Distress and default contagion in financial networks

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October 7, 2019

Abstract

We develop a new model for solvency contagion that can be used to quantify systemic risk in stress tests of financial networks. In contrast to many existing models it allows for the spread of contagion already before the point of default and hence can account for contagion due to distress and mark-to-market losses. We derive general ordering results for outcome measures of stress tests that enable us to compare different contagion mechanisms. We use these results to study the sensitivity of the new contagion mechanism with respect to its model parameters and to compare it to existing models in the literature. When applying the new model to data from the European Banking Authority we find that the risk from distress contagion is strongly dependent on the anticipated recovery rate. For low recovery rates the high additional losses caused by bankruptcy dominate the overall stress test results. For high recovery rates, however, we observe a strong sensitivity of the stress test outcomes with respect to the model parameters determining the magnitude of distress contagion.

Key words: Systemic risk, contagion, financial networks, stress testing, mark-to-market losses.

JEL code: C62, D85, G01, G21, G28, G33.

1 Introduction

Following the 2007-2009 financial crisis, stress tests have become an important tool to assess the stability of financial systems (Anderson, 2016). A particular concern of policy makers is to make these stress tests more *macroprudential* (Basel Committee on Banking Supervision, 2015b) by incorporating feedback and amplification mechanisms into the stress testing exercise (The Bank of England, 2015; Bardoscia et al., 2017). Modelling financial contagion lies at the heart of these efforts, see Glasserman & Young (2016) for a recent overview. An important aspect is to look more broadly at events that can trigger contagion in the first place.

Bankruptcy of an institution has been considered as the only potential trigger of a contagion mechanism in a large part of the literature on solvency contagion: this applies

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[†]Parts of this research were carried out while Luitgard Veraart was a George Fellow at the Bank of England whose hospitality and funding by the Houblon-Norman Fund are gratefully acknowledged.

to the stream of work building on the clearing approach by Eisenberg & Noe (2001) such as Cifuentes et al. (2005); Rogers & Veraart (2013); Weber & Weske (2017); Kusnetsov & Veraart (2019); Feinstein (2017) but also to several other default cascade models such as Furfine (2003); Gai & Kapadia (2010); Amini et al. (2016). We will refer to contagion that is triggered by default as *default contagion*.

As the 2007-2009 financial crisis has demonstrated, however, a large part of the losses was not due to defaults but due to mark-to-market losses. "Under Basel II, the risk of counterparty default and credit migration risk were addressed but mark-to-market losses due to credit valuation adjustments (CVA) were not. During the financial crisis, however, roughly two-thirds of losses attributed to counterparty credit risk were due to CVA losses and only about one-third were due to actual defaults", Basel Committee on Banking Supervision (2011). These mark-to-market losses occur already before the point at which an institution would declare bankruptcy. We will refer to any spread of losses prior to bankruptcy as *distress contagion*.

In line with these observations, regulators have recently started using models that allow for distress contagion, see Bardoscia et al. (2017). Examples of distress contagion models are the DebtRank model by Battiston et al. (2012) and extensions of it such as the models by Bardoscia et al. (2016, 2017). In all these existing models the magnitude of contagious losses is determined by the probability of default and the recovery rate. The probability of default is modelled as a function of the relative equity loss, which is linear in Battiston et al. (2012), non-linear in Bardoscia et al. (2016) and modelled in the spirit of the classical structural credit risk model by Black & Cox (1976) in Bardoscia et al. (2017). Usually a constant recovery rate is chosen which is often set to zero, see Bardoscia et al. (2016).

The main contribution of this paper is to develop a flexible and tractable framework for quantifying distress and default contagion in financial networks that is consistent with a scenario-based approach to stress testing. In particular our approach does not rely on historical estimates or default probabilities but is designed such that it can capture a wide range of possible contagion mechanisms. Furthermore, analytical ordering results are provided that enable us to compare the outcome of stress tests based on different contagion mechanisms. Regulators are starting to take a more simulation based approach to assessing financial risk, in particular to assess channels of risk transmission that might not have played a role in the past but might become relevant in a future financial crisis. For these situation no historical data would be available to build a statistical model. With reference to new types of risk, Alex Brazier, Executive Director for Financial Stability Strategy and Risk, Bank of England, pointed this out in his speech in September 2018. "These risks need monitoring. Not by asking whether they have appeared, but by asking whether they could. Don't wait. Simulate," Brazier (2018). Our new modelling framework is in line with this approach and enables us to assess a wide range of possible contagion mechanisms.

We use a network model to describe how contagion spreads through the system. The magnitude of contagious losses due to default is quantified using an approach that adheres to some stylized principles of insolvency law. We suggest a very general modelling framework for quantifying the magnitude of contagious losses that arise due to distress but in the absence of default. In contrast to some early distress contagion models, our framework allows for non-constant recovery rates in the case of default. In particular it extends the default contagion model by Rogers & Veraart (2013) to allow for distress contagion.

We characterize our contagion mechanism in terms of a small number of model parameters that have an intuitive economic interpretation. It can therefore be easily applied in a stress testing context. It reduces to some well-known contagion mechanisms such as the ones by (Eisenberg & Noe, 2001; Rogers & Veraart, 2013; Battiston et al., 2012) for special parameter choices.

Our new framework allows for a wide range of functional forms for contagious losses while still remaining analytically tractable. In particular, we provide analytical ordering results for some outcome measures of stress tests corresponding to different functional forms of the contagion mechanism. Importantly, our ordering results are independent of the underlying network structure. Based on these ordering results, we discuss sensitivities of outcome measures of stress tests with respect to the model parameters.

Furthermore, we discuss the relationship between modelling assumptions for default contagion and potential for distress contagion. In particular, we show that not every existing default contagion model can be generalized to account for distress contagion. Within our modelling framework the existence of bankruptcy costs is a necessary condition for the possibility of distress contagion. Hence, even though by definition bankruptcy costs only occur in the case of default, their pure existence can lead to an amplification of contagion in financial networks even prior to default.

Our approach is similar in spirit to a contagion mechanism mentioned in (Glasserman & Young, 2015, Section 6), but the modelling of the actual default and distress contagion is different. In particular, we provide a rich class of models for distress contagion that include a linear model as a special case.

Our framework is related to the idea of network valuation introduced in Barucca et al. (2016) but also has some fundamental differences. Barucca et al. (2016) consider the problem of determining the value of a network prior to the maturity date of the outstanding debt. Their main application is to consider a stochastic version of the Eisenberg & Noe (2001) model and they derive fixed points that determine the value of the network. This approach explicitly determines the stochastic dynamics for the underlying balance sheets (in particular, the external assets and the equity) and derives the corresponding default probabilities. It relies on two modelling choices: a function for the spread of losses in the case of default and a probability distribution for the external assets. The combination of these two modelling choices leads to a specific function whose fixed points are of interest. In a similar spirit, Banerjee & Feinstein (2018) also consider a stochastic generalization of the Eisenberg & Noe (2001) model. They derive formulas for the valuation of debt and equity of firms under comonotonic endowments.

We take a more direct approach and propose to model the function whose fixed points are of interest directly. We determine the value of a network as a function of a shock without accounting for the probability of the shock occurring. Our approach is therefore in line with a scenario-based approach to stress testing. In this approach one determines the outcome for a given scenario without accounting for the likelihood of the scenario itself. Therefore, we do not need to make probabilistic assumptions on shock distributions and can more easily conduct sensitivity studies of outcome measures of stress tests.

We concentrate on one channel of systemic risk only, by considering a network of exposures between financial institutions and the spread of losses in such a network. Our model could easily be used as a building block in a larger class of models that incorporates other channels of systemic risk, such as fire sales, liquidity and funding channels etc..

The structure of the paper is as follows. In Section 2 we introduce the model for the financial network and the stylized balance sheets for all financial institutions that constitute the nodes in the network. In Subsection 2.1 we discuss how the network can be *re-evaluated* if it is hit by a shock as part of a stress testing exercise. In particular, we show how the default contagion model by Eisenberg & Noe (2001) and a special case of the default contagion model by Rogers & Veraart (2013) can be written as a network re-evaluation in Subsection 2.2. We then provide general comparison results for different outcome measures of stress tests that correspond to different modelling assumptions for models allowing for both distress and default contagion in Subsection 2.3. In Section 3 we develop our new model for distress and default contagion. In Subsection 3.1 we introduce the functional form of the contagion mechanism and discuss the economic meaning of its model parameters in Subsection 3.2. In particular, we discuss the important role of bankruptcy costs when modelling distress and default contagion. We then highlight in Subsection 3.3 which features of the model or which parameter choices determine whether losses are spread, amplified or even damped. Finally, in Subsection 3.4 we discuss possible model extensions. In Section 4 we discuss how the new modelling framework can be applied in regulatory stress tests. In Subsection 4.1 we apply our new model to empirical data used in the 2011 stress test by the European Banking Authority and analyze the sensitivity of outcomes of stress tests to the model parameters. Finally, Section 5 concludes.

2 The framework

We consider a financial network consisting of N nodes which we refer to as banks with indices in $\mathcal{N} = \{1, \ldots, N\}$. The weighted directed edges between the banks describe the interbank liabilities and are denoted by the matrix $L \in [0, \infty)^{N \times N}$. In particular, L_{ij} denotes the interbank liability from bank i to bank j; i.e., bank i has to repay L_{ij} to bank j at the maturity date. Hence, L_{ij} is a loan from bank j to bank i and therefore an interbank asset of bank j. We assume that banks do not borrow from themselves and hence $L_{ii} = 0$ for all $i \in \mathcal{N}$. In addition to the interbank liabilities, we assume that banks can have external liabilities, i.e., liabilities to entities outside the interbank network and we denote them by $L^{\mathrm{e}} \in [0, \infty)^N$.

All contracts are established at time $t = 0$ and we denote the book value (time-0 value) of the total assets of bank i by $\bar{A}_i$, its total interbank assets by $\bar{A}_i^{\mathrm{B}}$, its total liabilities by $\bar{L}_i$ and its total interbank liabilities by $\bar{L}_i^{\mathrm{B}}$. Hence,

$$\begin{aligned}
\bar{A}_i &= A_i^{\mathrm{e}} + \sum_{j=1}^{N} L_{ji} = A_i^{\mathrm{e}} + \bar{A}_i^{\mathrm{B}}, \\
\bar{L}_i &= L_i^{\mathrm{e}} + \sum_{j=1}^{N} L_{ij} = L_i^{\mathrm{e}} + \bar{L}_i^{\mathrm{B}}, \\
w_i &= \bar{A}_i - \bar{L}_i,
\end{aligned}$$

where w_i denotes the net worth (or equity if positive) of bank i. We refer to a tuple $(L, L^{\mathrm{e}}, A^{\mathrm{e}})$ as a *financial system.*

In addition to the interbank liabilities matrix L we will also consider the relative interbank liabilities matrix $\Pi \in \mathbb{R}^{N \times N}$, which is given by

$$\Pi_{ij} = \begin{cases} \frac{L_{ij}}{\bar{L}_i}, & \text{if } \bar{L}_i > 0, \\ 0, & \text{if } \bar{L}_i = 0. \end{cases}$$

Note in particular that the row sums of Π are less or equal than 1 (and they are equal to 1 for those indices $i \in \mathcal{N}$ for which $L_i^{\mathrm{e}} = 0$ and $\bar{L}_i > 0$.

It will often be convenient to study the set of all nodes that have positive total liabilities. We denote it by

$$\mathcal{M} := \{i \in \mathcal{N} \mid \bar{L}_i > 0\}.$$

Throughout this paper we assume that all contracts have the same maturity date $T > 0$. For an analysis with multiple maturities we refer to Kusnetsov & Veraart (2019).

We assume that the external assets are subject to a deterministic shock $x = (x_1, \ldots, x_N)^\top$ where $x_i \in [0, A_i^{(\mathrm{e})}]$ as part of a stress testing exercise. In particular, bank i's shocked external assets are given by $A_i^{(\mathrm{e})} - x_i$ and its corresponding shocked net worth is given by $w_i - x_i$. Table 1 shows the corresponding balance sheet after the shock. In particular, we assume that the interbank liabilities, interbank assets and the external liabilities stay constant. In the following we provide a re-evaluation mechanisms for a financial network that is exposed to such a shock.

Table 1: Balance sheet of bank i after the shock.

Assets		Liabilities	
shocked external assets	$A_i^{(e)} - x_i$	external liabilities	$L_i^{(e)}$
interbank assets	$\sum_{j=1}^{N} L_{ji}$	interbank liabilities	$\sum_{j=1}^{N} L_{ij}$
		shocked net worth	$w_i - x_i$

2.1 Re-evaluation of the shocked network

One possible way to determine the value of the assets and liabilities in the network is to use the concept of a clearing vector as introduced by Eisenberg & Noe (2001) and extended to incorporate bankruptcy costs by Rogers & Veraart (2013) and considered under the aspect of re-evaluation after a shock by Glasserman & Young (2015). The ith component of a clearing vector characterizes the total amount that bank $i \in \mathcal{N}$ pays and this amount might potentially be less than its total nominal liabilities $\bar{L}_i$. All nodes pay their total nominal obligations if they have enough assets to do so. If they do not have enough assets they distribute their remaining assets according to the same proportion as their original debt was distributed and they pay a fixed proportion of their available assets as default costs. In particular, they never pay more than their total liabilities or more than their available assets. These assumptions are in line with the stylized principles of insolvency law of limited liabilities, priority of debt claims and proportionality, see Eisenberg & Noe (2001).

These concepts are captured by the following fixed point definition of a clearing vector which rewrites the definition by Rogers & Veraart (2013) in the spirit of Glasserman & Young (2015).

Definition 2.1. *For a financial system* (L, L^e, A^e) *a* clearing vector accounting for bankruptcy costs *for a shock realisation* $x = (x_1, \ldots, x_n)^\top \geq 0$ *is a vector* $L(x) \in [0, \bar{L}]$*, such that*

$$L(x) = \Psi^{RV}(L(x)),$$

where the function Ψ^{RV} *is given by*

$$\Psi^{RV}(L(x))_i = \begin{cases} \bar{L}_i, & \text{if } \sum_{j=1}^N \Pi_{ji} L_j(x) + A_i^e - x_i \geq \bar{L}_i, \\ \left(\beta \sum_{j=1}^N \Pi_{ji} L_j(x) + \alpha(A_i^e - x_i)\right)^+, & \text{else,} \end{cases} \tag{1}$$

where $\alpha, \beta \in [0, 1]$*. Here,* Π *denotes the relative liabilities matrix and* $y^+ = \max\{0, y\}$ *for* $0, y \in \mathbb{R}$*.*

If $\alpha = \beta = 1$ the definition reduces to the definition by Glasserman & Young (2015) of the Eisenberg & Noe (2001) model.

Definition 2.2. *For a financial system* (L, L^e, A^e) *a* clearing vector *for a shock realisation* $x = (x_1, \ldots, x_n)^\top \geq 0$ *is a vector* $L(x) \in [0, \bar{L}]$*, such that*

$$L(x) = \Psi^{EN}(L(x)),$$

where the function Ψ^{EN} *is given by*

$$\Psi^{EN}(L(x))_i = \min\left\{\bar{L}_i, \left(\sum_{j=1}^N \Pi_{ji} L_j(x) + A_i^e - x_i\right)^+\right\}. \tag{2}$$

Here, Π *denotes the relative liabilities matrix and* $y^+ = \max\{0, y\}$ *for* $0, y \in \mathbb{R}^N$*.*

Remark 2.3. Note that it will sometimes be convenient to consider the functions above using the set $\mathcal{M} = \{i \in \mathcal{N} \mid \bar{L}_i > 0\}$ rather than the relative liabilities matrix Π. In particular, for all $L \in \mathbb{R}^N$ it holds that

$$\sum_{j=1}^N \Pi_{ji} L_j = \sum_{j \in \mathcal{M}} \frac{L_{ji}}{\bar{L}_j} L_j.$$

The ith component of a clearing vector $L(x)$ characterizes the total payments that bank i makes. Each bank j receives a proportion $\Pi_{ij} L_i(x) \leq L_{ij}$ and bank i repays $(1 - \sum_{j=1}^N \Pi_{ij}) L_i(x) \leq L_i^{\mathrm{e}}$ of the external debt.

One can immediately see from the definition that any fixed point of Ψ^{RV} or Ψ^{EN} is in $[0, \bar{L}]$. Furthermore, for all $i \in \mathcal{N}$ the sum $\sum_{j=1}^N \Pi_{ji} L_j(x)$ are exactly the interbank assets that bank i has available assuming that every bank j in the network makes a total payment of $L_j(x)$ and $A_i^{\mathrm{e}} - x_i$ are the shocked external assets.

While the original concept of a clearing vector was introduced for payment systems and assumed that all payments are settled at the same time (the maturity) we can also use the clearing concept as a concept of re-evaluating the network potentially prior to the maturity date. In particular, then the clearing vector $L(x)$ no longer represents actual payments made, but a valuation of the possible payments in the light of a shock x, see (Glasserman & Young, 2015, p. 386).

Once we are concerned with the re-evaluation of a network after a shock there is no reason to assume that assets decline in value only after the net worth of a bank has fallen below zero. In practice, assets are marked to markets and therefore a decline in asset value can be caused prior to the actual default of a bank. To be able to develop this idea further it will be beneficial to rewrite the fixed point problem for a clearing vector of Definition 2.1 as a fixed point problem for a new quantity that we refer to as *re-evaluated equity*.

Definition 2.4 (Equity re-evaluation). *1. We refer to any function $\mathbb{V} : \mathbb{R} \to [0, 1]$ that is non-decreasing and right-continuous as an* admissible valuation function*.*

2. Let (L, L^e, A^e) be a financial system and let the shock vector x satisfy $x \in [0, A^{(e)}]$. Let $\mathbb{V}$ be an admissible valuation function and $\mathcal{E}(x) = [-\bar{L}, w - x]$ and $\mathcal{M} := \{j \in \mathcal{N} \mid \bar{L}_j > 0\}$. We refer to a function $\Phi = \Phi(\cdot; \mathbb{V}) : \mathcal{E}(x) \to \mathcal{E}(x)$, where for $i \in \mathcal{N}$

$$\Phi_i(E) = \Phi_i(E; \mathbb{V}) = A_i^e - x_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i, \tag{3}$$

as an equity valuation function*.*

3. Let Φ be an equity valuation function for a financial system (L, L^e, A^e) with shock vector $x \in [0, A^{(e)}]$. We refer to a vector $E \in \mathcal{E}(x)$ satisfying

$$E = \Phi(E) \tag{4}$$

as re-evaluated equity*.*

As one can see the re-evaluated equity is the difference between the re-evaluated assets and the nominal liabilities. Here the assets are re-evaluated based on an admissible valuation function $\mathbb{V}$. If one takes the positive part of the re-evaluated equity one would obtain what is usually considered as the equity of a company, see e.g., Eisenberg & Noe (2001).

Our approach here is similar to the idea of network valuation developed in Barucca et al. (2016) and the so-called *reduced form Eisenberg & Noe cascade algorithm* developed in Hurd (2016).

The main difference of our approach compared to the approach by Barucca et al. (2016) is that we only use one valuation function $\mathbb{V}$ to re-evaluate the whole network and parameterise $\mathbb{V}$ as a function of the ratio of re-evaluated assets divided by the total liabilities rather than as a function of the re-evaluated equity. This allows for a more parsimonious model description while still capturing a wide range of models for which Barucca et al. (2016) would need N valuation functions parameterised in terms of the re-evaluated equity.

The main difference between our approach and the approach by Hurd (2016) is that Hurd (2016) mainly considers a special case of the valuation function $\mathbb{V}$ (referred to as threshold function and parametrised as a function of the ratio of the re-evaluated equity divided by the total liabilities) that corresponds to the Eisenberg & Noe (2001) model. We allow for a far wider class of models than just the Eisenberg & Noe (2001) model. Our re-evaluated equity corresponds to what Hurd (2016) calls the *default buffer*.

Since $\mathcal{E}(x)$ is a complete lattice and Φ is non-decreasing one obtains from Tarski's fixed point theorem the existence of a greatest and least fixed point.

Theorem 2.5. *Let Φ be an equity valuation function for the financial system (L, L^e, A^e) with shock vector $x \in [0, A^{(e)}]$ and $\mathcal{E}(x) = [-\bar{L}, w - x]$. Then there exists a greatest fixed point E^* and a least fixed point E_*, such that for all solutions E to the fixed point problem (4) it holds that*

$$E_* \leq E \leq E^*.$$

A proof for this and all remaining results is given in Appendix A.

From an economic point of view the greatest fixed point of the equity valuation function is of particular interest, since it corresponds to the best possible outcome for the economy.

The greatest fixed point can be derived, using classical fixed point iteration and starting the iteration from the shocked net worth $w - x$.

Theorem 2.6 (Fixed point iteration for the greatest fixed point)**.** *Let Φ be an equity valuation function for the financial system (L, L^e, A^e) with shock vector $x \in [0, A^{(e)}]$ and $\mathcal{E}(x) = [-\bar{L}, w - x]$. Let $E^{(0)} = w - x$ and define recursively $E^{(\kappa+1)} = \Phi(E^{(\kappa)})$ for $\kappa \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Then,*

1. *$(E^{(\kappa)})_{\kappa \in \mathbb{N}_0}$ is a monotonically non-increasing sequence, i.e., $E^{(\kappa+1)} \leq E^{(\kappa)}\ \forall \kappa \in \mathbb{N}_0$.*
2. *The limit $\lim_{\kappa \to \infty} E^{(\kappa)}$ exists and $E^* = \lim_{\kappa \to \infty} E^{(\kappa)}$.*

If the valuation function is left-continuous, one can also start the iteration from $-L$ and then one obtains a non-decreasing sequence of equity values that converges to the least fixed point. To see why left-continuity is crucial, we refer to the discussion in Rogers & Veraart (2013).

Definition 2.7. *Let $\Phi = \Phi(\cdot; \mathbb{V})$ be an equity valuation function for the financial system (L, L^e, A^e) with shock vector $x \in [0, A^{(e)}]$ and let E^* be its greatest fixed point. Let $\mathcal{M} := \{j \in \mathcal{N} \mid \bar{L}_j > 0\}$.*

1. *We define the* relative system loss corresponding to $\Phi(\cdot) = \Phi(\cdot; \mathbb{V})$*, by*

$$\Lambda^{\mathbb{V}} = \frac{\sum_{i \in \mathcal{M}} \sum_{j=1}^{N} L_{ij}(1 - \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right))}{\sum_{i \in \mathcal{M}} \sum_{j=1}^{N} L_{ij}}.$$

2. *We refer to every node i with $E_i^* < 0$ as* in default under valuation $\Phi(\cdot) = \Phi(\cdot; \mathbb{V})$. *Furthermore we denote by $D^{\mathbb{V}} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{\{E*_i < 0\}}$ the proportion of defaulting banks under valuation $\Phi(\cdot) = \Phi(\cdot; \mathbb{V})$.*

2.2 Special cases from the literature rewritten as network re-evaluation

We now briefly describe how a special case of the clearing problem by Rogers & Veraart (2013) can be rewritten in terms of an equity valuation function. In the following we will consider the special case in which $\alpha = \beta$.

Lemma 2.8. *Let (L, L^e, A^e) be a financial system with shock vector $x \in [0, A^{(e)}]$ and let $\mathcal{M} := \{j \in \mathcal{N} \mid \bar{L}_j > 0\}$.*
Then, the function $\mathbb{V}^{RV} : \mathbb{R} \to [0,1]$ given by

$$\mathbb{V}^{RV}(y) = \begin{cases} 1 & \text{if } y \geq 1, \\ \beta y^+ & \text{if } y < 1, \end{cases} \tag{5}$$

is an admissible valuation function, and

$$\Phi_i^{RV}(E) = A_i^e - x_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}^{RV}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i, \tag{6}$$

is an equity valuation function.

Theorem 2.9. *Let (L, L^e, A^e) be a financial system with shock vector $x \in [0, A^{(e)}]$ and $\mathcal{M} = \{i \in \mathcal{N} \mid \bar{L}_i > 0\}$. Let $\alpha = \beta \in [0,1]$.*

1. *Let $L^*(x)$ be a fixed point of Ψ^{RV} defined in (1). Then, E^* given by*

$$E_i^* := A_i^e - x_i + \sum_{j \in \mathcal{M}} L_{ji} \frac{L_j^*(x)}{\bar{L}_j} - \bar{L}_i, \quad \forall i \in \mathcal{N} \tag{7}$$

   is a fixed point of Φ^{RV} defined in (6).

2. *Let $L^*(x)$ be the greatest fixed point of Ψ^{RV} defined in (1). Then, E^* defined in (7) is the greatest fixed point of Φ^{RV} defined in (6).*

3. *Let E^* be a fixed point of Φ^{RV} defined in (6), then $L^*(x)$ given by*

$$L_i^*(x) = \begin{cases} \mathbb{V}^{RV}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right) \bar{L}_i, & \text{if } i \in \mathcal{M}, \\ 0, & \text{if } i \in \mathcal{N} \setminus \mathcal{M}. \end{cases} \tag{8}$$

   is a fixed point of Ψ^{RV} defined in (1).

4. *Let E^* be the greatest fixed point of Φ^{RV} given in (6), then $L^*(x)$ given by (8) is the greatest fixed point of Ψ^{RV} given in (1).*

Since the Rogers & Veraart (2013) model reduces to the Eisenberg & Noe (2001) model if $\beta = \alpha = 1$ we immediately get the following results.

Corollary 2.10. *Let (L, L^e, A^e) be a financial system with shock vector $x \in [0, A^{(e)}]$ and let $\mathcal{M} := \{j \in \mathcal{N} \mid \bar{L}_j > 0\}$.*

1. *Then, the function $\mathbb{V}^{EN} : \mathbb{R} \to [0,1]$ given by*

$$\mathbb{V}^{EN}(y) = \left\{ \begin{array}{ll} 1 & \text{if } y \geq 1, \\ y^+ & \text{if } y < 1, \end{array} \right\} = \min\{y^+, 1\} = \min\{\max\{0, y\}, 1\} \tag{9}$$

   is an admissible valuation function, and

$$\Phi_i^{EN}(E) = A_i^e - x_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}^{EN}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i, \tag{10}$$

is an equity valuation function.

2. *Let $L^*(x)$ be a fixed point of Ψ^{EN} defined in (2). Then, E^* given by*

$$E_i^* := A_i^e - x_i + \sum_{j \in \mathcal{M}} L_{ji} \frac{L_j^*(x)}{\bar{L}_j} - \bar{L}_i, \quad \forall i \in \mathcal{N} \tag{11}$$

is a fixed point of Φ^{EN} defined in (10).

3. *Let $L^*(x)$ be the greatest fixed point of Ψ^{EN} defined in (2). Then, E^* defined in (11) is the greatest fixed point of Φ^{EN} defined in (10).*

4. *Let E^* be a fixed point of Φ^{EN} defined in (10), then $L^*(x)$ given by*

$$L_i^*(x) = \begin{cases} \mathbb{V}^{EN}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right)\bar{L}_i, & \text{if } i \in \mathcal{M}, \\ 0, & \text{if } i \in \mathcal{N} \setminus \mathcal{M} \end{cases} \tag{12}$$

is a fixed point of Ψ^{EN} defined in (2).

5. *Let E^* be the greatest fixed point of Φ^{EN} given in (10), then $L^*(x)$ given by (12) is the greatest fixed point of Ψ^{EN} given in (2).*

2.3 Ordering results

Which admissible valuation function should one now choose? The following result shows that if one admissible valuation function is bounded from below by another admissible valuation function in the sense that it always returns a larger value than the other one, the corresponding greatest fixed point of the re-evaluated equity under the larger valuation function is greater or equal than the corresponding greatest fixed point under the other admissible valuation function. Furthermore, the corresponding relative system losses and proportions of defaults corresponding to the valuation function that returns the higher values are smaller or equal than the ones that correspond to the valuation function that returns the smaller values.

Theorem 2.11. *Let (L, L^e, A^e) be a financial system with shock vector $x \in [0, A^{(e)}]$. Let $\mathbb{V}^A$ and $\mathbb{V}^B$ be two admissible valuation functions with corresponding equity valuation functions Φ^A and Φ^B, respectively. Suppose that for all $y \in \mathbb{R}$*

$$\mathbb{V}^A(y) \geq \mathbb{V}^B(y). \tag{13}$$

Let $E^{*,A}$ and $E^{*,B}$ be the greatest fixed points of Φ^A and Φ^B respectively. Then,

1. *the greatest fixed points satisfy*

$$E^{*,A} \geq E^{*,B}.$$

2. *The relative system losses satisfy*

$$\Lambda^{\mathbb{V}^A} \leq \Lambda^{\mathbb{V}^B}.$$

3. The proportions of defaulting banks satisfy

$$D^{\mathbb{V}^A} \leq D^{\mathbb{V}^B}.$$

Note that the ordering results presented in Visentin et al. (2016) are special cases of the above relationship.

3 Modelling distress and default contagion

We have seen that in classical clearing models such as Eisenberg & Noe (2001); Rogers & Veraart (2013) the valuation function is equal to one, which corresponds to the value of the assets being equal to their nominal value as long as the total amount of assets is greater or equal than the total liabilities. We now propose a valuation function that allows us to model the spread of contagion already prior to default. As in (Glasserman & Young, 2015, Section 6) we assume that there exists a so-called capital cushion modelled by a parameter $k \in [0, \infty)$ and as soon as the value of the total assets of a bank is below $(1+k)\bar{L}$ we assume a deterioration in the bank's asset value due to marking to markets. As long as the assets are greater or equal than $(1+k)\bar{L}$ the valuation function returns 1, i.e., the asset values coincide with the nominal values. In particular, we consider a valuation function $\mathbb{V}^{\text{Distress}} : \mathbb{R} \to [0,1]$ which has the following structure

$$\mathbb{V}^{\text{Distress}}(y) = \mathbb{I}_{\{y \geq 1+k\}} + \mathbb{I}_{\{y < 1+k\}} r(y), \tag{14}$$

where $r : \mathbb{R} \to [0,1]$ is non-decreasing and right-continuous and we will present its functional form below. Note that y represents the value of the assets of a bank divided by its total nominal liabilities. Then $\mathbb{V}^{\text{Distress}}$ is indeed an admissible valuation function.

In the classical models by e.g., Eisenberg & Noe (2001) and Rogers & Veraart (2013) the capital cushion parameter is $k = 0$ and hence no reduction in asset value occurs until a bank has reached the default point, i.e., has fewer assets than liabilities. It is important to note that the classical clearing models (Eisenberg & Noe, 2001; Rogers & Veraart, 2013) have been developed to describe a resolution mechanisms at the maturity date. Now we would like to take the perspective of evaluating the state of the network prior to the maturity date. Hence, we consider a valuation mechanism and not just a resolution mechanism. There are two approaches how this can be done in principle. The first approach is a probabilistic approach and has been developed by Barucca et al. (2016) and a special case of the methodology has been applied to real data from the UK interbank network in Bardoscia et al. (2017). They essentially use a classical resolution mechanism from the literature and assume stochastic external assets which implies a stochastic net worth and then compute expected clearing payments. In Barucca et al. (2016) they consider the case of Eisenberg & Noe (2001) as a resolution mechanism and assume that the external assets follow a geometric Brownian motion. In Bardoscia et al. (2017) they consider a Black & Cox (1976) model for the default probabilities and combine it with exogenous recovery rates in case of default. Hence, the probabilistic approach consists of choosing a resolution mechanism and a probability distribution for the external assets. This approach can lead to admissible valuation functions which return values strictly less than 1 prior to the point of default and hence account for distress contagion. In these models devaluation of asset values occur because of the possibility of a shock that leads to bankruptcy in the future.

In this paper we model distress contagion by proposing a functional form of an admissible valuation function that models the decline in asset value directly. In particular, any decline in asset value is modelled as a reaction to a shock without explicitly accounting for the probability of a shock in the future. This is in line with the scenario-based approach to stress testing. We therefore do not make any assumptions on the probability distribution of the quantities of interest.

Since there are no suitable data available that could be used to empirically estimate the functional form of the function r, we propose a functional form that can capture a wide range of possible shapes of the decline in asset value. This is in line with the point of view by Brazier (2018) that one should use simulation rather than rely purely on historical data to analyze possible shock transmission channels.

A major advantage of our proposed functional form is that it relies only on a small number of model parameters that all have an intuitive meaning. One can therefore study the sensitivities of the outcome measures of a stress test as a function of the (small) number of parameters and therefore can get a good understanding of the stability of the financial network under investigation. The parameters reflect bankruptcy law and accounting standards. We refer to Harris et al. (2013) for a discussion on how allowances for potential losses are made in accounting terms; in particular adjustments made are for losses "that represent the management's estimate of the outstanding balance that it is unlikely to collect given current information and events", (Harris et al., 2013, p. 937).

3.1 The functional form

Now we look at the functional form of the valuation function in more detail. We assume that there is a non-decreasing and right-continuous function r modelling the decline in asset value as soon as y which is the value of the assets divided by the total nominal liabilities is less than $1 + k$. As soon as $y < 1$ we have reached the classical definition for default as e.g., in Eisenberg & Noe (2001); Rogers & Veraart (2013). We will use their ideas to deal with this case. In particular, we assume that the decline in asset value is caused by satisfying certain stylized principles of bankruptcy law such as limited liabilities, proportionality etc. as described in detail in Eisenberg & Noe (2001); Rogers & Veraart (2013). The advantage of this modelling assumption is that it incorporates fundamental ideas from bankruptcy law into the modelling of default contagion. Furthermore, by keeping this part of the model consistent with existing approaches our model reduces to those models for special choices of the model parameters which is useful in sensitivity studies. This assumption implicitly assumes that marking-to-market is consistent with these fundamental principles of bankruptcy law. We will discuss some possible generalizations in Subsection 3.4.

Therefore from a modelling point of view we only need to find a model that describes the decline in asset value for $y \in [1, 1 + k)$. We propose the following functional form

$$r(y) = \begin{cases} 1 - (1 - R)F\left(\frac{1+k-y}{k}; a, b\right), & \text{if } 1 \leq y < 1 + k, \\ \beta y^+, & \text{if } y < 1, \end{cases} \tag{15}$$

where $y^+ = \max\{y, 0\}$ and F is the cumulative distribution function (cdf) of the Beta distribution with parameters $a > 0, b > 0$. Recall that the probability density function of the Beta distribution with parameters $a > 0$ and $b > 0$ is given by

$$f(x; a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\mathbb{I}_{\{0 \le x \le 1\}}$$

and the corresponding cumulative distribution function is

$$F(x; a, b) = \int_0^x f(y; a, b) dy$$

for $0 \le x \le 1$.

The reason for choosing the cumulative distribution function of the Beta distribution for modelling the distress contagion branch of the valuation function is that it enables us to model a wide range of possible declines in asset value using a very parsimonious set of model parameters. We will illustrate this in the next subsection.

The following Proposition shows that this indeed leads to an admissible valuation function.

Proposition 3.1. *Let $k \ge 0$, $R, \beta \in [0,1]$ with $R \le \beta$, $a > 0, b > 0$ and let $\mathbb{V}^{Distress} : \mathbb{R} \to [0,1]$ be given by*

$$\begin{aligned} \mathbb{V}^{Distress}(y) &= \mathbb{V}^{Distress}(y; k, \beta, R, a, b) \\ &= \mathbb{I}_{\{y \ge 1+k\}} + \mathbb{I}_{\{y < 1+k\}} r(y), \\ &= \begin{cases} 1 & \text{if } y \ge 1+k, \\ 1 - (1-R)F\left(\frac{1+k-y}{k}; a, b\right), & \text{if } 1 \le y < 1+k, \\ \beta y, & \text{if } 0 \le y < 1, \\ 0, & \text{if } y < 0. \end{cases} \end{aligned} \tag{16}$$

Then $\mathbb{V}^{Distress}$ is an admissible valuation function.

Hence, the model distinguishes between four different situations. If y modelling the asset value divided by the total nominal liabilities is above a certain level, i.e., higher than $1+k$ the asset values correspond to the nominal values. If $1 \le y < 1+k$ we are in the *distress contagion* branch of the valuation function, i.e., the bank has enough assets to satisfy its nominal payment obligations but still a decline in its asset value occurs. This captures the mark-to-market effects of losses suffered by the bank to its total assets. As soon as $y < 1$ we are in the *default contagion* branch of the valuation function, because now the value of its assets is strictly less than its nominal payment obligations. We assume here a proportional repayment of the remaining assets and proportional bankruptcy costs as in Rogers & Veraart (2013).

Remark 3.2 (Relationship between distress contagion and credit valuation adjustments)**.** Our distress contagion model is related to credit valuation adjustments (CVAs). "CVA is an adjustment to the fair value (or price) of derivative instruments to account for counterparty credit risk (CCR)," Basel Committee on Banking Supervision (2015a). Hence, the losses due to distress contagion can be interpreted as losses due to contagion that are caused by such adjustments.

It is common to model CVAs as a factor. In Hałaj & Kok (2015), CVAs are considered in the context of CVA-based capital charges in a network of interbank liabilities. They assume that a capital charge of $\gamma_j L_{ji}$ is applied to bank i, where γ_j is bank j's specific

CVA-factor. They provide explicit formulae for CVA-factors that are consistent with Basel III regulation. These formulae are based on the "advanced method of CVA calculation [which] involves the observed credit default spreads of the counterparts to infer the market-based probability of default and consequently, given the assumed exposures and their maturities, the expected loss on the portfolio of the interbank assets", (Hałaj & Kok, 2015, Appendix 1).

In our model, if node j is hit by a shock its re-evaluated equity declines and hence its creditworthiness deteriorates. Hence, our valuation assumes that it will no longer pay the full amount L_{ji} to node i but only $L_{ji}\mathbb{V}^{\text{Distress}}\left((E_j^* + \bar{L}_j)/\bar{L}_j\right) \le L_{ji}$, where E^* is the greatest re-evaluated equity. In that sense, the deterioration of j's credit quality is captured by the factor $\mathbb{V}^{\text{Distress}}\left((E_j^* + \bar{L}_j)/\bar{L}_j\right)$ which depends on j's re-evaluated assets and the size of the shock. This can be interpreted as a credit valuation adjustment.

Since E^* is a fixed point, the factor $\mathbb{V}^{\text{Distress}}\left((E_j^* + \bar{L}_j)/\bar{L}_j\right)$ in our model accounts for the re-evaluated assets of all nodes in the network and in particular captures higher-order network effects. Such higher-order network effects are currently not captured in more structural models for CVAs such as the one by Hałaj & Kok (2015) which build on the current Basel III regulation.

Hence, our choice of the functional form of $\mathbb{V}^{\text{Distress}}$ allows for a wide range of possible losses that are due to credit valuation adjustments while accounting for feedback effects.

It would be possible to include additional market fundamentals into the modelling of $\mathbb{V}^{\text{Distress}}$ as we will briefly discuss in Subsection 3.4.

3.2 The meaning of the model parameters

Our model consists of five model parameters that have the following interpretation:

- $k \in [0, \infty)$: parameter modelling the capital cushion. It determines at which asset level the bank starts suffering from contagion.

- $R \in [0, 1]$: parameter modelling the perceived exogenous recovery rate (and determining the perceived proportional default costs).

- $\beta \in [0, 1]$: parameter modelling the actual exogenous recovery rate (and determining the actual proportional default costs).

- $a, b \in (0, \infty)$: parameters modelling decline in asset value due to distress contagion by determining the shape of the cdf of the Beta distribution.

We refer to the parameters R and β as (perceived and actual) exogenous recovery rates. In addition to these exogenous recovery rates, the model also contains an endogenous (non-constant) recovery rate. This can be seen in (15) where the function r is also non-constant on the default branch, i.e., for $y < 1$.

Figure 1 illustrates the sensitivity of the valuation function $\mathbb{V}^{\text{Distress}}$ with respect to the model parameters. The left hand side of Figure 1 shows the effect of different exogenous recovery rate parameters (R, β) assuming $(a, b) = (1, 1)$. This choice of (a, b) corresponds to a linear function on the distress contagion branch of the valuation function. The default contagion branch of the valuation function is always a linear function. In this case, if $R = \beta$, the valuation function is piecewise linear and continuous. If $R > \beta$ the valuation function is still piecewise linear but discontinuous at $y = 1$.
The right hand side of Figure 1 shows the sensitivity of the valuation function with

respect to the parameters a, b for $(R, \beta) = (0.5, 0.2)$. Since we have chosen $R > \beta$ here, the valuation function is discontinuous at $y = 1$. We see that the cdf of the Beta function $F(;a, b)$ allows for a wide range of different shapes of the valuation function on the distress contagion branch of the valuation function. It includes the linear function and polynomials as special cases, in particular

$$\begin{aligned} F(x; 1, 1) &= x, \\ F(x; a, 1) &= x^a, \\ F(x; 1, b) &= 1 - (1 - x)^b. \end{aligned}$$

Furthermore, if we set $b = 1$ and let $a \to \infty$, the cdf of the Beta distribution models converges to the cdf of a probability distribution with point mass at 1. Hence, we see that if we choose $k > 0$ and an almost horizontal shape of the valuation function for large part of the distress contagion branch (obtained by setting $b = 1$ and letting $a \to \infty$) we obtain a valuation function that is very similar to choosing $k = 0$ (and arbitrary values of R, a, b). This overall shape would correspond to the Rogers & Veraart (2013) model.

In the following we derive ordering results for some outcome measures of a stress test for different choices of the model parameters.

Theorem 3.3. *Consider $\mathbb{V}^{Distress}(y) = \mathbb{V}^{Distress}(y; k, \beta, R, a, b)$ given in (16). Let $k \geq 0$, $\beta, R \in [0, 1]$ with $\beta \leq R$ and $a, b \in (0, \infty)$.*

1. *Sensitivity with respect to the capital cushion parameter: Let $k_1, k_2 \in [0, \infty)$ with $k_1 \leq k_2$.*

   (a) *Then for all $y \in \mathbb{R}$*

   $$\mathbb{V}^{k_1}(y) := \mathbb{V}^{Distress}(y; k_1, \beta, R, a, b) \geq \mathbb{V}^{Distress}(y; k_2, \beta, R, a, b) =: \mathbb{V}^{k_2}(y).$$

   (b) *Let E^{*,k_1} and E^{*,k_2} be the greatest fixed points of the to $\mathbb{V}^{k_1}$ and $\mathbb{V}^{k_2}$ corresponding equity valuation functions Φ^{k_1} and Φ^{k_2}. Then,*

      i. *the greatest fixed points satisfy*

      $$E^{*,k_1} \geq E^{*,k_2}.$$

      ii. *The relative system losses satisfy*

      $$\Lambda^{\mathbb{V}^{k_1}} \leq \Lambda^{\mathbb{V}^{k_2}}.$$

      iii. *The proportions of defaulting banks satisfy*

      $$D^{\mathbb{V}^{k_1}} \leq D^{\mathbb{V}^{k_2}}.$$

2. *Sensitivity with respect to the exogenous recovery rates β and R: Let $\beta_1, \beta_2, R_1, R_2 \in [0, \infty)$ with $\beta_1 \leq \beta_2 \leq R_1 \leq R_2$.*

   (a) *Then for all $y \in \mathbb{R}$*

   $$\mathbb{V}^{\beta_1, R_1}(y) := \mathbb{V}^{Distress}(y; k, \beta_1, R_1, a, b) \leq \mathbb{V}^{Distress}(y; k, \beta_2, R_2, a, b) =: \mathbb{V}^{\beta_2, R_2}(y).$$

(b) Let E^{*,β_1,R_1} and E^{*,β_2,R_2} be the greatest fixed points of the to $\mathbb{V}^{\beta_1,R_1}$ and $\mathbb{V}^{\beta_2,R_2}$ corresponding equity valuation functions Φ^{β_1,R_1} and Φ^{β_2,R_2}. Then,

i. the greatest fixed points satisfy

$$E^{*,\beta_1,R_1} \leq E^{*,\beta_2,R_2}.$$

ii. The relative system losses satisfy

$$\Lambda^{\mathbb{V}^{\beta_1,R_1}} \geq \Lambda^{\mathbb{V}^{\beta_2,R_2}}.$$

iii. The proportions of defaulting banks satisfy

$$D^{\mathbb{V}^{\beta_1,R_1}} \geq D^{\mathbb{V}^{\beta_2,R_2}}.$$

As an immediate consequence of part 1 of Theorem 3.3 (assuming $k_1 = 0 < k_2$) we see that allowing for distress contagion ($k_2 > 0$) will lead to worse or at best equal outcomes of the stress test compared to not allowing for distress contagion ($k_1 = 0$).

Similarly, from part 2 of Theorem 3.3 (assuming $\beta_1 < \beta_2 = 1 = R_1 = R_2$) we see that in particular stress test outcomes in the special case of the Rogers & Veraart (2013) model ($\beta_1 < 1$) are worse or at best equal to outcomes in the Eisenberg & Noe (2001) model ($\beta_2 = 1$).

3.3 Spread, containment and amplification of losses

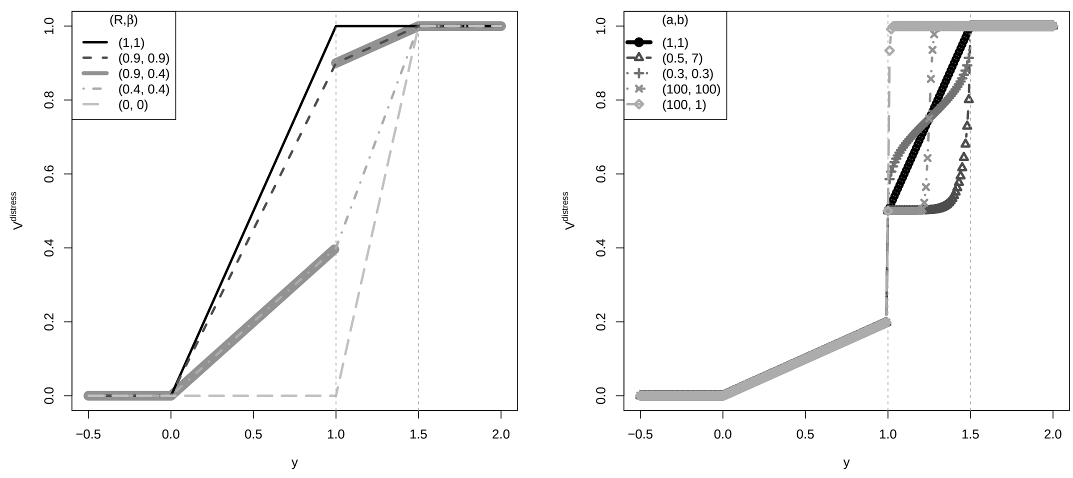
As already outline, choosing a capital cushion parameter $k > 0$ gives rise to a new class of models that account for distress and not just default contagion. From Theorem 3.3 it is clear that when keeping R, β, a, b fixed, higher levels of k will result in worse outcomes for the network. If $k = 0$ the model reduces to the model by Rogers & Veraart (2013) in which $\alpha = \beta$. If in addition $\beta = 1$ the valuation function reduces to the model by Eisenberg & Noe (2001). If $k = 0$ and $R = \beta = 0$, the model reduces to the default cascade model considered e.g., in Amini et al. (2016); Furfine (2003) with zero recovery rate.

Note that in the Furfine (2003); Amini et al. (2016); Rogers & Veraart (2013); Eisenberg & Noe (2001) models the net worth acts as a buffer that can absorb losses. Only if this buffer is depleted, i.e., if the net worth becomes negative, transmission of losses to other nodes in the network occurs. Since there are no default costs in the Eisenberg & Noe (2001) model, losses are not amplified in any way by the network. On the contrary, in the Eisenberg & Noe (2001) model losses can actually be contained due to the existence of the equity buffer that has to be depleted first. The analytical results derived in Glasserman & Young (2015) on the likelihood of contagion in the Eisenberg & Noe (2001) model reflect this. When we allow for $k > 0$ we reduce the ability of the network to absorb losses before they are transmitted.

Once losses are transmitted to other nodes in the network, the role of the parameters modelling the exogenous recovery rate and default costs is absolutely crucial. Note that we use two parameters $R, \beta \in [0, 1]$ with $\beta \leq R$ to model the perceived exogenous recovery rate (R) and the actual exogenous recovery rate (β). As soon as they do not coincide, i.e., $R > \beta$ the valuation function is discontinuous at the default point (but remains right-continuous throughout). The parameter $R \in [0, 1]$ determines the lower bound for the valuation function prior to the default event, i.e., a lower bound for the distress contagion branch of the valuation function. The parameter $\beta \in [0, R]$ acts as an upper bound of the default contagion branch of the valuation function.

The first important point to note is that if we do not introduce default costs there would be no way to model distress contagion in our setting. The reason why marking to markets reduces the value of the assets, is that implicitly there is the assumption that at default, the bank does not recover all of its assets. The valuation function corresponding to the Eisenberg & Noe (2001) model is a continuous function as we have seen in Corollary 2.10. Hence, there is no big difference between being just below the default threshold or above it. This means, if a bank is marginally below the default threshold it can repay almost all of its total nominal liabilities. Hence, from the point of view of marking to markets, there is no reason why the asset value of a bank that is close to default should be reduced in value if the default event itself is modelled as a continuous and soft threshold with no additional losses. As soon as the market participants think that there will be additional losses (due to bankruptcy costs modelled by R) assuming a lower asset value prior to default is reasonable. By assuming that perceived exogenous recovery rate R is greater or equal than the actual exogenous recovery rate β we ensure the monotonicity of the valuation function which keeps the model tractable.

Hence we see that despite the fact that bankruptcy costs by definition only occur in the case of bankruptcy, their existence can have consequences for the re-evaluation of the network even in the absence of bankruptcy. Therefore, bankruptcy costs can amplify losses in networks due to two different effects. The direct effect of bankruptcy costs is just the additional losses that occur in the case of default. The indirect effect of bankruptcy costs is that they imply mark to market losses prior to the default event itself, because the default point is modelled as a discontinuity of the valuation function. We are not aware that these twofold consequences of bankruptcy costs have been recognised before.



(a) $\mathbb{V}^{\text{distress}}$ for different choices of (R, β) (b) $\mathbb{V}^{\text{distress}}$ for different choices of (a, b)

Figure 1: Function $\mathbb{V}^{\text{distress}}$ for different choices of (R, β) with $(a, b) = (1, 1)$ (left) and for different choices of (a, b) with $(R, \beta) = (0.5, 0.2)$ (right) for $k = 0.5$.

3.4 Generalizations

We briefly would like to discuss three possible extensions of our valuation framework.

First, we could allow for node-specific model parameters. So far we have assumed that the model parameters k, R, β, a, b are the same for all nodes in the network. If we consider a heterogeneous network of financial institutions, there are good reasons to assume that some of these model parameters should be institution specific. In particular, some institutions might have much higher recovery rates in case of default since they have financial collateral in place and therefore their lending is much more secure. This would lead to making β and R institution specific. As we have already seen in our previous discussion, the default cost/recovery rate parameter is strongly linked to the capital cushion parameter k. In particular, one could argue that the higher the perceived exogenous recovery rate R, the lower should be the capital cushion parameter k. In this case also the parameter k should be institution specific. Since then the valuation function depends on the index j of the institution, it is no longer a valuation function according to our original definition, but all results can be easily generalized.

Theorem 3.4. *Let (L, L^e, A^e) be a financial system with shock vector $x \in [0, A^{(e)}]$ and let $\mathcal{M} := \{j \in \mathcal{N} \mid \bar{L}_j > 0\}$. Let for all $j \in \mathcal{N}$ $\mathbb{V}_j : \mathbb{R} \to [0,1]$ with*

$$\mathbb{V}_j^{General}(y) := \mathbb{V}^{Distress}(y; k_j, \beta_j, R_j, a_j, b_j)$$

with $\mathbb{V}^{Distress}$ as in (16).

Consider $\Phi^{General} : \mathcal{E}(x) \to \mathcal{E}(x)$, where $\mathcal{E}(x) = [-\bar{L}, w - x]$ and for $i \in \mathcal{N}$

$$\Phi_i^{General}(E) = A_i^e - x_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}_j^{General}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i. \tag{17}$$

Then,

1. *there exists a greatest fixed point E^{**} and a least fixed point E_{**} of $\Phi_i^{General}$ such that for all fixed points E of $\Phi_i^{General}$ it holds that*

$$E_{**} \leq E \leq E^{**}.$$

2. *Let $E^{(0)} := w - x$ and define recursively $E_i^{(\kappa+1)} = \Phi_i^{General}(E^{(\kappa)})$ for all $\kappa \in \mathbb{N}_0$ and for all $i \in \mathcal{N}$. Then*
   - *(a) $(E^{(\kappa)})_{\kappa \in \mathbb{N}_0}$ is a monotonically non-increasing sequence, i.e., $E^{(\kappa+1)} \leq E^{(\kappa)}$ $\forall \kappa \in \mathbb{N}_0$.*
   - *(b) The limit $\lim_{\kappa \to \infty} E^{(\kappa)}$ exists and $E^{**} = \lim_{\kappa \to \infty} E^{(\kappa)}$.*

We can also define the relative system loss and the proportion of defaulting banks corresponding to the greatest fixed point as before.

In the literature, there is one model that can be considered as using a institution specific capital cushion, namely the DebtRank model by Battiston et al. (2012). The DebtRank model with zero recovery rate can be rewritten in terms of our general model by setting $k_i = \frac{w_i}{\bar{L}_i}$, $a = b = 1$ and $R = \beta = 0$. This choice of the capital cushion parameter k_i implies that all shocks to a node are transmitted to other nodes in the network since there is no buffer left that can be depleted before shock transmission occurs. In practice, this is a strong assumption. We will discuss the effect of different choices for the model parameters including the capital cushion parameter in our case study in the next section.

Second, we could also consider relaxing the assumption on the default point which is currently assumed to be at $y = 1$, i.e., when the total assets are equal to the total liabilities and hence the net worth is equal to zero. This would give us a sixth model parameter $D \in [0, 1 + k]$ and would lead to the following admissible valuation function

$$\begin{aligned} \mathbb{V}^{\text{Distress2}}(y) &:= \mathbb{V}^{\text{Distress2}}(y; k, \beta, R, a, b) \\ &:= \begin{cases} 1 & \text{if } y \geq 1 + k, \\ 1 - (1 - R)F\left(\frac{1+k-y}{k}; a, b\right), & \text{if } D \leq y < 1 + k, \\ \beta y, & \text{if } 0 \leq y < D, \\ 0, & \text{if } y < 0. \end{cases} \end{aligned} \tag{18}$$

In practice, default will usually occur at $D > 1$, namely when regulatory capital requirements are no longer met. To model such a situation one would need to require that $D \leq \frac{R}{\beta}$ to guarantee the monotonicity of $\mathbb{V}^{\text{Distress2}}$.

From a mathematical point of view, the additional parameter D also allows for an application of the very flexible shape of the cdf of the Beta distribution over not just the distress contagion branch of the valuation function, but also over the default contagion branch by setting $D = 0$. This would then enable us to model situations in which the effects of marking-to-market are no-longer restricted to comply with the stylized principles of bankruptcy law established in Eisenberg & Noe (2001) on what we call the default contagion branch of the valuation function.

Third, it would be possible to extend our modelling framework such that the valuation function $\mathbb{V}^{\text{Distress}}$ depends explicitly on additional variables that describe the state of the market. In particular, we could make the model parameters a and b that describe the severity of decline due to distress contagion dependent on some market fundamentals. Our sensitivity analysis, however, will show that we can get many interesting insights about the state of the network already without adding this additional layer of structural modelling.

4 Applications to stress testing

In the following we will show how the new framework can be used in a regulatory stress test. The stress test would proceed in the following four steps.

First one collects the market data corresponding to the stylized balance sheet in Table 1, i.e., the value of the external assets, the interbank assets, the external liabilities and the interbank assets. This information is available from published balance sheets.

Second one needs to establish the network of interbank liabilities, i.e., the matrix L. If the interbank liabilities are not fully observable then one can reconstruct this matrix from its observable row and column sums. Several methods are available to do this, see e.g., the Bayesian approach by Gandy & Veraart (2017, 2019) and discussions on alternative approaches. These approaches do not rely on historical estimates of the financial network but reconstruct the financial network based on the partial information that is available on the current network.

Third one decides on a shock x or a selection of shocks used in the stress test. This would correspond to a choice of a scenario in a scenario based stress tests. Prior to the 2007-2008 financial crisis it was common to consider shocks that would correspond to some historic events. More recently there is a clear tendency to include hypothetical scenarios in stress testing, see e.g., Basel Committee on Banking Supervision (2009). Different shocks can be considered here. All institutions could be hit by a shock or only a selection of institutions could be hit by a shock. The number of fundamental defaults corresponding to different shocks can be read off directly from the stylized balance sheet in Table 1. All institutions whose shocked net worth is strictly less than zero are in fundamental default. One can use this information to choose shock sizes of interest.

Finally one uses the new contagion model to determine different outcome measures of the stress test. For example, one can compute which banks default as the result of the stress test or one can derive the corresponding system loss. Since these outcome measures will depend on the five model parameters, a sensitivity analysis should be performed to analysis the effects of the various model parameters. We will discuss how this can be done efficiently in Subsection 4.1. In particular we show how the analytical ordering results established in in Theorem 3.3 can be used in the sensitivity analysis.

This approach to stress testing does not rely on historical estimates of financial markets to perform the stress testing exercise. In particular, the contagion mechanism proposed incorporates ideas from bankruptcy law and accounting and does not rely on historical estimates of financial contagion processes. In that sense this approach to stress testing is not an econometric policy analysis and as such cannot be subject to the Lucas critique. Lucas (1976) argued that "given that the structure of all econometric model consists of optimal decision rules of economic agents, and that optimal decision rules vary systematically with changes in the structure of series relevant to the decision maker, it follows that any change in policy will systematically alter the structure of econometric models." This implies that models are useful for policy analysis only if the underlying parameters are policy-invariant. This is indeed the case in our framework. Our contagion mechanism takes the financial network as given and characterizes how asset values decline due to financial contagion. The modelling framework is so general that it can capture a wide variety of shapes of declines in asset value using a parsimonious set of parameters. By performing a sensitivity study as part of the stress testing exercise one can then understand the possible outcomes for the network if these parameters change. We will illustrate this in our empirical case study.

We also would like to point out that one of the main goals of an annual stress test is to assess the resilience of the financial system. If a stress test shows that some institutions would not be able to withstand a certain shock they might be required to take specific actions determined by the financial regulator such as adjusting capital buffers for example. While stress tests could be used to assess the effects of different regulatory policies this is usually not their main purpose. Still this would be possible to do in our setting. One could, for example, assess the consequences of requiring different leverage ratios for financial institutions, because our financial contagion mechanism is independent of such characteristics of the network. One could also, for example, assess consequences of large exposure constraints which would alter the underlying financial network. The contagion mechanism can then be applied to the network without large exposure constraints and to a corresponding network that satisfies some given constraints on the exposures. If one wanted to assess the consequences of changes in accounting practice or bankruptcy law for example, this could be achieved by adapting some of the model parameters. For example, one can investigate the effects of different legal frameworks which determine when the point of bankruptcy is reached as discussed in Subsection 3.4. Since the model parameters have a clear economic interpretation this can easily be done.

4.1 Empirical case study

We now apply the new framework for quantifying distress and default contagion based on the valuation function $\mathbb{V}^{\text{Distress}}$ to empirical data. We consider balance sheet data of 76 banks[1] that took part in the European Banking Authority's (EBA) 2011 stress test [2]. These data have been analyzed in Glasserman & Young (2015) and subsets of these data have also been used in Gandy & Veraart (2017) and Chen et al. (2016).

For each bank i the data contain its total assets $\bar{A}_i$, its total interbank assets $\bar{A}_i^{\text{B}}$ and the net worth w_i. From these observations we obtain the external assets as $A_i^{\text{e}} := \bar{A}_i - \bar{A}_i^{\text{B}}$. The total interbank liabilities $\bar{L}_i^{\text{B}}$ are not available. As in Gandy & Veraart (2017) we set them to be equal to a slightly perturbed version of $\bar{A}_i^{\text{B}}$. In particular, for $i \in \{1, 2, \ldots, N-1\}$ we set $\bar{L}_i^{\text{B}} := \text{Round}\left((\bar{A}_i^{\text{B}} + \epsilon_i)\frac{\sum_{j=1}^{N}\bar{A}_j^{\text{B}}}{\sum_{j=1}^{N}(\bar{A}_j^{\text{B}}+\epsilon_j)}\right)$ and $\bar{L}_N^{\text{B}} := \sum_{j=1}^{N}\bar{A}_j^{\text{B}} - \sum_{j=1}^{N}\bar{L}_j^{\text{B}}$, where Round$(\cdot)$ is the function that rounds to 1 decimal place and $\epsilon_1, \ldots, \epsilon_N$ are independent realisations from the normal distribution with mean 0 and standard deviation 100. We take one fixed realisations for the $\bar{L}_i^{\text{B}}$ for our analysis. We can then determine the external liabilities by setting $L_i^{\text{e}} := \bar{A}_i - \bar{L}_i^{\text{B}} - w_i$ for all $i \in \mathcal{N}$.

To re-evaluate the network we need to know the individual entries L_{ij}, where $i, j \in \mathcal{N}$, of the liabilities matrix. Since these are not available, we use the Bayesian approach to network reconstruction developed by Gandy & Veraart (2017, 2019) to reconstruct a matrix from its row and column sums. In particular, we use the *empirical fitness model* introduced in Gandy & Veraart (2019) and calibrate it to a network density of 0.4 as described in Gandy & Veraart (2019), i.e., 40% of the entries of the matrix are assumed to be non-zero. We obtain a sample of liabilities matrices $L^{(\nu)}$, which have the observed row and column sums. For the first part of the analysis we take one of the samples and treat it as the true liabilities matrix L. For the second part of the analysis we use all samples.

We choose a deterministic shock vector x where $x_i = 0.03A_i^{\text{e}}$ for all $i \in \mathcal{N}$. This leads to the fundamental default of ten banks, i.e., ten banks have a negative net worth $w - x$ even under the assumption that all banks satisfy their obligations in full.

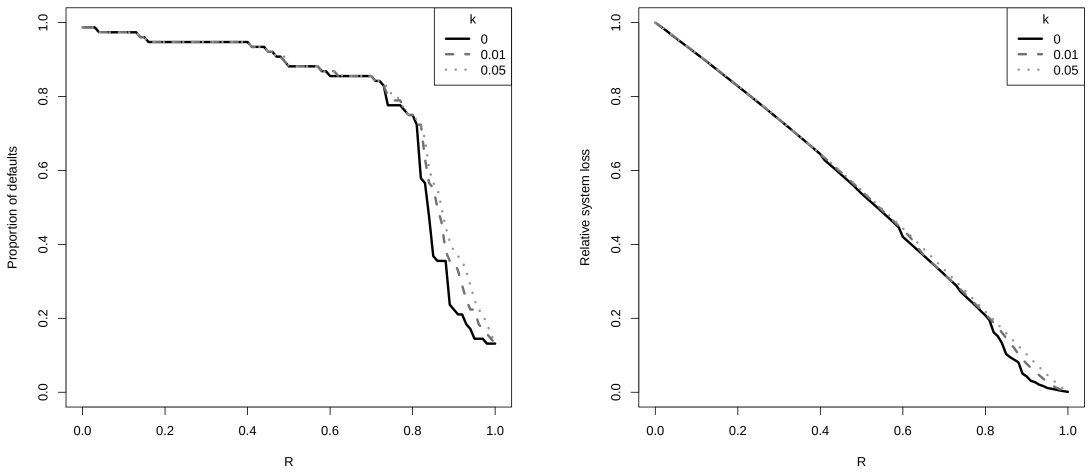
Now we conduct a sensitivity analysis of the proportion of defaults and the relative system loss with respect to the five model parameters. We first study the sensitivity of these measures with respect to two model parameters: the perceived exogenous recovery rate R and the parameter determining the start of the contagion process k. We assume that $\beta = R$ which implies that the valuation function is continuous. Furthermore, we assume that $a = b = 1$, i.e., the distress contagion branch of the valuation function is a linear function. Later we will investigate what happens if we assume $\beta < R$ and use different choices of a and b.

Figure 2 shows the proportion of defaults $D^{\mathbb{V}^{\text{Distress}}}$ (left hand side) and the relative system loss $\Lambda^{\mathbb{V}^{\text{Distress}}}$ (right hand side) as a function of the perceived exogenous recovery rate R for different choices of the capital cushion parameter k for one reconstructed network. As expected from Theorem 3.3 we see that smaller values of k (i.e., a later start of the contagion process) correspond to lower proportions of default and lower relative system losses in particular for large exogenous recovery rates R.

We can compute an upper bound $k_{\max}$ on the capital cushion parameter k that is of

[1] In total 90 banks took part in the stress test. Due to some problematic data with some of the smallest banks we excluded the ten smallest banks and any countries with only a single participating bank as in Glasserman & Young (2015). This results in 76 banks.

[2] See http://www.eba.europa.eu/risk-analysis-and-data/eu-wide-stress-testing/2011/results



(a) Proportion of defaults for different k (b) Relative system loss for different k

Figure 2: Proportion of defaults $D^{\mathbb{V}^{\text{Distress}}}$ (left) and relative system loss $\Lambda^{\mathbb{V}^{\text{Distress}}}$ (right) as a function of R for different values of the cushion parameter k using $\beta = R$ and $a = b = 1$ for one reconstructed network.

interest in a sensitivity analysis as follows. We assume that the asset value of bank i starts to decline due to marking-to-markets as soon as its assets are less than or equal to $(1 - k_i)\bar{L}$. Its maximum asset value is the notional amount listed on the balance sheet reduced by the shock x_i, i.e., this represents the shocked assets prior to the start of the contagion process. Therefore we determine $k_i \in \mathbb{R}$, $i \in \mathcal{N}$, by requiring that $\bar{A}_i - x_i = (1 - k_i)\bar{L}_i$. Now for all $i \in \mathcal{M} = \{i \in \mathcal{N} \mid \bar{L}_i > 0\}$ we can solve for k_i as follows[3]

$$k_i = \frac{\bar{A}_i - x - \bar{L}_i}{\bar{L}_i} = \frac{w_i - x_i}{\bar{L}_i} = \frac{1}{\frac{\bar{L}_i}{w_i}} - \frac{x_i}{\bar{L}_i}.$$

Hence we see how k_i is related to the debt-to-equity or leverage ratio $\bar{L}_i/w_i$. For our data set we find that the median and the mean of the k_i for $i \in \mathcal{N}$ is 0.015 and 0.016 respectively. Furthermore, we set

$$k_{\max} := \left(\max_{i \in \mathcal{M}} \left\{ \frac{\bar{A}_i - x - \bar{L}_i}{\bar{L}_i} \right\} \right)^+.$$

For our data set we obtain $k_{\max} \approx 0.08$, which means that $w_i - x_i = \bar{A}_i - x_i - \bar{L}_i \leq 0.08\bar{L}_i$ for all $i \in \mathcal{M}$. Hence, we see that for any sensitivity analysis choices of $k \in [0, k_{\max}]$ are potentially of interest.

As previously discussed, if the exogenous recovery rate satisfies $R = 1$, then the choice of k does not matter, since the distress contagion branch of the valuation function is just a horizontal line at 1, since it is bounded from below by $R = 1$. If R is slightly less than 1 than the choice of k seems to matter. If R is too small, however, the amplification effect and hence the total losses are so large that they dominate the overall behaviour. If

[3]Note that if $i \in \mathcal{N} \setminus \mathcal{M}$ then $\bar{L}_i = 0$. Hence, this bank can always repay its nominal liabilities of 0 in full and no losses due to distress or default can occur.

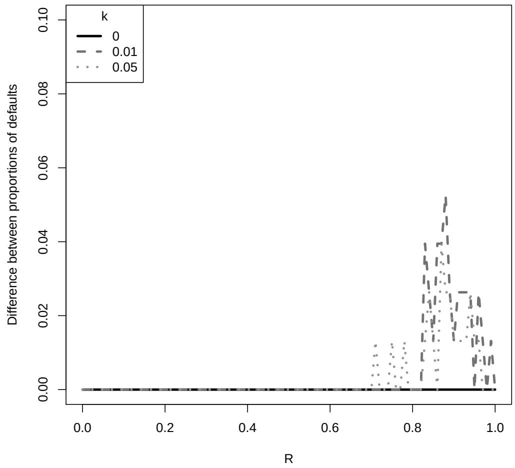
losses are too big, it is less important whether distress contagion starts slightly earlier or not. So we see that the choice of the exogenous recovery rate R will have a huge effect on the outcome of the contagion process. In our example, for $R = 1$, the proportion of defaulting banks corresponds to $10/76 \approx 0.13$ which corresponds to only the fundamental defaults, i.e., defaults that occur even if all banks are assumed to satisfy their payment obligations. For $R = 0$, however, the proportion of defaulting banks almost reaches 1, meaning that almost all banks have been wiped out. Note that we assume here that $R = \beta$, i.e., the perceived exogenous recovery rate R coincides with the actual exogenous recovery rate β.

From Theorem 3.3 we know that if $\beta < R$ then the outcome for the network is worse than for $R = \beta$. Hence, in that sense Figure 2 represents a best case scenario (for fixed parameters a, b, k, R) with respect to the parameter β. Further tests not reporter here show that the introduction of a discontinuity of the valuation function at the point of default by choosing $\beta < R$ only has a small (but worse) overall effect on the proportion of defaults and the relative system loss and again this only matters for rather large values of R. Here we find that for $k = 0.05$ and $R < 0.78$ more than 75 % of the banks have already defaulted. Hence, there is not much point in investigating what happens for $\beta < R < 0.75$ if already three quarters or more of the banks have defaulted under $\beta = R$. Hence, we see that in practice there is only a small range of values for R that would not lead to an almost complete collapse of the system and which would then require a further analysis of the effects of β.

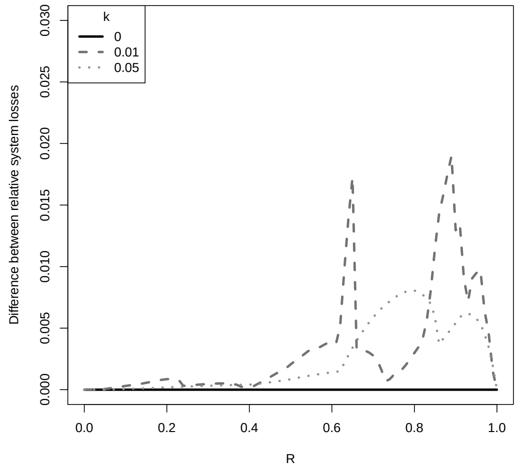
Next we investigate the effects of the parameters a and b modelling the decline due to distress contagion. Figure 3 shows that the choice of the parameters a, b only matters for large values of the recovery rate R. Here we compare the proportion of defaults and relative system losses for $(a, b) = (1, 1)$ which corresponds to a linear distress contagion model to $(a, b) = (0.5, 7)$ which corresponds to a model which has a strong decline in value close to the capital cushion and a flatter region closer to 1, the overall shape can be seen in Figure 3(c) where we used a much larger capital cushion $k = 0.5$ to make it easier to visualise the difference between the two curves. In line with the results of Theorem 2.11 we see that the linear decline in asset value always lead to a better outcome than the stronger than linear initial decline achieved by choosing $(a, b) = (0.5, 7)$. But again this only matters for rather high values of the exogenous recovery rate R.

These results show that distress contagion has a larger influence on the outcomes of the stress tests in financial networks with higher exogenous recovery rates. Higher exogenous recovery rates can for example be associated with more secured lending due to the use of financial collateral or higher seniority of the debt. Furthermore, the magnitude of any recovery rate is strongly linked to the duration of the actual recovery process. In particular, recovery rates over a long time horizon can be significantly higher than right at the point of default. If we allow for a longer time horizon, then there are also more opportunities for mark-to-market accounting and for distress contagion to unfold. So we see that accounting for distress contagion seems particularly important when assessing financial stability over a longer time horizon.

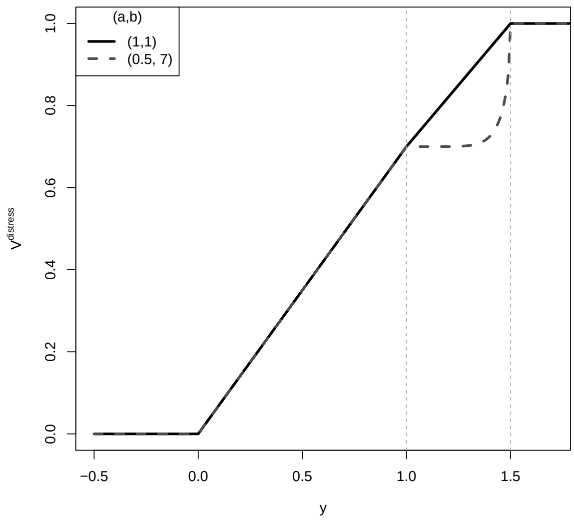
Next we investigate whether our results based on one reconstructed network carry over to a large sample of reconstructed networks. Figure 4 shows the minimum, mean and maximum of the MCMC sample of 10,000 reconstructed networks of the proportion of defaults (left) and relative system loss (right) as a function of R for $\beta = R$ and $a = b = 1$ with $k = 0$ (black) and $k = 0.05$ (gray). We again find that the proportion of defaults increases steeply if the exogenous recovery rate R falls below 1. Furthermore,



(a) Difference in proportion of defaults using $(a,b) = (1,1)$ and $(a,b) = (0.5,7)$

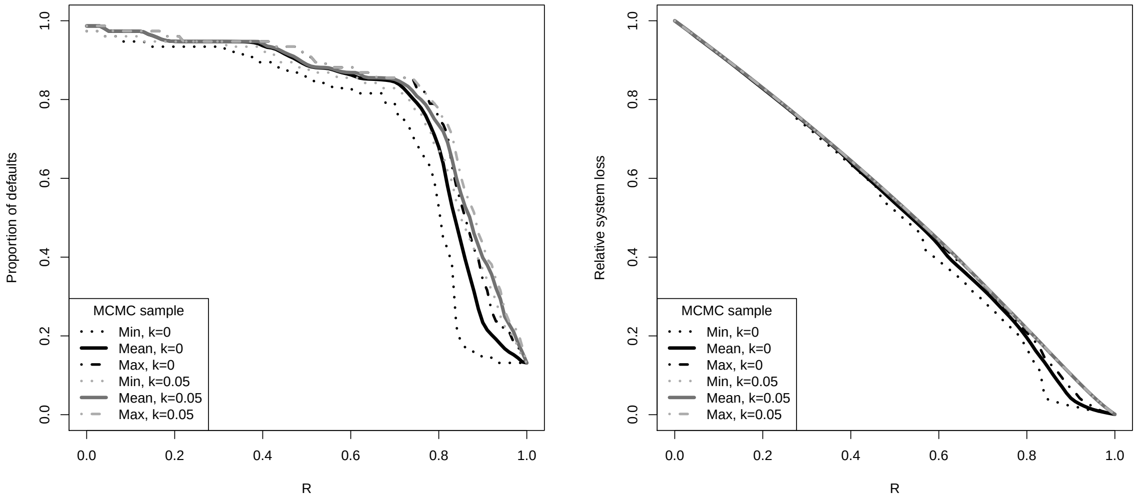


(b) Difference in relative system loss using $(a,b) = (1,1)$ and $(a,b) = (0.5,7)$



(c) Valuation function $\mathbb{V}^{\text{Distress}}$ for $(a,b) = (1,1)$ and for $(a,b) = (0.5,7)$ using $R = \beta = 0.7$ and $k = 0.5$.

Figure 3: Difference between the proportion of defaults $D^{\mathbb{V}^{\text{Distress}}}$ with $(a,b) = (1,1)$ and the proportion of defaults with $(a,b) = (0.5,7)$ (top left) and the difference between the relative system loss $\Lambda^{\mathbb{V}^{\text{Distress}}}$ (top right) with $(a,b) = (1,1)$ and the relative system loss with $(a,b) = (0.5,7)$ as a function of R for different values of the cushion parameter k using $\beta = R$ for one reconstructed network. Note that the figures use different scales for the y-axis. The lower picture shows the corresponding shape of $\mathbb{V}^{\text{Distress}}$ for $(a,b) = (1,1)$ and for $(a,b) = (0.5,7)$ using $k = 0.5$.



(a) Proportion of defaults

(b) Relative system loss

Figure 4: Minimum, mean and maximum of the MCMC sample of 10,000 reconstructed networks of the proportion of defaults (left) and relative system loss (right) as a function of R for $\beta = R$ and $a = b = 1$ with $k = 0$ (black) and $k = 0.05$ (gray).

the difference between the outcome measures for the different samples is very small, since the minimum and maximum values of the outcome measures are quite similar.

It seems that the exogenous recovery rate parameter R is clearly dominating the overall behaviour of the network. Only for large recovery rate parameters R does the choice of the capital cushion parameter k seem to matter in line with our results on only one reconstructed network. In particular if we allow for distress contagion (corresponding to $k = 0.05$) we have in general higher proportions of defaults and a higher relative system loss compared to having only default contagion ($k = 0$). This is in line with the theoretical results established in Theorem 3.3. We see that moving from $k = 0$ to a small but positive k can lead to different, i.e., worse outcome for the network. In our data, however, we observe this outcome only for rather large values of the anticipated exogenous recovery rate R.

5 Conclusion

We have developed a new model for distress and default contagion that can be used in macroprudential stress tests. Its basic form depends on only five model parameters: One parameter determines the start of the contagion process (k), one parameter determines a lower bound for the distress contagion branch (R), one parameter determines an upper bound for the default contagion branch (β) and two model parameters determine the overall shape of the distress contagion mechanism (a, b). We have shown how the new model reduces to some well-known contagion models for special choices of the model parameters.

We have provided ordering results for outcome measures of stress tests that correspond to different parameter choices. These results provide a useful tool to conduct stress tests together with an analytical sensitivity analysis and bounds on outcomes of stress tests.

We have discussed how different choices of the model parameter lead to amplification, containment or spread of losses. In particular, we have shown that bankruptcy costs can amplify losses in two ways. They amplify losses directly since every time a bank defaults additional losses occur. But they can also amplify losses indirectly, because they provide scope for distress contagion and therefore can lead to an earlier start of a contagion mechanism.

Our empirical case study showed that accounting for distress contagion is more important in models with higher recovery rates, since in models with low recovery rates the large losses due to bankruptcy costs dominate the overall outcome for the network.

A Proofs

Theorem 2.5 and Theorem 2.6 are special cases of Theorem 3.4 and hence we only prove Theorem 3.4 later.

Proof of Lemma 2.8. By definition $\mathbb{V}^{\mathrm{RV}} : \mathbb{R} \to [0,1]$ and it is clear that it is non-decreasing and right-continuous and hence an admissible valuation function. Hence, Φ^{RV} is an equity valuation function. □

Proof of Theorem 2.9. We will prove the statements 1. and 3. for general fixed points first, before we prove the results for the greatest fixed points 2. and 4..

1. Let $L^*(x)$ be the a fixed point of Ψ^{RV}, and set

$$E_i^* := A_i^{\mathrm{e}} - x_i + \sum_{j \in \mathcal{M}} L_{ji} \frac{L_j^*(x)}{\bar{L}_j} - \bar{L}_i, \quad \forall i \in \mathcal{N}.$$

To see that E^* is indeed a fixed point of Φ^{RV} we need to show that for all $i \in \mathcal{M}$ it holds that

$$\frac{L_i^*(x)}{\bar{L}_i} = \mathbb{V}^{\mathrm{RV}} \left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i} \right).$$

Let $i \in \mathcal{M}$ with $L_i^*(x) = \bar{L}_i$. Then by the definition of Ψ^{RV} it holds that

$$\begin{aligned} \bar{L}_i \leq A_i^{\mathrm{e}} - x_i + \sum_{j=1}^{N} \Pi_{ji} L_j^*(x) = A_i^{\mathrm{e}} - x_i + \sum_{j \in \mathcal{M}} L_{ji} \frac{L_j^*(x)}{\bar{L}_j} \\ \iff 0 \leq A_i^{\mathrm{e}} - x_i + \sum_{j \in \mathcal{M}} L_{ji} \frac{L_j^*(x)}{\bar{L}_j} - \bar{L}_i = E_i^* \\ \iff \frac{E_i^* + \bar{L}_i}{\bar{L}_i} \geq 1. \end{aligned}$$

Therefore,

$$\mathbb{V}^{\mathrm{RV}} \left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i} \right) = 1 = \frac{L_i^*(x)}{\bar{L}_i} = \frac{\bar{L}_i}{\bar{L}_i}.$$

Let $i \in \mathcal{M}$ with $L_i^*(x) = \beta(A_i^{\mathrm{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji} \frac{L_j^*(x)}{\bar{L}_j})^+$. Then by the definition of Ψ^{RV} it holds that

$$\begin{aligned} A_i^{\mathrm{e}} - x_i + \sum_{j=1}^{N} \Pi_{ji} L_j^*(x) \le \bar{L}_i \\ \iff 0 \ge A_i^{\mathrm{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji} \frac{L_j^*(x)}{\bar{L}_j} - \bar{L}_i = E_i^* \\ \iff \frac{E_i^* + \bar{L}_i}{\bar{L}_i} \le 1. \end{aligned}$$

Therefore,

$$\mathbb{V}^{\mathrm{RV}}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right) = \beta\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right)^+ = \frac{\beta}{\bar{L}_i}(E_i^* + \bar{L}_i)^+ = \frac{L_i^*(x)^+}{\bar{L}_i} = \frac{L_i^*(x)}{\bar{L}_i}.$$

Hence, $E^* = \Phi^{\mathrm{RV}}(E^*)$.

3. Let E^* be a fixed point of Φ^{RV}, hence for all $i \in \mathcal{N}$

$$E_i^* = A_i^{\mathrm{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji} \mathbb{V}^{\mathrm{RV}}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i.$$

Let

$$L_i^*(x) = \mathbb{V}^{\mathrm{RV}}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right)\bar{L}_i$$

for all $i \in \mathcal{M}$ and $L_i^*(x) = 0$ for all $i \in \mathcal{N} \setminus \mathcal{M}$.
Let $i \in \mathcal{M}$. Then,

$$\frac{L_i^*(x)}{\bar{L}_i} = \mathbb{V}^{\mathrm{RV}}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right)$$

and hence

$$E_i^* = A_i^{\mathrm{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji} \frac{L_j^*(x)}{\bar{L}_j} - \bar{L}_i.$$

Since $L_i^*(x) = 0$ for all $i \in \mathcal{N} \setminus \mathcal{M}$ we obtain that for all $i \in \mathcal{N}$

$$E_i^* = A_i^{\mathrm{e}} - x_i + \sum_{j=1}^{N} \Pi_{ji} L_j^*(x) - \bar{L}_i.$$

Let $i \in \mathcal{M}$ with $E_i^* \ge 0$ (and hence $A_i^{\mathrm{e}} - x_i + \sum_{j=1}^{N} \Pi_{ji} L_j^*(x) \ge \bar{L}_i$). Then,

$$\frac{L_i^*(x)}{\bar{L}_i} = \mathbb{V}^{\mathrm{RV}}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right) = 1$$

and hence $L_i^*(x) = \bar{L}_i$.

Let $i \in \mathcal{M}$ with $E_i^* < 0$ (and hence $A_i^{\mathrm{e}} - x_i + \sum_{j=1}^N \Pi_{ji} L_j^*(x) < \bar{L}_i$). Then,

$$\frac{L_i^*(x)}{\bar{L}_i} = \mathbb{V}^{\mathrm{RV}}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right) = \frac{\beta}{\bar{L}_i}(E_i^* + \bar{L}_i)^+ = \frac{\beta}{\bar{L}_i}\left(A_i^{\mathrm{e}} - x_i + \sum_{j \in \mathcal{M}} L_{ji} \frac{L_j^*(x)}{\bar{L}_j}\right)^+$$

and therefore

$$L_i^*(x) = \beta\left(A_i^{\mathrm{e}} - x_i + \sum_{j \in \mathcal{M}} L_{ji} \frac{L_j^*(x)}{\bar{L}_j}\right)^+.$$

Let $i \in \mathcal{N} \setminus \mathcal{M}$. Then, $\bar{L}_i = 0 = L_i^*(x)$ and in particular

$$A_i^{\mathrm{e}} - x_i + \sum_{j=1}^N \Pi_{ji} L_j^*(x) \geq \bar{L}_i = 0.$$

Combining these results we obtain that indeed

$$L_i^*(x) = \begin{cases} \bar{L}_i, & \text{if } A_i^{\mathrm{e}} - x_i + \sum_{j=1}^N \Pi_{ji} L_j^*(x) \geq \bar{L}_i, \\ \beta(A_i^{\mathrm{e}} - x_i + \sum_{j=1}^N \Pi_{ji} L_j^*(x))^+, & \text{else} \end{cases}$$

and hence $L^*(x)$ is a fixed point of Ψ^{RV} defined in (1).

2. It remains to show that E^* is the greatest fixed point of Φ^{RV}.

   Suppose there exists a vector $\tilde{E} \in [0, w - x]$ with $\Phi^{\mathrm{RV}}(\tilde{E}) = \tilde{E}$ and $\tilde{E} > E^*$, i.e., $\tilde{E}_i \geq E_i^*$ for all $i \in \mathcal{N}$ and there exists an $\nu \in \mathcal{N}$ such that $\tilde{E}_\nu > E_\nu^*$.

   We set $\tilde{L}_i := \mathbb{V}^{\mathrm{RV}}\left(\frac{\tilde{E}_i + \bar{L}_i}{\bar{L}_i}\right)\bar{L}_i$ for all $i \in \mathcal{N}$. By 3. this is a fixed point of Ψ^{RV}. We show that $\tilde{L} > L^*$ which is a contradiction to L^* being the greatest fixed point of Ψ^{RV}.

   By definition,

$$\begin{aligned} \tilde{E}_\nu &= A_\nu^{\mathrm{e}} - x_\nu + \sum_{j \in \mathcal{M}} L_{j\nu} \mathbb{V}^{\mathrm{RV}}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right), \\ E_\nu^* &= A_\nu^{\mathrm{e}} - x_\nu + \sum_{j \in \mathcal{M}} L_{j\nu} \mathbb{V}^{\mathrm{RV}}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right), \end{aligned}$$

   and hence

$$0 < \tilde{E}_\nu - E_\nu^* = \sum_{j \in \mathcal{M}} L_{j\nu}\left(\mathbb{V}^{\mathrm{RV}}\left(\frac{\tilde{E}_j + \bar{L}_j}{\bar{L}_j}\right) - \mathbb{V}^{\mathrm{RV}}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right)\right).$$

Since $\mathbb{V}^{\mathrm{RV}}$ non-decreasing and $\tilde{E} > E^*$, $\mathbb{V}^{\mathrm{RV}}\left(\frac{\tilde{E}_j+\bar{L}_j}{\bar{L}_j}\right) - \mathbb{V}^{\mathrm{RV}}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) \geq 0$. Since $0 < \tilde{E}_\nu - E_\nu^*$ there exists an $\mu \in \mathcal{M}$ with

$$L_{\mu\nu}\left(\mathbb{V}^{\mathrm{RV}}\left(\frac{\tilde{E}_\mu + \bar{L}_\mu}{\bar{L}_\mu}\right) - \mathbb{V}^{\mathrm{RV}}\left(\frac{E_\mu^* + \bar{L}_\mu}{\bar{L}_\mu}\right)\right) > 0$$

and hence

$$\mathbb{V}^{\mathrm{RV}}\left(\frac{\tilde{E}_\mu + \bar{L}_\mu}{\bar{L}_\mu}\right) > \mathbb{V}^{\mathrm{RV}}\left(\frac{E_\mu^* + \bar{L}_\mu}{\bar{L}_\mu}\right).$$

Furthermore,

$$\tilde{L}_\mu = \mathbb{V}^{\mathrm{RV}}\left(\frac{\tilde{E}_\mu + \bar{L}_\mu}{\bar{L}_\mu}\right)\bar{L}_\mu > \mathbb{V}^{\mathrm{RV}}\left(\frac{E_\mu^* + \bar{L}_\mu}{\bar{L}_\mu}\right)\bar{L}_\mu = L_\mu^*$$

and

$$\tilde{L}_i = \mathbb{V}^{\mathrm{RV}}\left(\frac{\tilde{E}_i + \bar{L}_i}{\bar{L}_i}\right)\bar{L}_i \geq \mathbb{V}^{\mathrm{RV}}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right)\bar{L}_i = L_i^*$$

for all $i \in \mathcal{N}$. Hence, $\tilde{L}$ is a fixed point of Ψ^{RV} by 3. and $\tilde{L} > L^*$ which is a contradiction to L^* being the greatest fixed point of Ψ^{RV}. Hence, E^* is indeed the greatest fixed point of Φ^{RV}.

4. Let E^* be the greatest fixed point of Φ^{RV} and let $L_i^*(x) := \mathbb{V}^{\mathrm{RV}}\left(\frac{E_i^*+\bar{L}_i}{\bar{L}_i}\right)\bar{L}_i$ for all $i \in \mathcal{N}$. We show that $L^*(x)$ is the greatest fixed point of Ψ^{RV}. From 2. we know already that $L^*(x)$ is a fixed point of Ψ^{RV}. We show that it is the greatest fixed point again by proof by contradiction. Suppose there exists a vector $\tilde{L}$ with $\Psi^{\mathrm{RV}}(\tilde{L}) = \tilde{L}$ and $\tilde{L} > L^*$, i.e., $\tilde{L}_i \geq L_i^*$ for all $i \in \mathcal{N}$ and there exists a $\nu \in \mathcal{N}$ for which $\tilde{L}_\nu > L_\nu^*$. Note that this implies $\nu \in \mathcal{M}$.

   We define

$$\tilde{E}_i := A_i^{\mathrm{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji}\frac{\tilde{L}_j}{\bar{L}_j} - \bar{L}_i \quad \forall i \in \mathcal{N}.$$

   By 1. this is a fixed point of Φ^{RV}. Since,

$$E_i^* := A_i^{\mathrm{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji}\frac{L_j^*(x)}{\bar{L}_j} - \bar{L}_i \quad \forall i \in \mathcal{N}.$$

   Then,

$$\tilde{E}_i - E_i^* = \sum_{j\in\mathcal{M}} \frac{L_{ji}}{\bar{L}_j}(\tilde{L}_j - L_j^*(x)) \geq 0 \quad \forall i \in \mathcal{N}.$$

   Since by assumption there exists a $\nu \in \mathcal{M}$ with $\tilde{L}_\nu > L_\nu^*$ there also exists a $\mu \in \mathcal{N}$

such that $L_{\nu\mu} > 0$ and hence $\tilde{E}_\mu > E^*_\mu$. This is a contradiction to E^* being the greatest fixed point of Φ^{RV} and hence such a $\tilde{L}$ does not exist.

□

Proof of Theorem 2.11. 1. Let $E^{(0),\nu} = M - x$ and define recursively $E^{(\kappa+1),\nu} = \Phi^\nu(E^{(\kappa),\nu})$ $\forall \kappa \in \mathbb{N}_0, \nu \in \{A, B\}$. By Theorem 2.6 we know that the limits $E^{*,\nu} = \lim_{\kappa\to\infty} E^{(\kappa),\nu}$, $\nu \in \{A, B\}$, exist and that they are the greatest fixed points.

We prove the statement by induction with respect to κ. Let $\kappa = 0$. Then $E^{(0),A} = M - x = E^{(0),B}$.

Suppose $E^{(\kappa),A} \geq E^{(\kappa),B}$ for a $\kappa \in \mathbb{N}_0$. Then for all $i \in \mathcal{N}$ it holds that

$$\begin{aligned} E_i^{(\kappa+1),A} := \Phi^A(E^{(\kappa),A})_i &= A_i^{\mathrm{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji} \mathbb{V}^A \left(\frac{E_j^{(\kappa),A} - \bar{L}_j}{\bar{L}_j} \right) - \bar{L}_i \\ &\geq A_i^{\mathrm{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji} \mathbb{V}^B \left(\frac{E_j^{(\kappa),A} - \bar{L}_j}{\bar{L}_j} \right) - \bar{L}_i \\ &\geq A_i^{\mathrm{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji} \mathbb{V}^B \left(\frac{E_j^{(\kappa),B} - \bar{L}_j}{\bar{L}_j} \right) - \bar{L}_i \\ &= \Phi^B(E^{(\kappa),B}) = E^{(\kappa+1),B}, \end{aligned}$$

where the last inequality follows from the induction hypothesis and the fact that $\mathbb{V}^B$ is an admissible valuation function and hence non-decreasing.

Hence, $E^{(\kappa),A} \geq E^{(\kappa),B}$ $\forall \kappa \in \mathbb{N}_0$ and hence

$$E^{*,A} = \lim_{\kappa\to\infty} E^{(\kappa),A} \geq E^{(\kappa),B} = \lim_{\kappa\to\infty} E^{*,B}.$$

2. The result is an immediate consequence of part 1 of this theorem.

3. The result is an immediate consequence of part 1 of this theorem.

□

Proof of Proposition 3.1. Since $F(\cdot; a, b)$ is the cumulative distribution function of the Beta distribution it is non-decreasing and continuous. Hence, $y \mapsto 1-(1-R)F\left(\frac{1+k-y}{k}; a, b\right)$ is continuous as well. Furthermore, it is non-decreasing for all $y \in [1, 1+k)$. One can see directly from the definition of $\mathbb{V}^{\mathrm{Distress}}$ that it is right-continuous on the other parts of the domain as well and non-decreasing. Hence, it is an admissible valuation function. □

Proof of Theorem 3.3. 1. Let $0 \leq k_1 \leq k_2$.

(a) We consider four cases.
First, let $y \in (-\infty, 1)$.

$$\mathbb{V}^{\mathrm{Distress}}(y; k_1, \beta, R, a, b) = \beta y^+ = \mathbb{V}^{\mathrm{Distress}}(y; k_2, \beta, R, a, b).$$

Second, let $y \in [1 + k_2, \infty)$. Then,

$$\mathbb{V}^{\mathrm{Distress}}(y; k_1, \beta, R, a, b) = 1 = \mathbb{V}^{\mathrm{Distress}}(y; k_2, \beta, R, a, b).$$

Third, let $y \in [1+k_1, 1+k_2)$. Then by definition

$$\mathbb{V}^{\text{Distress}}(y; k_1, \beta, R, a, b) = 1 \geq \mathbb{V}^{\text{Distress}}(y; k_2, \beta, R, a, b).$$

Fourth, let $y \in [1, 1+k_1)$. Then,

$$\begin{aligned}\mathbb{V}^{\text{Distress}}(y; k_1, \beta, R, a, b) &= 1 - (1-R)F\left(\frac{1+k_1-y}{k_1}; a, b\right),\\ \mathbb{V}^{\text{Distress}}(y; k_2, \beta, R, a, b) &= 1 - (1-R)F\left(\frac{1+k_2-y}{k_2}; a, b\right).\end{aligned}$$

Define, $G : [y-1, \infty) \to [R, 1]$, where

$$G(k) := 1 - (1-R)F\left(\frac{1+k-y}{k}; a, b\right).$$

Then, G is differentiable and

$$\begin{aligned}G'(k) &= -(1-R)F'\left(\frac{1+k-y}{k}; a, b\right)\frac{k-(1+k-y)}{k^2}\\ &= -\underbrace{(1-R)}_{\geq 0}\underbrace{f\left(\frac{1+k-y}{k}; a, b\right)}_{\geq 0}\underbrace{\frac{y-1}{k^2}}_{\geq 0} \leq 0.\end{aligned}$$

Hence, G is decreasing in k and therefore

$$\begin{aligned}\mathbb{V}^{\text{Distress}}(y; k_1, \beta, R, a, b) &= 1 - (1-R)F\left(\frac{1+k_1-y}{k_1}; a, b\right)\\ &\geq 1 - (1-R)F\left(\frac{1+k_2-y}{k_2}; a, b\right) = \mathbb{V}^{\text{Distress}}(y; k_2, \beta, R, a, b).\end{aligned}$$

(b) The results follow directly from Theorem 2.11 and part a).

2. The statement follows immediately from the definition of $\mathbb{V}^{\text{Distress}}$ and Theorem 2.11.

$\square$

In order to prove Theorem 3.4 we need the following Lemma.

Lemma A.1. *Let (L, L^e, A^e) be a financial system with shock vector $x \in [0, A^{(e)}]$ and let $\mathcal{M} := \{j \in \mathcal{N} \mid \bar{L}_j > 0\}$. Let $\Phi^{General} : \mathcal{E}(x) \to \mathcal{E}(x)$ be as in (17) and $\mathcal{E}(x) = [-\bar{L}, w-x]$. Then $\Phi^{General}$ has the following properties.*

1. *$\Phi^{General}$ is bounded from above by $w-x$ and bounded from below by $-\bar{L}$, i.e., for all $E \in \mathcal{E}(x)$ we have*
$$\Phi^{General}(E) \in [-\bar{L}, w-x].$$

2. *$\Phi^{General}$ is non-decreasing, i.e., for all $E, \tilde{E} \in \mathcal{E}(x)$ with $\tilde{E} \leq E$ it holds that $\Phi^{General}(\tilde{E}) \leq \Phi^{General}(E)$.*

Proof of Lemma A.1. 1. Since by definition $\mathbb{V}^{General}: \mathbb{R} \to [0,1]$ and the liabilities matrix $L \geq 0$ we obtain from the definition of Φ^{General} that for all $E \in \mathcal{E}(x)$

$$\begin{aligned}\Phi_i^{\text{General}}(E) &= A_i^{\text{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}_j^{\text{General}}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \leq A_i^{\text{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji} - \bar{L}_i \\ &= w_i - x_i\end{aligned}$$

and since $x \in [0, A^{\text{e}}]$

$$\Phi_i^{\text{General}}(E) = A_i^{\text{e}} - x_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}_j^{\text{General}}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \geq A_i^{\text{e}} - x_i - \bar{L}_i \geq -\bar{L}_i$$

for all $i \in \mathcal{N}$.

2. The result follows immediately from $\mathbb{V}^{\text{General}}$ being nondecreasing and $E \mapsto \frac{E+l}{l}$ being nondecreasing for all $l \in \mathbb{R}^N$ and $L \geq 0$.

□

Proof of Theorem 3.4. 1. By Lemma A.1 Φ^{General} is an non-decreasing function on $\mathcal{E}(x)$ to $\mathcal{E}(x)$. Furthermore, $\mathcal{E}(x)$ is a complete lattice with respect to the component wise partial order $\leq$.

Hence by Tarksi's fixed point theorem, see (Tarski, 1955, Theorem 1), the set of all fixed points of Φ^{General} is not empty and is a complete lattice with respect to $\leq$. Hence, there exists fixed points E_{**}, E^{**} of Φ^{General} such that for all fixed points E of Φ^{General}: $E_{**} \leq E \leq E^{**}$.

2. It remains to show that the fixed point iteration yields the greatest fixed point. We set $E^{(0)} := w - x$ and recursively $E^{(\kappa+1)} = \Phi^{\text{General}}(E^{(\kappa)})$ for all $\kappa \in \mathbb{N}_0$.

From Lemma A.1 Part 1. we have that $E^{(1)} \leq E^{(0)} = w - x$. We prove by induction that $E^{(\kappa+1)} \leq E^{(\kappa)}$ for all $\kappa \in \mathbb{N}_0$. Let $E^{(\kappa+1)} \leq E^{(\kappa)}$ for a $\kappa \in \mathbb{N}_0$. We show that $E^{(\kappa+2)} \leq E^{(\kappa+1)}$. By the definition of the sequence $(E^{(\kappa)})_{\kappa\in\mathbb{N}_0}$ it holds that $E^{(\kappa+2)} = \Phi^{\text{General}}(E^{(\kappa+1)})$ and $E^{(\kappa+1)} = \Phi^{\text{General}}(E^{(\kappa)})$. Furthermore,

$$E^{(\kappa+2)} = \Phi^{\text{General}}(E^{(\kappa+1)}) \leq \Phi^{\text{General}}(E^{(\kappa)}) = E^{(\kappa+1)},$$

since by the induction hypothesis $E^{(\kappa+1)} \leq E^{(\kappa)}$ and Φ^{General} is non-decreasing as shown in Lemma A.1 Part2.
Since by Lemma A.1 Part 1. the sequence $(E^{(\kappa)})$ is also bounded from below by $-\bar{L}$, there exists a monotone limit $\hat{E} := \lim_{\kappa\to\infty} E^{(\kappa)}$.

It remains to show that indeed $\hat{E} = E^{**}$. Note that $\hat{E}$ is a fixed point of Φ^{General}, since

$$\Phi^{\text{General}}(\hat{E}) = \Phi^{\text{General}}(\lim_{\kappa\to\infty} E^{(\kappa)}) = \lim_{\kappa\to\infty} \Phi^{\text{General}}(E^{(\kappa)}) = \lim_{\kappa\to\infty} E^{(\kappa+1)} = \hat{E}.$$

Here the second equality follows from the fact that $\Phi^{General}$ is right-continuous and $(E^{(\kappa)})$ is non-increasing and the third equality follows from the recursive definition of the $E^{(\kappa)}$.

Next we show that $E^{(k)} \geq E^{**}$ for all $\kappa \in \mathbb{N}_0$ by induction. Obviously, $E^{(0)} = w - x \geq E^{**}$. Suppose $E^{(\kappa)} \geq E^{**}$ for a fixed $\kappa \in \mathbb{N}_0$. Then,

$$E^{\kappa+1} = \Phi^{\text{General}}(E^{\kappa}) \geq \Phi^{\text{General}}(E^{**}) = E^{**},$$

where the first equality follows from the definition of the sequence $(E^{(\kappa)})_{k\in\mathbb{N}_0}$, the second equality follows from the monotonicity of Φ^{General} (Lemma A.1 Part 2.) and the induction hypothesis and the third equality holds because E^{**} is a fixed point of Φ^{General}.

Hence,

$$\hat{E} = \lim_{\kappa\to\infty} E^{(\kappa)} \geq E^{**}$$

and since $\hat{E} = \Phi^{\text{General}}(\hat{E})$, indeed $\hat{E} = E^{**}$.

□

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