruptcy probabilities. Because agents are recklessly borrowing and lending under CCP without netting, the probability of bankruptcy is very high. The OTC market case is much lower due to diversification but still the CCP with netting has the lowest bankruptcy rate. The velocity of collateral also follows a similar pattern.

I also test the effect of a CCP when the network is exogenously fixed as the decentral-

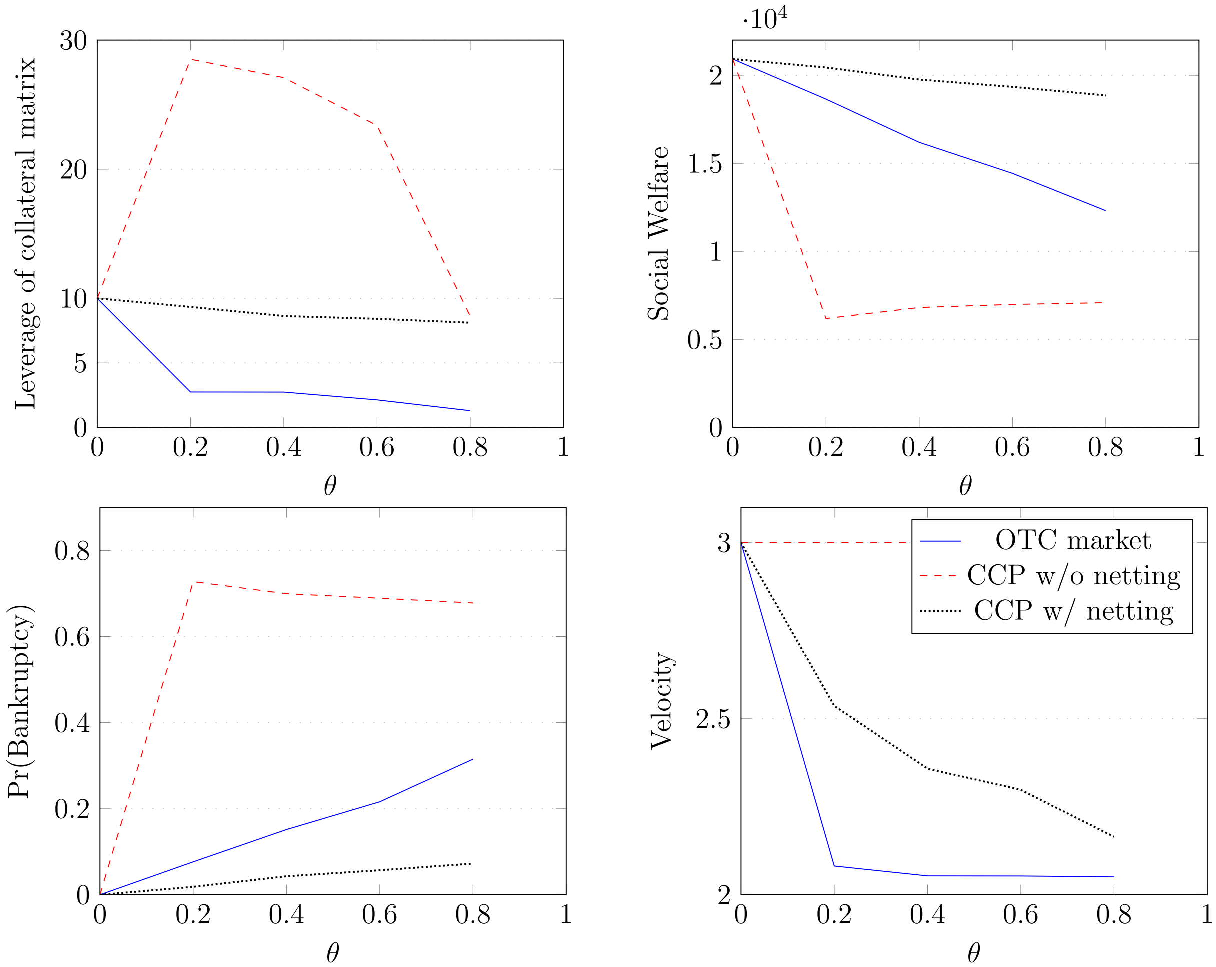

Figure 15: Numerical results when CCP can default

ized OTC market equilibrium. Suppose that even after the introduction of a CCP, agents still maintain the same links as before. Figure 16 plots social welfare of the three cases – OTC market, CCP without netting, and CCP with netting. Numerical results imply that CCP always increases social welfare if the network remains the same. Since netting reduces counterparty exposure, social welfare under CCP with netting is the highest as seen from the previous results. Figure 16 shows that the reversal of social welfare between the OTC market and the market under a CCP without netting in figure 14 and 15 comes from the endogenous change in network formation.

4.4. Policy Implications

The results in the previous subsection do not necessarily imply normative implication such as "introduction of a CCP is always bad." As we can see clearly from figure 15, the social

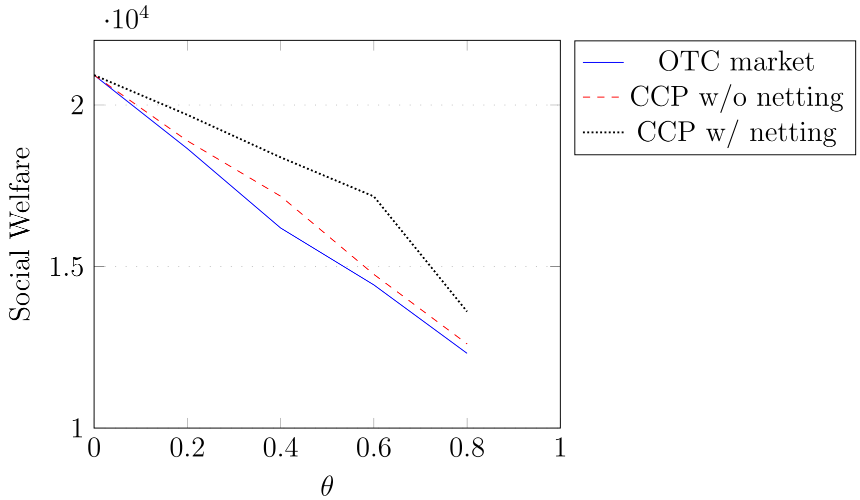

Figure 16: Social Welfare with Exogenous Network

welfare under CCP can be higher than the social welfare under the OTC market depending on the parameter values. The right way to interpret the results is that there can be an understudied or rather neglected cost (side-effect) of introducing a mandatory CCP. This new cost channel, which is a classic moral hazard problem under insurance, is amplified by the network contagion channels (price and counterparty channels), and the increased correlation of payoffs[26] creates a rather exacerbated externality problem. Therefore, introducing a CCP should be done after the cost and benefit analysis from pooling and netting. For example, the CDS market is already highly centralized, and the cost of centralizing such a market with a CCP could be less than the cost of centralizing well diversified markets with a CCP.

Another more direct regulation to solve for the diversification externality problem could be introducing a relevant leverage ratio restriction. In Basel III, there is Supplementary Leverage Ratio (SLR), which is effectively a tax on intermediation activity that is proportional to the size of an intermediary's balance sheet, defined as follows.

$$\frac{\text{Tier 1 Capital}}{\text{Total Leverage Exposure}} \geq 3\%$$

A slight modification of this ratio, Network Supplementary Leverage Ratio, can be used, and risk externality is included as weights of degree centrality in the denominator. Such restrictions provide marginal incentives to diversify and internalize second-order default and maintain borrower or lender discipline of agents and more effective than a crude measure of

[26]Note that the correlation problem was mitigated by liquidity holding incentives of each agent in the OTC market. If there is additional frictional period of liquidity resolution as in Gale and Yorulmazer (2013), then there could be even more problem.

single counterparty credit limit.

A supplementary policy is liquidity injection to the agent under distress according to its impact to the system as in Demange (2016). This injection or bail-out idea also faces side-effects from moral hazard in terms of network formation (Erol, 2018; Leitner, 2005). Markets under CCP will have even less ambiguity and uncertainty of such bail-out possibility and the resulting degree of concentration can be even greater as in the difference between the figures 14 and 15.

There is one crucial feature of the CCP which is absent in the model – increased transparency the CCP provides by having every trade centralized. The model is deliberately abstracted from any trading friction that stems from learning from prices and trading behavior and agents' strategic behavior due to information asymmetry. The crucial information benefits coming from the introduction of a CCP is absent in the model because all of the agents already have full information. Although this can usually be considered as a benefit of a CCP, opaqueness can provide benefits in terms of allocative efficiency as in Dang et al. (2017).

5. Conclusion

I constructed a general equilibrium model with collateral featuring endogenous leverage, endogenous price, and endogenous network formation. The model bridges the theory of financial networks and the theory of general equilibrium with collateral. Collateral generates an additional channel of contagion through asset price risk, the price channel, on top of the balance sheet risk through the debt network, the counterparty channel. Borrowers diversify their portfolios of lenders because of the possibility of lender defaults. However, lower counterparty risk comes at the cost of lower leverage. There are positive externalities from diversification because it reduces not only the individual counterparty risk, but also the systemic risk, by limiting the propagation of shocks and resulting price volatility. Because agents do not internalize these externalities, any decentralized equilibrium is only constrained inefficient. The key externalities here, arising from the tradeoff between counterparty risk and leverage, are absent in models with exogenous leverage or exogenous networks. The model also predicts the empirically observed changes in network structure, leverage (haircuts), asset price, and velocity of collateral during the financial crisis. Greater counterparty risk induces agents to diversify more, which lowers leverage and the velocity of collateral and increases the number of links. I performed a counterfactual analysis on the introduction of a CCP with this model. The loss coverage by CCP exacerbates the externality problems by eliminating individual agents incentives to diversify. Thus, the endogenous network change

	Securities	Derivatives	Repos	All three
Advanced economies	63.6	72.7	31.8	27.3
Europe	60.0	80.0	40.0	40.0
North America	50.0	50.0	25.0	0.0
Asia-Pacific	75.0	75.0	25.0	25.0
Emerging economies	84.6	76.9	61.5	53.8
Asia	100.0	87.5	87.5	25.0
Latin America	60.0	60.0	20.0	87.5
Total	**71.4**	**74.3**	**42.9**	**37.2**

Table 2: Product types cleared by CCPs in 2014 (Source: Bank for International Settlements)

after the introduction of a CCP creates additional systemic risk that exogenous leverage or exogenous network models do not capture.

A. Appendix

A.1. Institutional Details on Central Clearing

In the aftermath of the Lehman Brother's collapse, G20 reform of the over-the-counter (OTC) derivatives market mandated the clearing of standardized derivative contracts by CCPs. One of the principal risks in the financial system that CCPs seek to address is the counterparty credit risk. CCPs enable multilateral netting, and an empirical analysis in Cecchetti et al. (2009) shows that CCP can reduce gross notional exposures by approximately 90 percent. There are also benefits of transparency from CCPs. In a bilateral market, parties know their cross exposures to counterparties but they do not know their counterparties' exposures to third parties. The lack of transparency could significantly increase the systemic risk both by misaligned risk management (Tirole, 2011)—for example, excessive sales of CDS without proper collateral—and the increase in uncertainty. If we consider the risk from derivative contracts sold to unregulated counterparties, such as foreign financial institutions, as counterparties, it creates further risk exposures (Cecchetti et al., 2009). Both private and public sector responses to failures of large financial institutions could be even more complicated. In addition CCP may decrease transaction costs significantly (Case et al., 2013). Multilateral netting also reduces collateral requirements as shown in Duffie et al. (2015). Standardized products by CCPs make it easier to adjust for appropriate margin calls and allow supervisors to monitor the solvency of CCPs. With the concentrated counterparty risk to CCPs, the central clearing market usually pools the "default fund" from all the clearing members or more sophisticated loss-absorbing predetermined contingent equity resources termed "rights of assessment" which mutualizes the aggregate shock to all the clearing members. Furthermore, CCPs require initial margins as collateral from clearing members to recover idiosyncratic counterparty shock, which is usually the operating cost from novating the contract to another potential buyer or seller of the contract. In summary, the reduction of counterparty risks through netting, pooling (mutualization), and orderly distribution of losses are the key differences between trades that are centrally cleared compared with non-cleared transactions.

However, failure of a large CCP could act as a channel of contagion. CCPs actions may have 'procyclical' effects by adjusting initial margin demands, and strict requirements upon its members cause limited access to the market to members with adequate financial and technical resources. Historically, there have been few incidences of CCPs failing, but when these incidents have happened, the impacts on financial markets have been significant. In 1974, the Caisse de Liquidation failed because of trades put forward by members without the consent of their clients and high volatility in Paris White Sugar Market, leading to large margin calls that participants were unable to meet. More recently, the Kuala Lumpur Commodity Clearing House failed in 1983 after massive defaults on palm oil contracts following a market squeeze. The Hong Kong Futures Guarantee Corporation failed in the aftermath of the stock market crash of 1987, which led to the closure of stock and futures exchanges in Hong Kong for four days (Rehlon and Nixon, 2013).

Through intermediation of OTC counterparties, CCPs face a great amount of concentrated counterparty risk, which make CCPs as systemically important financial institutions. It might even be the case that a CCP, while solvent, cannot meet immediate demands for the return of clearing member collateral as well (Singh, 2010). Thus, CCPs demand collateral (initial margin) from their counterparties. CCPs may decrease the systemic risk by reducing the impact of a default of bilateral clearing, while they may increase the systemic risk by increasing margin requirements during financial turmoil, which exacerbates procyclicality. CCPs may be considered as risk pooling and sharing mechanisms through the mutualization of the default funds. Hence, central clearing may reduce the overall margin requirements that are required in bilateral trades (Duffie et al., 2015).

A.2. Proofs

The following lemma is useful for the proofs of the next two results.

Lemma 5. *For a given financial network that satisfies collateral constraints, the effective demand* $[m_j(p)]^+$ *is increasing in* p *for any* $j \in N$.

Proof of Lemma 5. It is enough to show that $m_j(p)$, which is

$$e_1^j - \epsilon_j + h_{j,1} + \sum_{k \in N \setminus \{j\}} c_{jk} \min\{p, y_{jk}\} - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij})[p - y_{ij}]^+,$$

is increasing in p. Since $\min\{y_{ij}, p\} \leq p$, both $\min\{p, y_{ij}\}$ and $\min\{y_{jk}, p\}$ are increasing in p. For any value of promise $\hat{y}$,

$$\sum_{\substack{i \in N \setminus \{j\} \\ y_{ij} \geq \hat{y}}} c_{ij} \min\{y_{ij}, p\} \leq \sum_{\substack{k \in N \setminus \{j\} \\ y_{jk} \geq \hat{y}}} c_{jk} \min\{y_{jk}, p\} + h_{j,1}$$

by intermediation order. Therefore, the sum of the payments from other agents will always exceed the sum of payments that j has to pay to others.[27] Also, by $\zeta(c) \leq c$, the total sum of coefficients for p will always be nonnegative. For fixed $B(\epsilon)$, each $m_j(p)$ is increasing in p. Therefore, for any $p' < p$, $B(\epsilon) \subseteq B(\epsilon')$ and the indicator function for the bankruptcy cost is decreasing in p. ■

Proof of Proposition 1. If $p = s$, then we automatically have an equilibrium that satisfies inequality (3) or otherwise p cannot be s. Now suppose $p < s$. The equilibrium equation can be represented as

$$(M, p) = \left([m_j(p)]_{j \in N}, \frac{\sum_{i \in N} [m_i(p)]^+}{\sum_{j \in N} h_{j,1}} \right) \equiv \mathcal{M}[(M, p)].$$

Consider an ordering $\succeq$ such that $(M, p) \succeq (M', p')$ when $M \geq M'$ and $p \geq p'$. Then an infimum under $\succeq$ can always be defined for any subset of $\mathbb{R}^{n+1}$. By the assumption $(M(s), s) \geq \mathcal{M}[(M(s), s)]$. Since the denominator of the price equation is constant and $h_i^2(p)$ and $[m_i(p)]^+$ are increasing in p by lemma 5, the function $\mathcal{M}$ is an order-preserving function. Then, by Knaster-Tarski's fixed point theorem, there exists a fixed point (M, p), and the set of (M, p) that satisfies the equilibrium condition has a maximal point.

Now suppose that the maximal fixed point price $\bar{p}$ is greater than s, and we will show that either there exists a price $0 < p \leq s$ that is also a fixed point or $p = s$ satisfies equilibrium condition (3). If equation (2) is true when $p = 0$, then we already have a fixed point with $p \leq s$. If equation (2) is not true when $p = 0$, then that implies at least some $m_j(0)$ is positive for $j \in N$ after subtracting the counterparty bankruptcy costs.

[27]This is, in fact, the reason why there is a collateral constraints. It guarantees the agent to have non-negative amount of cash from all the payments netted out so that they can actually pay the debt at any price level of the given collateral.

Therefore, $\frac{\sum_{i \in N}[m_i(p)]^+}{\sum_{j \in N} h_{j,1}} \geq 0$. This implies that as p increases, the difference between the p and $\frac{\sum_{i \in N}[m_i(p)]^+}{\sum_{j \in N} h_{j,1}}$ will be eventually closed out at $\bar{p}$ by intermediate value theorem. Therefore, the two functions either meet for some $p \leq s$, or the gap between them does not close out even when $p = s$ so equation (3) holds. ■

Proof of Proposition 2. For the proof, suppress the ϵ term in bankruptcy sets. If no agent is going to bankrupt at any price $p \in [0, s]$, then the equilibrium price is trivially and uniquely determined as $p = s$. Now suppose some agents go bankrupt at a liquidity constrained price $\tilde{p}$—that is, $B(\tilde{p}) \neq \varnothing$. Denote $\mathcal{V}_l$ as the set of agents such that there is a link between l and i for any $i \in \mathcal{V}_l$. Suppose that $l \notin B(\tilde{p})$ and $\mathcal{V}_l \cap B(\tilde{p}) \neq \varnothing$. Thus, at least at some price close to (or equal to) $\tilde{p}$, the agent l will bear some bankruptcy cost and may go bankrupt. If there is no agent l that satisfies $z^l \equiv e_1^l - \epsilon_l < \sum_{i \in \mathcal{V}_l \cap B(\tilde{p})} \zeta(c_{il})[\tilde{p} - y_{il}]^+$ from $\tilde{p} = 0$ all the way up to s, then $B(p) = B(p')$ for any $p, p' \in [0, s]$ and in fact there is unique equilibrium since there will be no jumps in $\mathcal{M}$.

Now suppose that for some price $\tilde{p}$ and some agent l, $z^l < \sum_{i \in \mathcal{V}_l \cap B(\tilde{p})} \zeta(c_{il})[\tilde{p} - y_{il}]^+$ is satisfied. Then, there exists p^* less than p (due to monotonicity of $m_l(p)$) such that $\forall p' < p^*$, $m_l(p') < 0$ and suppose l be the only one who goes bankrupt due to the price decline from p to $p' < p^*$ without loss of generality. The left-hand side of the market clearing condition, the sum of effective money, can be decomposed as

$$\sum_{j \in N}[m_j(p)]^+ = \sum_{j \in N} e_1^j + \sum_{j \in N} h_{j,1}p - \sum_{j \in N} \sum_{i \in B(p)} \zeta(c_{ij})[p - y_{ij}]^+$$
$$- \sum_{j \in N} \min\left\{\epsilon_j, e_1^j - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(p)} \zeta(c_{ij})[p - y_{ij}]^+ + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\}\right\}.$$

Since the term is the same as the supply side of the equation, price is determined by the remaining cash from $t = 0$ and the amount of aggregate liquidity shock to the demand, bounded by its entire position, and the counterparty default costs. We can rewrite the market clearing condition into loss-coverage with remaining cash equality as

$$\begin{aligned}\sum_{j \in N} e_1^j = \sum_{i \in B(p)} \sum_{j \in N} \zeta(c_{ij})[p - y_{ij}]^+ + \sum_{j \in N} \min\Bigg\{&\epsilon_j, e_1^j + h_{j,1}p \\ &- \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(p)} \zeta(c_{ij})[p - y_{ij}]^+ + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\}\Bigg\}\end{aligned} \tag{10}$$

Then, there can be a price $\hat{p}$ such that the additional jump in bankruptcy cost $\beta_l(p) \equiv \sum_{j\in N}\zeta(c_{lj})[p-y_{lj}]^+$ coincides with the amount of decrease in losses from bankrupt agent's endowments and counterparty costs—that is,

$$\begin{aligned}\beta_l(\hat{p}) =& \epsilon_l + \sum_{j\in B(p)}\Bigg[\sum_{i\neq j}\left(c_{ij} - \mathbb{1}\{i\in B(p)\}\zeta(c_{ij})\right)\left(\mathbb{1}\left\{p>\hat{p}\geq y_{ij}\right\}(p-\hat{p})\right.\\ &\left.+\mathbb{1}\left\{p\geq y_{ij}>\hat{p}\right\}(p-y_{ij})\right)+\zeta(c_{lj})[\hat{p}-y_{ij}]^+\\ &+\sum_{k\in N}c_{jk}\left(\mathbb{1}\left\{y_{jk}>p>\hat{p}\right\}(p-\hat{p})+\mathbb{1}\left\{p\geq y_{jk}>\hat{p}\right\}(y_{jk}-\hat{p})\right)\Bigg]\\ &-\left[e_1^l-\sum_{i\neq l}c_{il}\min\{\hat{p},y_{il}\}-\sum_{i\in B(p)}\zeta(c_{il})[\hat{p}-y_{il}]^+ +\sum_{k\in N}c_{lk}\min\{\hat{p},y_{lk}\}\right].\end{aligned} \tag{11}$$

Therefore, $\hat{p}$ is also an equilibrium price. ■

Proof of Proposition 3. For a fair price, there exists unique equilibrium price no matter what happens in shocks and bankruptcies. Now focus on liquidity constrained prices. When $\zeta(c) = 0$ for any $c \geq 0$, equation (10), the market clearing condition with loss-coverage, becomes

$$\sum_{j\in N}e_1^j = \sum_{j\in N}\min\left\{\epsilon_j, e_1^j + h_{j,1}p - \sum_{i\in N\setminus\{j\}}c_{ij}\min\{p,y_{ij}\}+\sum_{k\in N\setminus\{j\}}c_{jk}\min\{p,y_{jk}\}\right\},$$

and by intermediation order, the right-hand side is increasing in p. Also the right-hand side is bounded below by $\sum_{j\in N}\min\{\epsilon_j, e_1^j\}$, when $p = 0$. By intermediate value theorem, there exists a unique equilibrium price p between $[0, s]$ that satisfies the market clearing condition above.

For the second statement of the proposition, first note that the nominal wealth with no lender default cost is

$$m_j(p) = e_1^j + h_{j,1}p - \epsilon_j - \sum_{i\in N\setminus\{j\}}c_{ij}\min\{p,y_{ij}\}+\sum_{k\in N\setminus\{j\}}c_{jk}\min\{p,y_{jk}\}.$$

The sum of nonnegative nominal wealth is

$$\sum_{j\in N}[m_j(p)]^+ = \sum_{j\in N} e_1^j + \sum_{j\in N} h_{j,1}p$$
$$- \sum_{j\in N}\min\left\{\epsilon_j, e_1^j - \sum_{i\in N\setminus\{j\}} c_{ij}\min\{p, y_{ij}\} + \sum_{k\in N} c_{jk}\min\{p, y_{jk}\}\right\},$$

which can be re-written as the sum of indegrees and outdegrees as below.

$$\sum_{j\in N}[m_j(p)]^+ = \sum_{j\in N} e_1^j + nh_0 p - \sum_{j\in N}\min\left\{\epsilon_j, e_1^j - \sum_{i\in N\setminus\{j\}} c_{ij}x_{ij} + \sum_{k\in N} c_{jk}x_{jk}\right\},$$

which will have the same value with a network with

$$\sum_{i\in N\setminus\{j\}} c_{ij}x_{ij} = \sum_{i\in N\setminus\{j\}} \hat{c}_{ij}\hat{x}_{ij}$$
$$\sum_{k\in N} c_{jk}x_{jk} = \sum_{k\in N} \hat{c}_{jk}\hat{x}_{jk},$$

so networks (C, X) and $(\hat{C}, \hat{X})$ have the same equilibrium price and final asset holdings. ■

Proof of Lemma 1. For each agent $i \in N$, the maximum cash he can hold for $t = 1$ is by saving all the cash while not lending any cash because borrowing requires collateral and no arbitrage condition will prevent anyone from making positive cash from borrowing. The price of the asset at $t = 0$ cannot exceed the most optimistic agent's fair value since there is always a possibility of liquidity constrained underpricing in $t = 1$. Thus, $e_0 + h_0 s^1$ is always the upper bound of the maximum amount of cash each agent can hold by selling all the asset endowments and not borrowing from or lending to anyone. Since G is differentiable with full support of $[0, \bar{\epsilon}]$, any agent can go bankrupt regardless of how much cash they hold in $t = 0$ because $G([e_0 + h_0 s^1, \bar{\epsilon}])$ is positive. Now suppose that agent j has zero cash holdings—that is, $e_1^j = 0$. Agent j's nominal wealth depends on asset price p_1, which becomes zero if $p_1 = 0$. By equation (10), this implies that if every other agent $i \neq j$ goes bankrupt because of liquidity shocks, which happens with probability greater than $[G([e_0 + h_0 s^1, \bar{\epsilon}])]^{n-1}$, while agent j is not, which happens with positive conditional probability, the price of the asset becomes zero while agent j is not bankrupt. Marginal utility of cash in such a state becomes $\lim_{p_1\to 0}\frac{s^j}{p_1}$ which is infinity. Hence, expected marginal utility of holding cash in $t = 0$ becomes infinity as well and agent j would like to hold a positive amount of cash for any $j \in N$. If

$e_1^j > 0$, then the only state that with infinite marginal utility of cash is when $\epsilon_j = e_1^j$ which happens with zero probability measure by differentiability of G. Thus, in an equilibrium, $e_1^j > 0$ for any $j \in N$. ■

Proof of Lemma 2. The proof is done by contradiction. Suppose that $E_j\left[\frac{s^j}{p_1}\right] \leq E_k\left[\frac{s^k}{p_1}\right]$ for $j < k$. If both j and k are simply holding cash exclusively, then they have the same cash holdings and it is trivially $E_j\left[\frac{s^j}{p_1}\right] > E_k\left[\frac{s^k}{p_1}\right]$. Therefore, at least agent k should be investing in something other than cash. Suppose that agent k is borrowing from i and lending to l. Then her return from this intermediation is

$$\begin{aligned}
&\frac{E_k\left[\min\left\{s^k, y'\frac{s^k}{p_1}\right\} - \min\left\{s^k, y\frac{s^k}{p_1}\right\} - \zeta'(c_{ik})\left[s^k - y\frac{s^k}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{q_k(y') - q_i(y)} \\
=&\frac{s^k E_k\left[\min\left\{1, \frac{y'}{p_1}\right\} - \min\left\{1, \frac{y}{p_1}\right\} - \zeta'(c_{ik})\left[1 - \frac{y}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{q_k(y') - q_i(y)} = E_k\left[\frac{s^k}{p_1}\right].
\end{aligned}$$

The last equality holds because the return should be equal to the return from holding cash because of positive cash holding by lemma 1. Now consider an agent j who deviates from her equilibrium portfolio decision. Agent j can mimic the investment portfolio of agent k and obtain the return of

$$\frac{s^j E_j\left[\min\left\{1, \frac{y'}{p_1}\right\} - \min\left\{1, \frac{y}{p_1}\right\} - \zeta'(c_{ik})\left[1 - \frac{y}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{q_k(y') - q_i(y)} \leq E_j\left[\frac{s^j}{p_1}\right],$$

with the last inequality coming from optimality of agent j's original portfolio decision. In other words, she would have already done the intermediation more if it exceeded the return from her cash holdings (which is again positive by lemma 1). If agent j is mimicking k's portfolio exactly the same, the two agents will have the same cash holdings and also the same counterparty risks (or even less if j was the lender). Then, inequalities

$$\begin{aligned}
&E_j\left[\min\left\{1, \frac{y'}{p_1}\right\} - \min\left\{1, \frac{y}{p_1}\right\} - \zeta'(c_{ik})\left[1 - \frac{y}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right] \\
\geq&E_k\left[\min\left\{1, \frac{y'}{p_1}\right\} - \min\left\{1, \frac{y}{p_1}\right\} - \zeta'(c_{ik})\left[1 - \frac{y}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right],
\end{aligned}$$

and $s^j > s^k$ imply

$$E_j\left[\frac{s^j}{p_1}\right] \geq \frac{s^j E_j\left[\min\left\{1,\frac{y'}{p_1}\right\} - \min\left\{1,\frac{y}{p_1}\right\} - \zeta'(c_{ik})\left[1-\frac{y}{p_1}\right]\mathbb{1}\{i\in B(\epsilon)\}\right]}{q_k(y')-q_i(y)}$$
$$> \frac{s^k E_k\left[\min\left\{1,\frac{y'}{p_1}\right\} - \min\left\{1,\frac{y}{p_1}\right\} - \zeta'(c_{ik})\left[1-\frac{y}{p_1}\right]\mathbb{1}\{i\in B(\epsilon)\}\right]}{q_k(y')-q_i(y)} = E_k\left[\frac{s^k}{p_1}\right],$$

that is, $E_j\left[\frac{s^j}{p_1}\right] > E_k\left[\frac{s^k}{p_1}\right]$, which contradicts the initial assumption $E_j\left[\frac{s^j}{p_1}\right] \leq E_k\left[\frac{s^k}{p_1}\right]$. The same method could be applied to any other possible investment strategy of agent k – lending without leverage or buying the asset with or without leverage. Therefore, $E_j\left[\frac{s^j}{p_1}\right] > E_k\left[\frac{s^k}{p_1}\right]$ holds for any equilibrium. ■

The following lemma shows that whenever leveraging is profitable for certain investment, the same leverage makes other investment more profitable than not leveraging.

Lemma 6. *Suppose* $\frac{a-p}{b-q} = \pi = \frac{c-p}{d-q} = \frac{e}{f}$ *and* $\frac{a}{b} < \frac{a-p}{b-q}$ *for* $a,b,c,d,e,f,p,q,\pi > 0$. *Then,* $\frac{c}{d} \leq \frac{c-p}{d-q}$ *and* $\frac{e}{f} < \frac{e-p}{f-q}$.

Proof of Lemma 6. Since $\frac{a-p}{b-q} = \pi$, $a-p = b\pi - q\pi$. By $\frac{a}{b} < \frac{a-p}{b-q}$, we obtain $a < b\pi$. By combining the previous equation and inequality, we have $p < q\pi$. Now suppose that $\frac{c}{d} > \frac{c-p}{d-q}$. Then, $\frac{c-p}{d-q} = \pi$ implies $c > d\pi$. Combining this with $p < q\pi$, we get $\frac{c-p}{d-q} > \pi$, which is a contradiction. Therefore, $\frac{c}{d} \leq \frac{c-p}{d-q}$. Similarly, suppose $\frac{e}{f} \geq \frac{e-p}{f-q}$. Then, from $e = f\pi$, we obtain $e-p \leq f\pi - q\pi$, which implies $q\pi \leq p$, which is again a contradiction. Thus, $\frac{e}{f} < \frac{e-p}{f-q}$. ■

Proof of Lemma 3.

From the return equation (7), we immediately get $y' > y$, and $q_j(y') > q_i(y)$ should hold for agent j's decision optimality and no arbitrage.[28] Similarly, from the positive cash holding and optimality we know that

$$q_i'(y) = \frac{E_i\left[\frac{1}{p_1}\middle| p_1 > y\right]\Pr_i(p_1 > y)}{E_i\left[\frac{1}{p_1}\right]},$$

[28]No arbitrage prevents the case of $y' < y$ and $q_j(y') < q_i(y)$.

which is zero for any $y > s^i$. The (semi) partial derivative for agent j's decision on the contract promise choice y to agent i is

$$
\begin{aligned}
& s^j E_j\left[-\frac{c_{ij}}{p_1}+\zeta(c_{ij})\frac{1}{p_1}\mathbb{1}\left\{i\in B(\epsilon)\right\}\Big|p_1>y\right]\Pr_j(p_1>y)+\lambda c_{ij}q_i'(y)\\
=& s^j E_j\left[-\frac{c_{ij}}{p_1}\Big|p_1>y\right]\Pr_j(p_1>y)+s^j E_j\left[\zeta(c_{ij})\frac{1}{p_1}\mathbb{1}\left\{i\in B(\epsilon)\right\}\Big|p_1>y\right]\Pr_j(p_1>y)\\
&+s^j E_j\left[\frac{1}{p_1}\right]c_{ij}\frac{E_i\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_i(p_1>y)}{E_i\left[\frac{1}{p_1}\right]},
\end{aligned}
$$

where λ is the Lagrangian multiplier for the budget constraint, $\lambda = s^j E_j[1/p_1]$, from lemma 1 and envelope theorem. First, if $y > s^i$, then the last term is zero. Since $c_{ij} > \zeta(c_{ij})$, the first-order derivative (right-semi differential if $y = s^i$) is negative for any $y > s^i$. Now consider $y \le s^i$. Even if the counterparty risk is zero, we can show that the above first-order derivative is positive by showing the following inequality for any $y \le s^i$,

$$
\frac{E_j\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_j(p_1>y)}{E_j\left[\frac{1}{p_1}\right]}<\frac{E_i\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_i(p_1>y)}{E_i\left[\frac{1}{p_1}\right]}. \tag{12}
$$

Suppose that the above inequality does not hold—that is,

$$
\frac{E_j\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_j(p_1>y)}{E_j\left[\frac{1}{p_1}\right]}\ge\frac{E_i\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_i(p_1>y)}{E_i\left[\frac{1}{p_1}\right]}. \tag{13}
$$

From lemma 2,

$$
\begin{aligned}
E_j\left[\frac{s^j}{p_1}\right]=& s^j\left(E_j\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_j(p_1>y)+E_j\left[\frac{1}{p_1}\Big|p_1\le y\right]\Pr_j(p_1\le y)\right)\\
>& s^i\left(E_i\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_i(p_1>y)+E_i\left[\frac{1}{p_1}\Big|p_1\le y\right]\Pr_i(p_1\le y)\right)=E_i\left[\frac{s^i}{p_1}\right],
\end{aligned}
$$

which can be rearranged as

$$
\begin{aligned}
&\frac{1}{s^j\left(E_j\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_j(p_1>y)+E_j\left[\frac{1}{p_1}\Big|p_1\le y\right]\Pr_j(p_1\le y)\right)}\\
<&\frac{1}{s^i\left(E_i\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_i(p_1>y)+E_i\left[\frac{1}{p_1}\Big|p_1\le y\right]\Pr_i(p_1\le y)\right)}.
\end{aligned}
\tag{14}
$$

By the assumption (13),

$$
\begin{aligned}
&\frac{s^jE_j\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_j(p_1>y)}{s^j\left(E_j\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_j(p_1>y)+E_j\left[\frac{1}{p_1}\Big|p_1\le y\right]\Pr_j(p_1\le y)\right)}\\
\ge&\frac{s^iE_i\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_i(p_1>y)}{s^i\left(E_i\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_i(p_1>y)+E_i\left[\frac{1}{p_1}\Big|p_1\le y\right]\Pr_i(p_1\le y)\right)},
\end{aligned}
$$

which implies that

$$
\frac{s^jE_j\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_j(p_1>y)}{s^iE_i\left[\frac{1}{p_1}\Big|p_1>y\right]\Pr_i(p_1>y)}>\frac{s^jE_j\left[\frac{1}{p_1}\right]}{s^iE_i\left[\frac{1}{p_1}\right]}.
$$

Since the upper bound for price under agent j's perspective, s^j, is higher than that under agent i's perspective, s^i, previous inequality holds only if $\Pr_j(p_1>y)$ is much larger than $\Pr_i(p_1>y)$. However, then $\Pr_i(p_1\le y)>\Pr_j(p_1\le y)$ and $1/p_1$ is larger when $p_1\le y$ than $1/p_1$ when $p_1>y$. Therefore,

$$
\frac{s^jE_j\left[\frac{1}{p_1}\right]}{s^iE_i\left[\frac{1}{p_1}\right]}<1,
$$

which violates (14). Therefore, the assumption (13) is false, and (12) holds, which implies the first-order derivative (left-semi differential) is positive for any $y\le s^i$. Hence, agent j promises s^i and maximizes her leverage.

In the second part of the lemma, we apply the result from the first part of the lemma and fix the contracts with promises of expected asset payoffs of the lenders. Suppose agent

j is borrowing both from i and k with the same probability of bankruptcy and $i < k$, for the same amount of contracts—that is, $c \equiv c_{ij} = c_{ik}$. The marginal returns from both leveraged positions for j are

$$R_j^i \equiv \frac{s^j}{q_j(s^j) - q_i(s^i)} E_j \left[\min\left\{1, \frac{s^j}{p_1}\right\} - \min\left\{1, \frac{s^i}{p_1}\right\} - \zeta'(c) \left[1 - \frac{s^i}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\} \right]$$
$$R_j^k \equiv \frac{s^j}{q_j(s^j) - q_k(s^k)} E_j \left[\min\left\{1, \frac{s^j}{p_1}\right\} - \min\left\{1, \frac{s^k}{p_1}\right\} - \zeta'(c) \left[1 - \frac{s^k}{p_1}\right]^+ \mathbb{1}\left\{k \in B(\epsilon)\right\} \right]$$

Since q_k' is increasing at the left limit of s^k and j maximizes over (s^k, s^i) at s^i, relative increase in the amount of borrowing (or decrease in down payment) should exceed the relative decrease in expected payoff at $t = 1$ under no arbitrage condition. Therefore, agent j prefers to borrow more from i over k. ■

Proof of Proposition 4.

By lemmas 3 and 4, agents form a chain of intermediation: Agent 1 borrows from 2, who borrows from 3, who borrows from 4, and so on. There will be no missing chain because of lemma 3 and the property of lender cost function ζ—that is, at least some positive amount of borrowing occurs through the lending chain linking the agents in the order of optimism. Also, in the equilibrium, $q_{i+1}(y) > q_i(y)$ for any $y \leq s^{i+1}$ for any $i \in N, i < n$. This is true because, if i can leverage and maximize return for some other contract such as lending to agent $i-1$, then she can also increase her return from lending at y by leveraging from agent $i+1$ with the same y. Thus, because of the possible counterparty risk, which is positive due to lemma 3, the marginal return from this intermediation is

$$\frac{-\zeta'(c_{i+1,i}) E_j \left[\left[1 - \frac{y}{p_1}\right]^+ \mathbb{1}\left\{i+1 \in B(\epsilon)\right\} \right]}{q_i(y) - q_{i+1}(y)},$$

and the sign of $q_i(y) - q_{i+1}(y)$ should be negative to make the return match agent i's other returns[29]. Hence, all the contract prices are determined by the subsequent lender. In other words, competitive contract prices for $y \in [s^{j+1}, s^j]$ are determined by j.

[29]The inequality of $q_i(y) < q_{i+1}(y)$ will be clear in the contract pricing formula (15) as well.

From equation (8), we have j's contract pricing formula as follows.

$$q_j(y) = q_{j+1}(s^{j+1}) + \frac{E_j\left[\min\left\{1, \frac{y}{p_1}\right\} - \min\left\{1, \frac{s^{j+1}}{p_1}\right\} - \zeta'(c_{ij})\left[1 - \frac{s^{j+1}}{p_1}\right]^+ \mathbb{1}\left\{_{j+1 \in B(\epsilon)}\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}. \tag{15}$$

Since $q_{j+1}(s^{j+1})$ is determined by the perspective of $j+1$, the only relevant factor is the second term. As y increases, the relevant lower bound of price for borrower default increases. Obviously, s^j is the maximum price in j's perspective, and $q_j'(y) = 0$ at $y = s^{j+}$—that is, the right semi-derivative is zero. On the other hand, $y = s^{j+1}$ provides no additional value and simply becomes $q_j(s^{j+1}) = q_{j+1}(s^{j+1}) - \zeta'(c_{j+1,j})\omega_{j+1,j}(y)$, and again we find $q_j(y) < q_{j+1}(y)$ at $y = s^{j+1}$.

Now we compute the derivatives. By Leibniz integral rule, for any $y \in [s^{j+1}, s^j)$,

$$q_j'(y) = \frac{E_j\left[\frac{1}{p_1}\middle| p_1 > y\right]\mathrm{Pr}_j(p_1 > y)}{E_j\left[\frac{1}{p_1}\right]} > 0$$

$$q''(y) = -\frac{1}{E_j\left[\frac{1}{p_1}\right]}\frac{f_j(y)}{y} < 0,$$

where f_j is the density function of F_j, which is the distribution function of the asset price in $t = 1$ that comes from the convolution of shock distributions. Thus, $q_j(y)$ is concavely increasing in y. Denote κ_j as the inverse function of $q_j(y)$ which is well defined in the domain of $y \in [s^{j+1}, s^j)$ since $q_j'(y) > 0$ in the domain and $q_j'(s^j) = 0$. Suppress the subscript for q, κ for the rest of the proof.

By inverse function theorem of first and second-order derivatives, for any $q(y)$ in the range of original function, we obtain

$$\kappa'(q(y)) = \frac{1}{q'(y)} > 0$$

$$\kappa''(q(y)) = -\frac{q''(y)}{(q'(y))^3} > 0.$$

Now denote the gross interest rate function as $\delta(q) \equiv \frac{\kappa(q)}{q}$, where q is in the range of $q(y)$.

The first derivative of the gross interest rate function becomes

$$\delta'(q) = \frac{\kappa'(q)q - \kappa(q)}{q^2} = \frac{\dfrac{q(y)}{q'(y)} - y}{q(y)^2},$$

where $\kappa(q) = y$. The numerator of the term can be rearranged as $q(y) - yq'(y)$ and this is positive because

$$\begin{aligned} q_j(y) &= q_{j+1}(s^{j+1}) + \frac{E_j\left[\min\left\{1, \dfrac{y}{p_1}\right\} - \min\left\{1, \dfrac{s^{j+1}}{p_1}\right\} - \zeta'(c_{ij})\left[1 - \dfrac{s^{j+1}}{p_1}\right]^+ \mathbb{1}\left\{j+1 \in B(\epsilon)\right\}\right]}{E_j\left[\dfrac{1}{p_1}\right]} \\ &> \frac{E_j\left[\dfrac{y}{p_1}\middle| p_1 > y\right] Pr_j(p_1 > y)}{E_j\left[\dfrac{1}{p_1}\right]}, \end{aligned}$$

where the last inequality is positive by lemma 6. Therefore, the gross interest rate is increasing in y. The second derivative of the gross interest rate function becomes

$$\delta''(q) = \frac{1}{q^4}\left[q^2\left(\kappa''(q)q + \kappa'(q) - \kappa'(q)\right) - 2q\left(\kappa'(q)q - \kappa(q)\right)\right],$$

and the numerator is

$$\begin{aligned} \kappa''(q)q^3 - 2q^2\kappa'(q) + 2q\kappa(q) &= -q''(y) + 2q(y)\left[y - q(y)\kappa'(q(y))\right] \\ &= \frac{f_j(y)/y}{E_j\left[\dfrac{1}{p_1}\right]} - 2q(y)\left[q(y)/q'(y) - y\right] \\ &= \frac{f_j(y)/y}{E_j\left[\dfrac{1}{p_1}\right]} - 2q(y)\left[q(y)\frac{E_j\left[\dfrac{1}{p_1}\right]}{E_j\left[\dfrac{1}{p_1}\middle| p_1 > y\right]\mathrm{Pr}_j(p_1 > y)} - y\right], \end{aligned}$$

which is negative because $q(y) > yq'(y)$ as shown previously. Also $q(y)/q'(y) - y > 1$ implies the inequality to be trivial, and $q(y)/q'(y) - y \leq 1$ also means the first term is negligible compared to the conditional expectation in $q(y)$ of the second term. Thus, $y/q(y)$ is concavely increasing in the interval of $q(y) \in [q(s^{j+1}), q(s^j))$.

Now we need to check for the kink points and the whole graph. Because $q_j'(s^j) = 0$, $\delta_j'(q)$ goes to infinity, that is why $q_1'(s^1)$ is infinity. A unique property of the pricing of equation

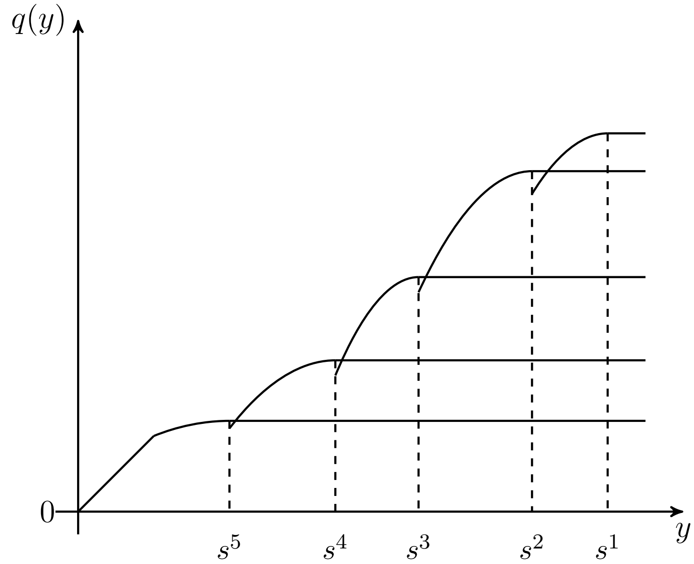

Figure 17: Graph of Contract Prices

(8) is that y close to s^{j+1} will make $q_j(y) < q_{j+1}(s^{j+1})$ coming from the left limit of $q_j(s^{j+1})$. Therefore, there are intersections around each point of s^j for $j \in N$ as can be seen in the figure 17. Since the borrowers would rather prefer to borrow from low y for higher $q(y)$, the market price function for $q(y)$ will take the upper envelope of the functions q defined for each interval $(s^{j+1}, s^j]$ for $j = 1, 2, \ldots, n-1$. Hence, the inverse function of q, κ will have jumps at each point of $q(s^j)$ for $j \neq 1, n$ and the right derivative is greater than the left derivative of each point. Finally, since the upper envelope of functions q are continuous because above s^j there is a point that borrowers prefer to simply borrow from j at a constant price rate up to the point that $j-1$ becomes the preferred lender when $q(y)$ is greater than or equal to $q(s^j)$. Therefore, both the upper envelope function of market price $q(y)$ is continuous, and the interest rate function is also continuous. ■

Proof of Lemma 4. Suppose agent $j > 1$ is buying the asset while agent 1 is not buying, then agent 1 will have an even larger amount of cash holdings. If agent 1's cash holding e_1^1 is large, then j's return of cash is large. Return from the asset purchase for agent j is s^j/p_0. By lemma 1, agent j should equate the returns from cash and asset as

$$\frac{s^j}{p_0} = E_j\left[\frac{s^j}{p_1}\right].$$

But then, $\dfrac{s^j}{p_0} < \dfrac{s^1}{p_0} < E_1\left[\dfrac{s^1}{p_1}\right]$. Agent j can sell the asset lower than the market price to agent 1 and accumulate more cash because of the gap between the two returns. This implies agent j would rather sell her asset to agent 1 and both make profitable trades. The same inference can be done with levered purchases, as both agents can do the same borrowing from the same set of lenders and simply change the price as the down payment such as $p_0 - q(s^i)$. The second statement holds with the similar argument as the problem becomes isomorphic by substituting the asset with the promise of s^2 (which is coming from lemma 3) from agent 1 and so forth. ■

Proof of Proposition 5. By lemma 3, fix the equilibrium contract matrix Y as $y_{ij} = s^i$ for any $i > j$. From the contract pricing equation from equation (8),

$$q_j(s^j) - q_{j+1}(s^{j+1}) = \frac{E_j\left[\min\left\{1, \frac{s^j}{p_1}\right\} - \min\left\{1, \frac{s^{j+1}}{p_1}\right\} - \zeta'(c_{j+1,j})\left[1 - \frac{s^{j+1}}{p_1}\right]^+ \mathbb{1}\left\{j+1 \in B(\epsilon)\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}$$

$$= \frac{E_j\left[1 - \min\left\{1, \frac{s^{j+1}}{p_1}\right\} - \zeta'(c_{j+1,j})\left[1 - \frac{s^{j+1}}{p_1}\right]^+ \mathbb{1}\left\{j+1 \in B(\epsilon)\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}.$$

Agent j makes a positive return out of this margin purchase only if $p_1 > s^{j+1}$. The denominator of the equation is

$$E_j\left[\frac{1}{p_1}\right] = \int \frac{1}{p_1} dG_\Sigma(\epsilon),$$

while the numerator without the counterparty risk becomes

$$E_j\left[1 - \min\left\{1, \frac{s^{j+1}}{p_1}\right\}\right] = \int_{p_1 > s^{j+1}} \frac{p_1 - s^{j+1}}{p_1} dG_\Sigma(\epsilon).$$

As j decreases—that is, becomes more optimistic agent—the probability of $p_1 > s^{j+1}$ becomes smaller as agents agree upon the distribution of liquidity shocks and underpricing. Also the maximum return from the leveraged purchase $\frac{s^j - s^{j+1}}{s^j}$ is (weakly) increasing with

j as well because the belief is harmonically dispersed and

$$
\begin{aligned}
s^j s^{j+2} &\leq (s^{j+1})^2 \\
s^j s^{j+1} + s^j s^{j+2} &\leq s^j s^{j+1} + (s^{j+1})^2 \\
s^j s^{j+1} - (s^{j+1})^2 &\leq s^j s^{j+1} - s^j s^{j+2} \\
\frac{s^j - s^{j+1}}{s^j} &\leq \frac{s^{j+1} - s^{j+2}}{s^{j+1}}.
\end{aligned}
$$

Each agent's cash holding becomes

$$e_1^j = e_0 + h_0 q_1(s^1) - \left(q_j(s^j) - q_{j+1}(s^{j+1})\right) c$$

for all $j \in N$ where $q(s^{n+1}) = 0$. Difference of cash holdings between agent j and $j+1$ is

$$e_1^j - e_1^{j+1} = \left(q_{j+1}(s^{j+1}) - q_{j+2}(s^{j+2})\right) - \left(q_j(s^j) - q_{j+1}(s^{j+1})\right) > 0$$

for any $j < n$, so $e_1^1 > e_1^2 > \cdots > e_1^n$. ■

The following lemma characterizes the properties of the inverse of equilibrium price, especially with respect to indegree of the bankrupt agents. It will be used to prove proposition 6.

Lemma 7 (Convexity of Inverse Price)**.** *Consider a class of debt networks $(N, C, Y, e_1, h_1, \epsilon, s, \zeta)$ with $C > 0$ that is under intermediation order. Suppose that $j \in B(\epsilon)$. Then, the inverse of the asset price $\frac{1}{p}$ is convexly decreasing in c_{ij} and convexly increasing in c_{jk} for any i and k in N. The convexity of inverse of the price with respect to c_{ij} and c_{jk} is strict up to the point $p = y_{ij}$ and $p = y_{jk}$, respectively.*

Proof of Lemma 7.

For prices $p = s$ and $p = 0$, the result is trivially true. Now consider the intermediate case of $p = \pi(p)$. Recall that

$$\frac{1}{p} = \frac{\sum_{j \in B(\epsilon)} \sum_{\substack{k \in N \\ p < y_{jk}}} c_{jk} - \sum_{j \in B(\epsilon)} \sum_{\substack{i \neq j \\ p < y_{ij}}} c_{ij}}{\sum_{i \notin B(\epsilon)} (e_1^i - \epsilon_i) + \sum_{j \in B(\epsilon)} \sum_{\substack{i \neq j \\ p \geq y_{ij}}} c_{ij} y_{ij} - \sum_{j \in B(\epsilon)} \sum_{\substack{k \in N \\ p \geq y_{jk}}} c_{jk} y_{jk} - \sum_{j \in B(\epsilon)} \sum_{\substack{k \notin B(\epsilon) \\ p \geq y_{jk}}} \zeta(c_{jk})}.$$

Denote $\frac{1}{p} = \frac{(num)}{(den)}$. Suppose $j \in B(\epsilon)$ and we differentiate the inverse price with respect to

c_{ij}, which will become

$$\frac{\partial(1/p)}{\partial c_{ij}} = \begin{cases} -\dfrac{1}{(den)} < 0 \ , & \text{if } p < y_{ij} \text{ and } i \notin B(\epsilon) \\ -\dfrac{(num)y_{ij}}{(den)^2} < 0 \ , & \text{if } p \geq y_{ij} \text{ and } i \notin B(\epsilon) \\ 0 \ , & \text{if } i \in B(\epsilon), \end{cases}$$

and differentiating with respect to c_{ij} once more gives

$$\frac{\partial^2(1/p)}{\partial c_{ij}^2} = \begin{cases} 0 \ , & \text{if } p < y_{ij} \text{ or } i \in B(\epsilon) \\ 2\dfrac{(num)y_{ij}^2}{(den)^3} > 0 \ , & \text{if } p \geq y_{ij} \text{ and } i \notin B(\epsilon). \end{cases}$$

Thus, $\dfrac{1}{p}$ is convexly decreasing in c_{ij} with strict convexity up to the point $p = y_{ij}$. Now differentiate inverse price with respect to lending of bankrupt agent j, c_{jk}.

$$\frac{\partial(1/p)}{\partial c_{jk}} = \begin{cases} \dfrac{1}{(den)} > 0 \ , & \text{if } p < y_{jk} \text{ and } i \notin B(\epsilon)\backslash\{j\} \\ \dfrac{(num)(y_{jk} + \zeta'(c_{jk}))}{(den)^2} > 0 \ , & \text{if } p \geq y_{jk} \text{ and } i \notin B(\epsilon)\backslash\{j\} \\ 0 \ , & \text{if } i \in B(\epsilon)\backslash\{j\} \end{cases}$$

The second derivative becomes zero for the case of $p < y_{jk}$ and $i \in B(\epsilon)\backslash\{j\}$. In the case of $p \geq y_{jk}$ and $i \notin B(\epsilon)\backslash\{j\}$, the numerator of the second derivative becomes

$$(den)^2(num)\zeta''(c_{jk}) + 2(den)\left(y_{jk} + \zeta'(c_{jk})\right)^2,$$

which is again positive. Therefore, the inverse of price is convexly increasing in indegree and strict convexity holds up to the point $p = y_{jk}$. ■

The following lemma is also used to prove proposition 6.

Lemma 8 (Counterparty Risk Order)**.** *For any network equilibrium and any agent $j \in N$, $\zeta(c_{ij})\omega_{ij} \geq \zeta(c_{kj})\omega_{kj}$ for any $j < i < k$.*

Proof of Lemma 8. If $c_{ij} > 0$ and $c_{kj} = 0$ or $c_{ij} = c_{kj} = 0$, then the result holds trivially. Suppose that $c_{ij} > 0$ and $c_{kj} > 0$. Consider the return equations. For $c_{ij} = c_{kj} = c$, $R_j^i > R_j^k$

as shown in lemma 3 where

$$R_j^i \equiv \frac{s^j}{q_j(s^j) - q_i(s^i)} E_j \left[\min\left\{1, \frac{s^j}{p_1}\right\} - \min\left\{1, \frac{s^i}{p_1}\right\} - \zeta'(c) \left[1 - \frac{s^i}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\} \right]$$
$$R_j^k \equiv \frac{s^j}{q_j(s^j) - q_k(s^k)} E_j \left[\min\left\{1, \frac{s^j}{p_1}\right\} - \min\left\{1, \frac{s^k}{p_1}\right\} - \zeta'(c) \left[1 - \frac{s^k}{p_1}\right]^+ \mathbb{1}\left\{k \in B(\epsilon)\right\} \right]$$

and agent j will borrow more from i and c_{ij} will increase. In other words, agent j has the higher return when she borrows from the more optimistic lender, agent i. Agent k should have lower counterparty risk in the persepctive of agent j in order to make the indifference condition $R_j^i = R_j^k$ hold. Therefore, $\zeta(c_{ij})\omega_{ij} \geq \zeta(c_{kj})\omega_{kj}$ for $j < i < k$ in any network equilibrium. ■

Proof of Proposition 6. Since every belief is bounded above by s^j for each $j \in N$, a decrease in expectation of p_1 and an increase in the expected sum of default costs implies an increase in volatility. Suppose that (s, ϵ) is realized and $j \in B(\epsilon)$, which happens with positive probability because of the distribution of ϵ_j. By lemma 7, c_{ij} convexly decreases the inverse price, and c_{jk} convexly increases the inverse price for $i, k \in N$. By lemma 3, the debt network is under intermediation order. Also from the proof of lemma 7 and the intermediation order, the slope from c_{jk} dominates the slope from c_{ij}. Thus, any inverse of price $p_1 \geq y_{ij}$ will be convexly increasing in c_{jk}.

Suppose that C^* is uniformly less indebted than C. Holding the bankruptcy state realizations the same, this decrease in debt decreases volatility directly from the previous argument of lemma 7 for each realization of (s, ϵ) involving bankruptcy will have a smaller impact on increase in inverse price and price decline. Also the decrease will generate fewer states of bankruptcy as every agent becomes less susceptible to price as in the wealth equation

$$m_j(p) = e_1^j - \epsilon_j - \sum_{i \in N \setminus \{j\}} c_{ij} \min\{p, y_{ij}\} - \sum_{i \in B(\epsilon)} \zeta(c_{ij})[p - y_{ij}]^+ + \sum_{k \in N} c_{jk} \min\{p, y_{jk}\},$$

which has smaller coefficients on prices and also the bankruptcy of lenders have smaller impact and less second-order bankruptcy will occur for the same state realizations.

Now, suppose that C^* is a diversification of j from C. From the application of lemma 7 in the beginning, the direct price effect from diversification is always positive, and the states that incur bankruptcy are fewer by lemma 5. In order to consider the effect from the counterparty channel, consider the simplest case of three agents, 1, 2, and 3, in a network. Suppose agent 1 is borrowing more from agent 2 than from agent 3—that is, $c_{21} > c_{31}$. By diversification of agent 1, $\zeta(c_{21}^*) + \zeta(c_{31}^*) < \zeta(c_{21}) + \zeta(c_{31})$ by convexity of ζ. Also, agent 2 has less collateral from agent 1 to reuse. Lower collateral makes agent 2's borrowing from agent 3 less, so $c_{32} \geq c_{32}^*$ because of collateral constraint. Even though agent 1's promise becomes smaller by $y_{21} > y_{31}$, which implies that it is more susceptible to lender bankruptcy, the reduction of rehypothecation means the susceptibility is only replaced by the identity of the agent, from 2 to 1.

The only case left is that diversification happens, and it does not affect any change in intermediation—that is, the rehypothecation constraint is not binding. Now the conditions are

$$\begin{gathered}
\zeta(c_{21}) > \zeta(c_{31}) \\
\zeta(c_{21}^*) > \zeta(c_{31}^*) \\
\zeta(c_{21})\omega_{21} > \zeta(c_{31})\omega_{31} \\
\zeta(c_{21}^*)\omega_{21} > \zeta(c_{31}^*)\omega_{31} \\
\zeta(c_{21}) + \zeta(c_{31}) > \zeta(c_{21}^*) + \zeta(c_{31}^*) \\
\zeta(c_{21}) > \zeta(c_{21}^*) \\
\zeta(c_{31}) < \zeta(c_{31}^*),
\end{gathered}$$

with $0 \leq \omega_{21}, \omega_{31} < 1$, where the third and fourth inequalities come from lemma 8 and the condition of diversification. By rearranging the inequalities, we obtain

$$\zeta(c_{21})\omega_{21} + \zeta(c_{31})\omega_{31} > \zeta(c_{21}^*)\omega_{21} + \zeta(c_{31}^*)\omega_{31}.$$

Thus, the expected default cost is lower under diversification. Also, even the bankruptcy probability change goes in the same direction. By the distributional assumption on G and because the second-order bankruptcy of agent 2 is now even more likely when agent 3 is bankrupt, $\omega_{21|C^*} - \omega_{21|C} > \omega_{31|C^*} - \omega_{31|C}$. Thus,

$$\zeta(c_{21})\omega_{21|C} + \zeta(c_{31})\omega_{31|C} > \zeta(c_{21}^*)\omega_{21|C^*} + \zeta(c_{31}^*)\omega_{31|C^*},$$

and the increased case of greater default cost from 3 is dominated by the decrease of default cost from a more likely occurrence of agent 2's bankruptcy. Thus, the counterparty channel also decreases the aggregate expected deadweight loss and increases expected price. Therefore, diversification in this case decreases aggregate expected deadweight loss, increases expected price, and decreases volatility.

Finally, we can extend this argument of three agents to any general number of agents. For any $j \in N$, $c_{L_j j} > c_{L_j j}^*$ while keeping $\sum_{i \in N \setminus \{j\}} c_{ij} = \sum_{i \in N \setminus \{j\}} c_{ij}^*$ implies there is an agent $i > L_j$ such that $c_{ij} < c^*_{ij}$. Using the same argument for agent 1, 2, and 3 on agent j, L_j, and i will provide the same result. If agent j is diversifying even further, then that will divide $c_{L_j j}$ into even further diversification, and convexity will make it an even lower aggregate expected default cost. Thus, any diversification increases expected price and decreases aggregate expected default cost and volatility. ■

Proof of Theorem 1. The first and second properties come directly from proposition 4 and lemmas 3 and 4. The third property comes from the indifference equation for borrower j, who has to be indifferent between borrowing cash from i and k if j is borrowing from the two in a positive amount. The fourth property is again derived from lemma 4, and the fifth property is simply from the budget constraint and contract prices.

Now we show that an equilibrium that satisfies those properties exists. Define $Z \equiv C \circ Y$. Consider a class of networks $\mathcal{Z}$ such that every $Z \in \mathcal{Z}$ satisfies the intermediation order for fixed Y s.t. $y_{ij} = s^i$ for any $i, j \in N$. Now use the matrix order to compare the total amount of promises—that is, $Z > Z'$ implies $Z_{ij} \geq Z'_{ij}$ for all $i, j \in N$ and at least one element has strict inequality. Similarly, $Z \geq Z'$ can be defined allowing equality for every entry. Note that this ordering is only a partial ordering among $\mathcal{Z}$. There can be networks $Z, Z' \in \mathcal{Z}$ with neither $Z \geq Z'$ nor $Z' \geq Z$ is true. However, $(\mathcal{Z}, \geq)$ forms a complete lattice, because for any subset $\mathcal{Z}' \subseteq \mathcal{Z}$, the least upper bound $\overline{Z}$ with $\overline{Z}_{ij} = \sup\limits_{Z \in \mathcal{Z}'} Z_{ij}$ and the greatest upper bound $\underline{Z}$ with $\underline{Z}_{ij} = \inf\limits_{Z \in \mathcal{Z}'} Z_{ij}$ exist because each element is from a subset of Euclidean space. Fix the norm $\|\cdot\|$ of matrices as the Frobenius norm (or any other $L_{p,q}$ norm with $p, q \geq 1$). If $\|Z\|$ increases, then there is more aggregate borrowing in the economy which generates greater probability of bankruptcy and default costs as shown by proposition 6.

Let $V : \mathcal{Z} \to \mathcal{Z}$ be a function from network to network—that is, given the price and counterparty risk distribution of the first network in $t = 1$, V generates the agents' optimal network formation decisions as best responses. Now I show that V is monotonous in $(\mathcal{Z}, \geq)$. Let $\underline{Z}$ be the network with $\|\underline{Z}\| = 0$—that is, no risk of counterparty bankruptcy and dispersion of cash holdings. Under $\underline{Z}$, return from cash holding is minimized by lemma 7 in the proof of proposition 6. By intermediation order, $V(\underline{Z}) \leq \overline{Z}$ where $\overline{Z}$ denotes the maximum leverage network—that is, the single-chain network with full borrowing as defined in the proof of proposition 5. Similarly, $V(\overline{Z}) \geq \underline{Z}$ because of the zero lower bound. Therefore, the range of V is compact.

Since the network is under intermediation order, lemma 3 and proposition 6 imply that $Z \in \mathcal{Z}$ with a large $\|Z\|$ has a lower degree of diversification and larger average default costs relative to $Z' \in \mathcal{Z}$ with lower $\|Z'\|$. Then, an increase in $\|Z\|$ has two effects to the return calculation. First, it increases counterparty exposure $\omega_{ij}(Z)$ and the default cost, which implies $E_j\left[\mathbb{1}\{i \in B(\epsilon)\}\right]$, and $\beta_i(p_1)$ increase for each $i, j, \in N$. Second, the state in which the liquidity is constrained is exactly the state in which the optimists are under liquidity shock—that is, when they would have really wanted to have additional liquidity. The marginal value of cash in such a state is even greater. Thus, p_1 is lower and more volatile under higher $Z \in \mathcal{Z}$ by lemma 7. Then, the return on cash $E_j[s^j/p_1]$ becomes greater for each $j \in N$ under greater Z,—that is, the return of cash holding is greater, and the agent's return on leverage goes down. Thus, any increase in Z (under the two possible directions restricted by intermediation order) will make the optimal response to the given distribution of Z to be lowering $\|Z\|$. In other words, a large Z makes the agents diversify or reduce borrowing or lending in general. The equilibrium portfolio decision holds as

$$
\begin{aligned}
E_j[1/p_1] =& \frac{s^j}{q(s^j) - q(s^i)} E_j\left[1 - \min\left\{1, \frac{s^i}{p_1}\right\} - \frac{\zeta'(c_{ij})}{p_1}\mathbb{1}\left[1 > \frac{s^i}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right] \\
=& \frac{s^j}{q(s^j) - q(s^k)} E_j\left[1 - \min\left\{1, \frac{s^k}{p_1}\right\} - \frac{\zeta'(c_{kj})}{p_1}\mathbb{1}\left[1 > \frac{s^k}{p_1}\right]\mathbb{1}\left\{k \in B(\epsilon)\right\}\right],
\end{aligned}
$$

as in the proof of lemma 3, and the equality condition holds only at a greater diversification or lower overall collateral exposure. Thus, $V(Z)$ decreases as Z increases. Then, V is a monotonic function on a complete lattice, and there exists a fixed point network Z^* such that $Z^* = V(Z^*)$ by the Knaster-Tarski fixed point theorem. Therefore, there exists a network equilibrium, and the set of equilibria is also a complete lattice.

Now the rest of the proof is simply applying the results and $q(y)$ from proposition 4 into market clearing conditions. Combining lemmas 1 and 4 with lemma 3, we can conclude that $q(s^1) = p_0$. Also, the nominal wealth are determined by the combination of budget constraints and market clearing conditions. ■

Proof of Theorem 2. As discussed in the description

$$
\frac{\partial}{\partial c_{ik}} \sum_{j \in N} E_j\left[m_j(\epsilon)\frac{s^j}{p_1(\epsilon)}\right] \neq \frac{\partial}{\partial c_{ik}} E_j\left[m_j(\epsilon)\frac{s^j}{p_1(\epsilon)}\right]
$$

and due to counterparty externality and price externality being positive coming from the arguments in proposition 6, the direction of inefficiency is coming from under-diversification.
■

Proof of Proposition 7.

1. Suppose that s^i increases to $s^i + \eta$ for every $i \in N$. As shown in proposition 4, $q(y)$ is increasing in y for any $y \in [s^i, s^i + \eta]$ and $q'(y) < 1$ by the lower bound of y in the numerator.

By equation (8), the function for contract price becomes

$$q(y) = q(s^{j+1}) + \frac{E_j\left[\min\left\{1, \frac{y}{p_1}\right\} - \min\left\{1, \frac{s^{j+1}}{p_1}\right\} - \zeta'(c_{j+1,j})\left[1 - \frac{s^{j+1}}{p_1}\right]^+ \mathbb{1}\left\{j+1 \in B(\epsilon)\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}.$$

Any change in the terms related to $q(s^j)$ has a direct effect of increase in $q(s^i)$ in linear terms for any $i < j$ by the recursive equation

$$q(s^i) = q(s^j) + \sum_{k=i+1}^{j-1} \frac{E_k\left[1 - \min\left\{1, \frac{s^{k+1}}{p_1}\right\} - \zeta'(c_{k+1,k})\left[1 - \frac{s^{k+1}}{p_1}\right]^+ \mathbb{1}\left\{k+1 \in B(\epsilon)\right\}\right]}{E_k\left[\frac{1}{p_1}\right]}.$$

As in the argument in the proof of proposition 4, for any agent $k < j$, prices relevant to cashflow of the leveraged contracts are bounded below by the subject belief of the lender $k+1$, s^{k+1} as in

$$s^k E_k\left[1 - \min\left\{1, \frac{s^{k+1}}{p_1}\right\} - \zeta'(c_{k+1,k})\left[1 - \frac{s^{k+1}}{p_1}\right]^+ \mathbb{1}\left\{k+1 \in B(\epsilon)\right\}\right].$$

However, the return from cash holdings, $s^k E_k\left[1/p_1\right]$ is not bounded by any prices. The ratio between the changes of the two terms is increasing in k as the lower bound of the price distribution becomes smaller—that is,

$$\begin{aligned} &\frac{\Delta E_k\left[1 - \min\left\{1, \frac{s^{k+1}}{p_1}\right\} - \zeta'(c_{k+1,k})\left[1 - \frac{s^{k+1}}{p_1}\right]^+ \mathbb{1}\left\{k+1 \in B(\epsilon)\right\}\right]}{\Delta E_k\left[\frac{1}{p_1}\right]} \\ <&\frac{\Delta E_{k+1}\left[1 - \min\left\{1, \frac{s^{k+2}}{p_1}\right\} - \zeta'(c_{k+2,k+1})\left[1 - \frac{s^{k+2}}{p_1}\right]^+ \mathbb{1}\left\{k+2 \in B(\epsilon)\right\}\right]}{\Delta E_{k+1}\left[\frac{1}{p_1}\right]}. \end{aligned}$$

Thus, a direct increase in s^i dominates the changes in the denominator and in the expectations of the return equation

$$R_i^{i+1} \equiv \frac{s^i}{q(s^i) - q(s^{i+1})} E_{i+1}\left[\min\left\{1, \frac{s^i}{p_1}\right\} - \min\left\{1, \frac{s^{i+1}}{p_1}\right\} - \zeta'(c)\left[1 - \frac{s^{i+1}}{p_1}\right]^+ \mathbb{1}\left\{i+1 \in B(\epsilon)\right\}\right].$$

Hence, higher counterparty risk can be justified as the leverage return for agent i increases. Agent i will increase $c_{i+1,i}$ more, which implies fewer links (intensively and extensively), if i was diversifying. Also, the velocity of collateral (weakly) increases by the increase in $c_{i+1,i}$ as well as relaxing collateral constraints for the subsequent agents $i+1, i+2, \ldots, n$.

Also changes in $q(s^j)$ have indirect effects by the induced borrowing pattern, changing the relative distribution of prices F_i for given liquidity shocks ϵ and the return on cash holdings $E_i\left[\frac{s^i}{p_1}\right]$ as well as changing the probability of bankruptcy of the lenders. First, there will be a change in price distribution of $\tilde{p}_1$, which influences both the denominator and the numerator of equation (8). The increase in agents' debts will increase the price volatility by proposition 6. The effect from the indirect increase in bankruptcy probability is confined by the increase in $E_k\left[\frac{s^k}{p_1}\right]$, because now the underpricing is more likely due to the increase in s^k. And the increase in second-order bankruptcy probability $G_i\left(\left[\zeta(c_{i+1,i}), \zeta(\hat{c}_{i+1,i})\right] | i+1 \in B(\epsilon)\right)$ is always lower than the increase in first-order bankruptcy probability, which is taken into account by agent i. Thus, the direct effect $E_k\left[\min\{1, \frac{\eta}{p_1}\}\right] / E_k\left[\frac{1}{p_1}\right]$ always dominates the indirect effect. Hence, $q(s^i)$ and leverage increase, and R_i^j increases for all $i < j$, which implies the velocity of collateral increases.

The last thing to check is whether the change will affect the agents with beliefs below agent i. Note that the increase in c_{lk} for any $k, l \leq j$ does not affect the expected sum of lender default costs of each agent in $\{j+1, j+2, \ldots, n\}$, because any promise between agents $k, l \leq j$ is going to be defaulted no matter what in their perspective of the upper bound of the asset price $s^{j+1} > s^{j+2} > \cdots > s^n$. Thus, the debt amount or even the change in price distribution is irrelevant to these pessimistic agents. The only change for them comes from the increase in asset price $p_0 = q(s^1)$ that increases their nominal value of endowments which incentivizes them to increase borrowing and increase the reuse of collateral—that is, the velocity of collateral.

2. Suppose θ_j decreases by η for all $j \in N$. Then R_i^{i+1} increases again because of the lower probability of default costs and $c_{i+1,i}$ increases. The rest of the argument goes the same as in the previous case. In this case, it is even more simple because there is a reduction of counterparty risk in every link that offsets the indirect change. ■

Proof of Proposition 8. From equation 9, an individual agent does not care about the terms of γ and $\dfrac{m_0(\epsilon|p_1)}{\sum_{i\in N}\mathbb{1}\{i\notin B(\epsilon)\}}$, since they are determined by the macro variables and the agent considers herself as a price-taker. Under the case of 1 and 2, the term ω_{ij} equals to zero for any $i,j\in N$. Therefore, each agent does not have any incentive to diversify and lower leverage and will maximize their leverage. The equilibrium network under CCP has a collateral matrix C_{ccp}, which has a greater debt than the debt of decentralized equilibrium network C, by being more indebted (the opposite of less indebted) and less diversified (the opposite of diversification) maximizing concentration of the network. By proposition 6, this equilibrium network maximizes the systemic risk by maximizing the sum of expected default costs. Even if γ is not large and CCP can go bankrupt in some states, agent j's perceived risk of borrowing from agent i,

$$E_j\left[[1-y_{ij}/p_1]^+\mathbb{1}\{0\in B(\epsilon)\ \&\ i\in B(\epsilon)\}\right]$$

is always smaller than

$$\omega_{ij}=E_j\left[[1-y_{ij}/p_1]^+\mathbb{1}\{i\in B(\epsilon)\}\right]$$

under decentralized equilibrium, and the debt of the network becomes larger either by more indebted or less diversification. As argued in the proof of theorem 2, the positive externality becomes even less incorporated into the agent's individual decisionmaking, and the systemic risk is always greater under C_{ccp} than the systemic risk under C. ■

Proof of Proposition 10. Suppose only one contract y is available in the market. As in lemma 4, agent 1 will buy the asset and borrow cash from agents who has $s^j\geq y$ with equal weights as diversification. If agent 1's endowment e_0 is not enough to purchase all the assets with the downpayment, then agent 2 also joins the buyer side and borrows from another pool of lenders. This can be repetitively done for agent 3, 4, and so forth. Similarly, if the demand for cash is too high, then the price of the contract $q(y)$ will decrease, and even agents with $s^j<y$ can become a lender, similar to the argument in lemma 4. Since the maximization problem and the budget constraints with down payments are all monotone, there is always an equilibrium. The resulting network becomes a complete bi-partite network for the given component of market participants. Since agents have no tradeoff between choice of counterparties and choice of leverage, they have no incentives to change their network formation behavior even after eliminating the counterparty risk concerns ω_{ij} for each $i,j\in N$. Since all the walks in the network have a length of 1, there will be no effect from netting as well. ■

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