Tail Risk Targeting:
Target VaR and CVaR Strategies

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Abstract

We present dynamic trading strategies that target a predefined level of risk measured by volatility, Value-at-Risk (VaR) or Conditional-Value-at-Risk (CVaR). Recent studies have shown that volatility targeting increases the risk-adjusted performance and heightens utility gains for mean-variance investors. We find that downside risk targeting outperforms volatility targeting in terms of a higher Sharpe Ratio, better drawdown protection and higher utility gains for mean-variance, CRRA and loss-averse investors. In particular, a loss-averse investor is not willing to pay a positive fee to switch from a static portfolio to a volatility managed strategy, whereas this investor would pay a fee of 18% per year to have access to the downside risk managed strategy. We also find that the performance of risk targeting can further be enhanced by switching between volatility and CVaR targeting based on estimates of whether the market will be in a bull or bear regime.

Keywords: Volatility; Value at Risk; Conditional Value at Risk; Risk targeting; Extreme Value Theory; Dynamic trading strategies

JEL classification: C53; G11; G17

1 Introduction

During financial crises, due to an increase of correlations, diversification fails as a risk management tool. Especially when financial markets exhibit huge downturn periods correlations significantly increase and thus lower the benefit of diversification just when it is most needed (Ang and Bekaert, 2002, Butler and Joaquin, 2002, Chabi-Yo et al., 2018, Guidolin and Timmermann, 2008, Karolyi and Stulz, 1996, Patton, 2004, Poon et al., 2004). Investors typically

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overestimate the benefits of diversification in bear markets and underestimate return potentials in bull markets. This leads to too high equity exposures in bear markets whereas the equity allocation is too low in bull markets (Ang and Chen, 2002, Longin and Solnik, 2001).

For that reason more tactical tools, such as volatility targeting, have become popular in the financial industry and academic literature.[1] The aim of volatility targeting is to build a portfolio consisting of a risky and a riskless asset, that has a (predetermined) constant level of portfolio volatility over time. In order to achieve this constant level of portfolio volatility, the target volatility strategy allocates money between the risky and the riskless asset, based on a forecast of the risky asset's volatility: if the risky asset's volatility is expected to be high, the weight of the risky asset is decreased and vice versa (see Bollerslev et al. (2018) for example). The economic value of volatility timing in terms of significant utility gains of investors who allocate their money among several risky assets has been examined extensively by Fleming et al. (2001), Fleming et al. (2003), Han (2005), Kirby and Ostdiek (2012) and Taylor (2014).[2] Marquering and Verbeek (2004), Bollerslev et al. (2018) and Moreira and Muir (2017) examine the economic value of volatility timing in a single asset scenario and find vast utility gains of volatility timing and that volatility timing is superior to return timing. Moreira and Muir (2019) assess the economic value of volatility timing for long-horizon investors and find that even long-horizon investors should time short-term volatility, supporting the finding of Benartzi and Thaler (1995) that long-horizon investors have short evaluation periods. Busse (1999) examines the impact of volatility timing for the institutional fund industry and concludes that "funds that reduce systematic risk when conditional market volatility is high earn higher risk-adjusted returns" and that funds who time volatility the most are associated with higher Sharpe Ratios (Busse, 1999, p. 1010 and 1027).

Barroso and Santa-Clara (2015), Moreira and Muir (2017) and Barroso and Maio (2016) examine volatility targeting by managing the volatility of different portfolio strategies, and find

[1] For academic research on target volatility strategies see Hocquard et al. (2013), Benson et al. (2014), Barroso and Santa-Clara (2015), Bollerslev et al. (2018), Moreira and Muir (2017), Barroso and Maio (2016), Moreira and Muir (2019), Cederburg et al. (2019) among others. For institutional research see e.g. Banerjee et al. (2016).

[2] Fleming et al. (2001), Fleming et al. (2003), Han (2005), Kirby and Ostdiek (2012) and Taylor (2014) assess the economic value of volatility timing in a multivariate setting. This approach is slightly different to volatility targeting but demonstrates that investment decisions relying on volatility (or more precisely covariances) solely work well in empirical applications.

significant improvements in risk-adjusted performance.[3] This highlights a nice characteristic of volatility targeting. Volatility targeting can be used for any underlying investment strategy, i.e. volatility targeting can be separated from the fund manager's asset allocation choice, where the asset allocation is chosen first and is then overlayed by a volatility targeting strategy as market timing tool (Hocquard et al., 2013, Zakamulin, 2015).

So far, studies on risk targeting focus on volatility as a risk measure: the weight of the risky asset is a function of the risky asset's volatility. However, since asset returns are typically skewed, fat-tailed and non-normally distributed, the choice of volatility as a measure of market risk is not appropriate (see Szegö (2002), Poon and Granger (2003), Kuester et al. (2006, p. 56) and Bali et al. (2009)). Xiong and Idzorek (2011), Guidolin and Timmermann (2008), Jondeau and Rockinger (2006), Jondeau and Rockinger (2012) and Ghysels et al. (2016) examine the impact of skewness and fat-tails on the asset allocation and show that incorporating higher moments, as done by adequately measuring downside risk, is beneficial compared to mean-variance optimization. Further, most investors have preferences for higher skewness and lower kurtosis (see Kraus and Litzenberger (1976), Scott and Horvath (1980), Guidolin and Timmermann (2008) among others). Kelly and Jiang (2014) find that an increase of tail risk predicts higher kurtosis and lower (or more negative) skewness. Hence, investors who dislike negative skewness and high kurtosis should better manage a portfolio's downside risk. Similarly, investors are more concerned about downside risk instead of volatility (Bollerslev et al., 2015, Kelly and Jiang, 2014, Lee and Rao, 1988). Most investors weight losses higher than gains, which implies that avoiding huge losses is crucial to increase a loss-averse investor's utility (Aït-Sahalia and Brandt, 2001, Ang et al., 2005, 2006a, Benartzi and Thaler, 1995). Further, avoiding crashes is important since investors are crash averse and have a demand for portfolio insurance, especially in times of extremely negative returns (Bollerslev and Todorov, 2011,

[3] Barroso and Santa-Clara (2015) successfully use a target volatility strategy to manage the risk of the momentum portfolio and show that targeting a constant level of volatility extremely reduces the drawdowns of the momentum portfolio, the so called "momentum crashes", and translates into a superior risk-adjusted performance (see also Daniel and Moskowitz (2016)). Moreira and Muir (2017) use a volatility timing strategy for different factor portfolios and show that the risk-adjusted performance of the volatility managed portfolios is superior to the non-managed portfolios. This finding is most pronounced for the momentum strategy. Barroso and Maio (2016) use volatility targeting for several factor strategies and find huge improvements of the volatility targeting strategies for all strategies except for the size factor. The best results are found for the momentum strategy and the "Betting against Beta" strategy of Frazzini and Pedersen (2014).

Chabi-Yo et al., 2018). Therefore, timing an asset's downside risk instead of volatility fits better to most investors' preferences. Furthermore, Benson et al. (2014) find that the superior performance of volatility targeting does result from mitigating drawdowns. Similarly, Harvey et al. (2018) find that volatility targeting reduces the likelihood of extreme (negative) returns which is an important source of the outperformance of volatility targeting. Consequently, if drawdown protection is a main driver of the risk-adjusted performance of risk targeting, choosing the risky asset's weight based on a forecast of the risky asset's downside risk should be more successful in mitigating drawdowns and hence should result in a superior risk-adjusted performance.

In this paper we show how the idea of volatility targeting can be extended to targeting a constant level of tail risk, measured by Value at Risk (VaR) or Conditional Value at Risk (CVaR). These strategies aim to keep the VaR or CVaR of the portfolio constant over time by shifting money between the risky and the riskless asset, based on a forecast of the risky asset's tail risk. This approach translates into a strategy that increases the weight of the risky asset if the risky asset's tail risk is expected to be low and vice versa. Basak and Shapiro (2001), Alexander and Baptista (2004), Cuoco et al. (2008), Agarwal and Naik (2004) and Wang et al. (2012) demonstrate the benefits of managing downside risk instead of volatility in an asset allocation context. To compare the economic of volatility and downside targeting we follow the literature and assess the economic value of risk targeting for a mean-variance investor. Further, to incorporate preferences for higher moments like skewness and kurtosis we asses the economic value for a CRRA investor (Jondeau and Rockinger, 2012). Finally, since most investors weight losses higher than gains we assess the economic value of risk targeting for loss averse investors. We find that risk targeting strategies deliver high utility gains compared to a static portfolio allocation. This is in line with Cuoco et al. (2008) who find that frequently reallocating portfolio weights based on estimates of downside risk is superior to static portfolio allocations. In particular, an investor should manage portfolio risk based on a conditional risk model using a dynamic volatility model like the GARCH(1,1) or EWMA model. Simple risk estimation models like Historical Sample Deviation, as used in Moreira and Muir (2017), Barroso and Maio (2016) and Barroso and Santa-Clara (2015), or Historical Simulation typically fail to significantly increase an investor's utility and produce lower Sharpe Ratios than the conditional approaches. Moreover, we find that the economic value of CVaR timing is significantly higher than the economic value of volatility timing, especially when investors are highly risk or loss averse and in times of bear markets. Further, even mean-variance investors should manage CVaR instead of volatility. For example, we find that a mean-variance investor is willing to pay a fee of about 0.8% per year to have access to a volatility targeting strategy and even 4.253% for the CVaR managed strategy. In contrast, a loss averse investor is not willing to pay a positive fee for volatility targeting, but the same investor would pay up to 18% per year to have access to the CVaR targeting strategy.

Since estimating downside risk is more sophisticated than estimating volatility, we additionally show how the target VaR and target CVaR strategies can be approximated by a target volatility strategy. Further, we demonstrate how the accuracy of the target volatility, target VaR and target CVaR strategies can be backtested. For assessing the accuracy of volatility targeting we resort to the approaches of Diebold and Mariano (1995), White (2000), Hansen (2005), Romano and Wolf (2005), Hansen et al. (2003), Hansen et al. (2011), Hsu et al. (2010), Barras et al. (2010) and Bajgrowicz and Scaillet (2012) that test for equal or superior predictive ability. For assessing the accuracy of VaR and CVaR targeting we use the VaR backtest of Christoffersen (1998) and the CVaR backtests of McNeil and Frey (2000) and Embrechts et al. (2005). With these backtests in hand, we assess the accuracy of approximating a target VaR or target CVaR strategy by a target volatility strategy, i.e. we answer the question if controlling volatility is sufficient when downside risk is targeted. We find that for investors who are interested in targeting a constant VaR or CVaR over time, controlling volatility is not sufficient. Similarly, for targeting a constant level of volatility an investor should manage volatility directly instead of downside risk. Generally, risk should be managed by a dynamic risk model, based on a dynamic volatility model like EWMA or GARCH(1,1). In contrast, using a static risk model like Historical Standard Deviation or Historical Simulation fails to target the portfolio risk at a constant level, achieves a worse risk-adjusted performance and lower utility gains. In line with Bollerslev et al. (2018) we find a positive relation between forecasting accuracy, and hence a more constant portfolio risk, and risk-adjusted performance and utility gains.

Finally, we use strategies that switch between volatility and CVaR targeting, based on an estimate of the market regime. If the market is expected to be in a down-market CVaR targeting is used whereas the portfolio's risk is managed by volatility if an up-market is expected. To determine up- and down-markets we use technical trading rules (Bajgrowicz and Scaillet, 2012, Moskowitz et al., 2012) and the asset's expected volatility. We find that these switching strategies further increase the risk-adjusted performance and utility gains of risk targeting. For example, a mean-variance investor is willing to pay 5.667% per year to switch to a strategy that dynamically switches between volatility and CVaR targeting. Further, a loss-averse investor is even willing to pay 21.82% per year to have access to this strategy. Over the last 88 years a 100\$ investment in the market would result into a portfolio value of $357{,}591$\$. By using the volatility targeting strategy this amount can be raised to $4{,}420{,}160$\$. However, by switching between volatility and CVaR targeting, the wealth would even increase to $28{,}313{,}411$\$.

This paper is structured as follows. In Section 2 we present the target volatility framework and review the literature on volatility targeting. Section 3 presents the target VaR and CVaR strategies and shows how VaR and CVaR are estimated. Furthermore, we show how the target VaR and CVaR strategies can be approximated by a target volatility strategy. Section 4 demonstrates how the accuracy of volatility, VaR and CVaR targeting can be tested. Section 5 shows the empirical results and Section 6 concludes the paper.

2 Target Volatility Strategy

Throughout the paper, we consider a risky asset, e.g. an equity index, with price process $\{S_t\}_{t\in\{0,\dots,T\}}$ over the period $[0,T], T \in \mathbb{N}$ and we define the return of the risky asset over the period $[t-1,t]$, representing one day, as

$$R_t := \frac{S_t}{S_{t-1}} - 1. \tag{1}$$

Further, we consider a riskless asset with returns $\{R_t^f\}_{t\in\{0,\dots T\}}$. R_t^f describes the return of the riskless asset over the period $[t-1,t]$ and we assume that R_t^f is known at time $t-1$.[4] The day

[4]More formally, we assume that R_t^f is measurable with respect to $\mathcal{F}_{t-1}$, where $\mathcal{F}_{t-1}$ is the σ-algebra generated by the variables that are observed up to time $t-1$ (Hansen and Lunde, 2005, p. 875). Hence, $\mathcal{F}_{t-1}$ contains all relevant information available at time $t-1$.

t return R_t^P of the portfolio that invests a weight w_t in the risky asset and $1 - w_t$ in the riskless asset is then given by

$$R_t^P := w_t \cdot R_t + (1 - w_t) \cdot R_t^f. \tag{2}$$

The aim of the target volatility strategy is to determine the weight w_t for each day t such that the portfolio volatility is constant over time and equals a predefined value. We denote the *portfolio volatility*, i.e. the (conditional) standard deviation of the portfolio return R_t^P conditioned on the information $\mathcal{F}_{t-1}$ available at time $t-1$, by $\sigma_t^P := \sqrt{\operatorname{var}(R_t^P \mid \mathcal{F}_{t-1})}$, where the (conditional) portfolio variance is denoted by $\operatorname{var}\big(R_t^P \mid \mathcal{F}_{t-1}\big)$ (see Hansen and Lunde (2005, p. 875)). In order to achieve a constant volatility level σ^{target} for σ_t^P over time, the weight of the risky asset has to be chosen as

$$w_t = \frac{\sigma^{\text{target}}}{\sigma_t}, \tag{3}$$

where σ^{target} is the desired volatility target and $\sigma_t := \sqrt{\operatorname{var}(R_t \mid \mathcal{F}_{t-1})}$ is the (conditional) volatility of the risky asset at day t (see Bollerslev et al. (2018, p. 2757) for example). By construction, the day t weight w_t is known at day $t-1$ since σ_t is $\mathcal{F}_{t-1}$-measurable. The use of volatility targeting has several advantages which are summarized in Appendix A.

To implement a target volatility strategy, the volatility of the risky asset σ_t in Equation (3) is needed, which is unobservable in practice. Therefore, the volatility for day t has to be forecasted, based on the information available at time $t-1$.[5] We denote this (one-step ahead) forecast by $\hat{\sigma}_t$. Based on this volatility forecast the weight w_t of the risky asset is given by

$$w_t = \frac{\sigma^{\text{target}}}{\hat{\sigma}_t}. \tag{4}$$

Consequently, the success of the target volatility strategy strongly depends on the quality of the volatility forecast.[6] Benson et al. (2014) show that a target volatility strategy with perfect foresight, i.e. a strategy that knows the next period's volatility, outperforms the benchmark by more than 10% per year with a lower volatility and is successful in delivering a constant volatility

[5]See Bollerslev et al. (1992), Taylor (2005, Sec. 2), Poon and Granger (2003) and Hansen and Lunde (2005) for surveys on volatility forecasting.

[6]Obviously, the volatility of the target volatility strategy is only constant over time and equals σ^{target} if and only if the volatility forecast $\hat{\sigma}_t$ equals the true (ex-post) realized volatility σ_t on each day t.

indicated by an almost zero volatility of volatility (see also Bollerslev et al. (2018)). Marquering and Verbeek (2004) find that periods where volatility can be predicted well correspond to periods where volatility timing generates high utility gains. Similarly, Moreira and Muir (2017) and Bollerslev et al. (2018) show that using advanced volatility forecasting models in a volatility targeting strategy improves the risk-adjusted performance and heightens utility gains compared to simple and less accurate forecasting models. Taylor (2014) and Fleming et al. (2003) find a similar observation in a multivariate volatility timing strategy. In particular, Bollerslev et al. (2018) find a positive relation between forecasting accuracy, and hence a constant portfolio volatility, and risk-adjusted performance and utility gains.[7] Moreover, Dopfel and Ramkumar (2013, p. 31) find that high volatility regimes concurrently occur with negative returns and significantly lower Sharpe Ratios compared to regimes with normal volatility, but this result reverses when returns of regimes with a high or normal volatility in the *previous* period are compared.[8] Similarly, Dachraoui (2018) finds a negative relation between σ_t and R_t but no relation between σ_{t-1} and R_t. This result highlights that accurately *forecasting* future volatility is crucial when volatility should be managed, since simply measuring today's volatility is not sufficient to determine tomorrow's weight of the risky asset. Therefore, an accurate forecasting model is important for the target volatility strategy to achieve an enhanced risk-return profile. For that reason, we present methods to test the accuracy of different target volatility strategies in Section 4.1.

For practical implementations, simple forecasting methods, like Historical Sample Deviation (HSD) or Exponential Weighted Moving Average (EWMA) proposed by the RiskMetrics™ group, can be used. Nevertheless, more advanced – and potentially more accurate – methods, like the GARCH(1,1) model proposed by Bollerslev (1986), could be interesting e.g. for fund managers. In this paper we use these three volatility models, where the HSD statically measures today's volatility used as a forecast for tomorrow's volatility and hence does not con-

[7] Similarly, in a cross-sectional setting, Baltussen et al. (2018) find that assets with a high volatility of volatility (vol-of-vol) underperform assets with a more constant volatility. Further, higher vol-of-vol assets also exhibit higher downside risk. This especially holds during down markets when high vol-of-vol assets underperform low vol-of-vol assets by 0.83% per month.

[8] Interestingly, although returns of periods following a high volatility period are higher than returns following a low volatility period, Sharpe Ratios are slightly higher for periods following a low volatility period. Thus, the higher volatility is not compensated by an adequate higher return (Moreira and Muir, 2017).

sider the aforementioned issue of forecasting next day's volatility. In contrast, the EWMA and GARCH(1,1) models dynamically forecast next day's volatility and thus should result in a more constant portfolio volatility and a higher risk-adjusted performance.[9] All three models have several advantages and disadvantages, therefore a possible extension could be to combine several forecasting model as suggested by Taylor (2014).

The day t volatility using HSD is estimated by

$$\hat{\sigma}_t = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(R_{t-i} - \hat{\mu}_t)^2}, \tag{5}$$

where $\hat{\mu}_t = \frac{1}{m}\sum_{i=1}^{m} R_{t-i}$ is an estimate of the expected mean return. For the EWMA and the GARCH(1,1) models it is assumed that the day t return of the risky asset can be described by

$$R_t = \sigma_t \cdot Z_t, \tag{6}$$

where Z_t is iid with mean zero, variance one and cumulative distribution function F_Z (see McNeil and Frey (2000, p. 275)). As usual, when working with daily returns we assume that the expected mean return is zero. This is a quite weak assumption, since (absolute) daily returns are close to zero. Further, an accurate estimate of the expected daily return is not feasible (see Merton (1980), Fleming et al. (2001, p. 332), Fleming et al. (2003, p. 476), Kirby and Ostdiek (2012) among others). Christoffersen and Diebold (2006) show that the conditional mean is not forecastable, since returns R_t conditioned on $\mathcal{F}_{t-1}$ do not fluctuate over time. Further, Hansen and Lunde (2005) compare different mean specifications and find that all lead to an almost identical performance of the volatility models.

For the EWMA model the volatility forecast $\hat{\sigma}_t$ is given by

$$\hat{\sigma}_t = \sqrt{(1-\lambda)\cdot R_{t-1}^2 + \lambda \cdot \hat{\sigma}_{t-1}^2}, \tag{7}$$

where λ is typically chosen as 0.94 when working with daily returns (Christoffersen, 2012, p. 70). The advantage of the EWMA model is that no parameters have to be estimated, what

[9]In the EWMA and GARCH(1,1) model past negative and positive returns have the same impact on future volatility. A well-known stylized fact, the so-called *leverage effect*, states that past negative returns influence future volatility more than past positive returns. We also used the GJR-GARCH model of Glosten et al. (1993) and the EGARCH model of Nelson (1991) that account for the leverage effect, but results were quite similar to the results of the EWMA and GARCH(1,1) model. This is in line with Taylor (2014) who comes to the same conclusion in a multivariate setting. See also Poon and Granger (2003) and Hansen and Lunde (2005) for a comparison of different volatility forecasting models.

makes this model interesting for practical applications (Halbleib and Pohlmeier, 2012). However, frequently re-estimating the model parameters as in the GARCH(1,1) model should also result in a more accurate volatility forecast. The volatility forecast in the GARCH(1,1) model is given by

$$\hat{\sigma}_t = \sqrt{\hat{\omega} + \hat{\alpha}\, R_{t-1}^2 + \hat{\beta}\, \hat{\sigma}_{t-1}^2}, \tag{8}$$

where the parameters $\hat{\omega}, \hat{\alpha}, \hat{\beta}$ are estimated via Quasi Maximum Likelihood, i.e. we assume that the innovations Z_t in Equation (6) are iid standard normally distributed.[10]

Another field of current research that could be of high interest in the context of target volatility strategies is forecasting volatility based on the theory of *realized volatility* that measures volatility using high-frequency-data (see Andersen et al. (2001) for example). Due to its simplicity the Heterogeneous Autoregressive model of Realized Volatility (HAR-RV) proposed by Corsi (2009) fits well to the target volatility framework (see Taylor (2014) who finds good results of the HAR model in a multivariate setting). Bollerslev et al. (2018) extend the HAR-RV model in several directions and use these modifications in a volatility targeting framework. The authors find good results in using these models compared to models that rely on daily data. For example, an investor using a volatility targeting strategy would pay an annualized fee of 0.46% to switch from a simple strategy to a high frequency data based strategy. This again demonstrates that the quality of a volatility targeting strategy strongly depends on the accuracy of the inherent volatility forecasting model. Similarly, Fleming et al. (2003) examine the economic value of high-frequency-data based estimates of daily volatility. They find that using high-frequency-data based volatility measures instead of daily data based measures can substantially increase the economic value of volatility timing in a multivariate mean-variance context (Fleming et al., 2003, p. 495-496).[11]

[10]The GARCH(p,q) model is defined for any lag order p and q. Bollerslev et al. (1992, p. 22) state that small lag orders are sufficient to model the volatility of equity returns in empirical applications (see also Kellner and Rösch (2016) in the context of VaR and CVaR forecasting). Since target volatility strategies are of high interest for practical implementations, we restrict ourselves to the lag orders $p = 1$ and $q = 1$.

[11]The authors use a mean-variance framework with a constant mean which essentially translates into a volatility timing strategies, i.e. the weights of the assets are determined by estimates of conditional volatility and correlation solely. Although the authors use a multivariate setting – based on stocks, bonds, gold and cash – their findings, i.e. that volatility timing adds economic value, is highly related to our approach using stocks and cash.

For the implementation of the target volatility strategy we follow Barroso and Santa-Clara (2015) who use an annualized volatility target σ^{target} of 12%. Typically, target volatility levels used in the literature range from 5% to 40% as annualized volatility target. Clearly, the higher the volatility target the higher the exposure to the risky asset. Therefore, risk-averse or loss-averse investors will prefer a lower target volatility level whereas risk-seeking investors will choose a high volatility target.[12] Bollerslev et al. (2018) show how the volatility target can be derived as a function of the investor's risk aversion. An appealing alternative to choosing a fixed target volatility level was introduced by Wang et al. (2012) in a slightly different setting. The authors propose to switch between two target levels based on whether the market is expected in a high risk or low risk regime.

Since the volatility of the risky asset is usually not constant over time, the weight of the risky asset has to be rebalanced every day, which leads to high transaction costs. Several possibilities are usually used in the literature to lower the turnover and, as a consequence, the transaction costs (see for example Kirby and Ostdiek (2012), Moreira and Muir (2017) and Bollerslev et al. (2018)).[13] Moreira and Muir (2017) and Bollerslev et al. (2018) find that volatility targeting is beneficial even after realistic transaction costs. Similarly, Harvey et al. (2018, Exhibit 8) find that transaction costs hardly influence the Sharpe Ratio of volatility targeting. Marquering and Verbeek (2004) find that transaction costs only marginally impact the utility gains of dynamic trading strategies if short sales and leverage in the risky asset are not allowed. By definition, risk targeting is a long-only strategy and by choosing a moderate target volatility level the strategy

[12] Since some investors have a threshold for the allocation in the risky asset, the weight w_t is often capped by a maximum allowed weight. Strub (2013), Moreira and Muir (2017) and the S&P Dow Jones Risk Control Indices use a cap of 150% (Banerjee et al., 2016). Das and Uppal (2004) and Liu et al. (2003) find that investors should face potential jump risk by not leveraging the risky asset, i.e. they should choose an equity cap of 100% (see also Poon et al. (2004)). In this paper we restrict ourselves on uncapped target risk strategies since we are also interested in the accuracy of different forecasting methods. Using an equity cap would distort this examination. Besides, by choosing a quite low volatility target, the strategy usually does not need weights above 100%. See Moreira and Muir (2017) on how a equity cap of 100% and 150% affects the utility gains of a mean-variance investor compared to the unconstrained strategy. The authors find substantial utility gains of volatility timing even after a tight equity cap is set.

[13] One possibility is to reallocate the weight less frequently, e.g. monthly or quarterly. Moreira and Muir (2017) find a superior performance of the volatility timing strategy when portfolio weights are adjusted monthly. Moreover, in an earlier version of their paper, Bollerslev et al. (2018) find a trade-off between forecasting accuracy and transaction costs and conclude that it may be better not to trade every change in the optimal weight. The authors find better utility gains for the strategies that adjust the weight less frequently, especially when transaction costs are high and/or when models induce high day-to-day changes in the optimal weight. Taylor (2014) presents a method to decrease the changes in the optimal weight in order to lower transaction costs.

is seldom leveraged. In this paper we will reallocate the weight on a daily basis to better assess the accuracy of different risk models.[14]

3 Targeting a Constant Level of Tail Risk: Target VaR and CVaR Strategies

3.1 Managing Volatility versus Managing Tail Risk

As motivated in the previous section due to the risk-averse nature of most investors, the demand for risk-managed investment strategies is very high. Risk management emerged as a major topic within the financial industry and is becoming more important for portfolio managers (Berkowitz and O'Brien, 2002, Christoffersen and Diebold, 2000). We have summarized several justifications and advantages of volatility targeting as a tool to manage the risk of a portfolio of risky assets in Appendix A. However, managing volatility does not necessarily mean managing risk (Poon and Granger, 2003, Szegö, 2002).

Return distributions are typically skewed and fat-tailed (see Farinelli et al. (2008) among others).[15] A negative skewed return distribution implies a higher probability of extreme negative returns, whereas a positive skewed return distribution coincides with a higher probability of extreme positive returns. A fat-tailed distribution implies that extreme (positive or negative) returns are more common than would be expected if returns were normally distributed (see for example Campbell and Hentschel (1992)). Gormsen and Jensen (2017) find that skewness becomes more negative when kurtosis increases, making extreme negative returns more likely. Managing volatility can thus lead to too low weights in times of huge positive returns and too high weights in times of extreme negative returns. Harvey and Siddique (2000, p. 1293) suggest that instead of a mean-variance framework, a mean-variance-skewness framework should be used in an asset allocation analysis (see also Ghysels et al. (2016)). Guidolin and Timmermann

[14] We also used a reallocation buffer of 5% and found a similar risk-adjusted performance to the strategies that are rebalanced daily.

[15] Campbell and Hentschel (1992) explain the existence of negatively skewed and fat-tailed return distributions by the volatility feedback effect and the arrival of news. That is, positive and negative news increase volatility and thus lower stock prices. Negative news additionally cause a stock decline whereas positive news dampen the volatility feedback induced stock decline. Hence, combining these effects produces negative skewness and excess kurtosis.

(2008) show that accounting for higher moments like skewness and kurtosis strongly affects the investor's asset allocation, and hence should be incorporated in asset allocation decisions (see also Patton (2004), Ang et al. (2006a) and Jondeau and Rockinger (2012)). This is also confirmed by Xiong and Idzorek (2011) who highlight that accounting for skewness and kurtosis is crucial and superior to mean-variance optimization especially in times of extreme negative returns. Farinelli et al. (2008) show that maximizing the Sharpe Ratio, i.e. maximizing the mean-variance trade-off, leads to a lower portfolio performance than maximizing the mean-downside risk trade-off. Jarrow and Zhao (2006) compare mean-variance portfolios with mean-downside risk portfolios and find huge differences in both portfolios when asset return distributions are non-normally distributed. Similarly, Agarwal and Naik (2004) compare a mean-downside risk framework to the mean-variance framework using hedge fund data and demonstrate that the mean-variance framework significantly underestimates the downside risk and produces much higher losses during downturn periods. Managing volatility is only suitable if asset returns are normally distributed or investors have quadratic preferences (see Agarwal and Naik (2004) and Bali et al. (2009) and references therein). This is confirmed by Jondeau and Rockinger (2006) who show that mean-variance portfolios and portfolio allocations that account for higher moments are nearly indistinguishable if returns are approximately normally distributed. However, both approaches produce significantly diverse allocations for non-normally distributed returns. Packham et al. (2017) use the difference between Value at Risk (VaR) forecasts using a normality assumption and distributions that account for fat tails and skewness to manage tail risk, and find huge improvements compared to buy-and-hold and other risk-protection strategies. Further, Campbell and Hentschel (1992), Jondeau and Rockinger (2003), Harvey and Siddique (1999) and Bali et al. (2008) show that conditional skewness and kurtosis are time-varying. Hence continuously reallocating the risky asset's weight based on an estimate of the current downside risk, and hence incorporating time-variation in higher moments, is crucial. Managing volatility or simply using static allocations fail to incorporate time-varying higher moments. Cuoco et al. (2008) demonstrate the importance of dynamically managing tail risk and considering actual information on the return distribution compared to static models. Further, Jondeau and Rockinger (2012) demonstrate that incorporating time-variation in skewness and kurtosis is crucial in portfolio selection problems and that higher moment timing outperforms volatility timing.

Besides the existence of skewed and fat-tailed return distributions and the importance of incorporating this observation in asset allocation decisions, Scott and Horvath (1980) theoretically show that, under some assumptions, investors have preferences for higher (or positive) skewness and lower kurtosis (see also Guidolin and Timmermann (2008) and Bali et al. (2009)).[16] Typically, investors have a preference for odd moments, e.g. higher returns and positive skewness, but dislike even moments like variance and kurtosis. Bali et al. (2009) show that higher downside risk predicts lower future skewness. Similarly, Kelly and Jiang (2014) show that an increase of tail risk predicts higher kurtosis and lower skewness of future returns, i.e. an investor exhibiting preferences as in Scott and Horvath (1980) should lower the exposure to the risky asset if tail risk – not necessarily volatility – is high. Generally, downside risk measures increase if the return distribution is leptokurtic or negatively skewed (Bali et al., 2009, Ghysels et al., 2016). By managing downside risk instead of volatility a higher kurtosis and/or a more negative skewness of the risky asset's return distribution induces a lower weight of the risky asset and fits better to a typical investor's preferences.

Bollerslev and Todorov (2011, p. 2187) find that the compensation of tail risk – called "crash-o-phobia" by the authors – is extremely high and much higher than the compensation for volatility, i.e. investors fear tail risk much more than volatility (see also Bollerslev et al. (2015) and Chabi-Yo et al. (2018)). This is also confirmed by the earlier work of Lee and Rao (1988) who find that investors are more concerned about downside risk and that managing volatility is only sufficient when asset returns follow a symmetric distribution (see also Szegö

[16]See also Kraus and Litzenberger (1976), Harvey and Siddique (2000) and Patton (2004) on the preference of positive skewness. Kraus and Litzenberger (1976) extend the traditional CAPM to a three moment CAPM including mean, variance and skewness. Harvey and Siddique (2000) extend this model to a conditional version. See also Section I.C in Harvey and Siddique (2000) on the geometry of the three moment efficient portfolios, where investors demand higher expected returns for holding negatively skewed assets. Guidolin and Timmermann (2008) examine optimal asset allocation under four-moment preferences and regime switching and demonstrate that the asset allocation under four-moment preferences differs from the asset allocation of a mean-variance investor. See also Jondeau and Rockinger (2006), Jondeau and Rockinger (2012) and Lempérière et al. (2017) and references therein on preferences for higher moments and implications on asset allocation decisions.

(2002) and Strub (2013)).[17] Investors are not concerned about return deviations from a mean but more about extreme negative returns which are described by higher moments and rare tail events (see Lempérière et al. (2017) and references therein). Similarly, in a utility based setting Bali et al. (2009, p. 892) find that "investors dislike VaR". Further, most investors are loss averse, i.e. they weight losses higher than gains (Benartzi and Thaler, 1995). Loss-averse investors have a high demand for portfolio insurance methods that avoid huge losses and seek for risk reduction especially in times of high market downturns (Aït-Sahalia and Brandt, 2001, Ang et al., 2006a, Bollerslev and Todorov, 2011, Chabi-Yo et al., 2018). Consequently, for loss averse investors controlling downside risk instead of volatility, i.e. controlling negative returns instead of return deviation, is crucial to increase their utility (see Aït-Sahalia and Brandt (2001, p. 1298), Ang et al. (2005), Ang et al. (2006a) and references therein). Aït-Sahalia and Brandt (2001, p. 1315) state that the theory of loss aversion is related to the literature on downside risk-based investment decisions. Jarrow and Zhao (2006) motivate that loss-averse investors should manage downside risk instead of volatility when asset return distributions are non-normally distributed. Timing downside risk instead of volatility also fits better to safety-first investors which are concerned about avoiding disasters (see Bali et al. (2009) and references therein).

As mentioned in Appendix A, diversification fails as a risk-management tool due to the increase of correlations in bear markets. This renders the benefits of diversification just when it is most needed (Ang and Chen, 2002, Butler and Joaquin, 2002, Karolyi and Stulz, 1996, Longin and Solnik, 2001, Poon et al., 2004). Chabi-Yo et al. (2018) find a stronger asymptotic dependence in the left tail of stocks than in the right tail, i.e. stocks tend to crash simultaneously. In particular, the left tail dependence increases in periods of market crashes (Chabi-Yo et al., 2018, Figure 2). Therefore, lowering the exposure to the risky asset in bear markets is needed to manage the risk of a portfolio. However, Longin and Solnik (2001) show, by using Extreme Value Theory (EVT) and thus measuring tail risk, that increases in correlations do not necessarily coincide with increases in volatility, but with huge negative returns, i.e. the portfolio risk in bear markets should be better managed by tail risk measures instead of volatility. This is line

[17] For most investors "risk" is associated with low or even negative returns. Describing risk by volatility does not differentiate between positive or negative returns (see Lee and Rao (1988, p. 452) or Poon and Granger (2003, p. 480)).

with Poon et al. (2004) and Kelly and Jiang (2014) who find that volatility standardized returns still exhibit significantly tail dependency and tail risk. Similarly, Gormsen and Jensen (2017) find that skewness and kurtosis typically co-move, i.e. when skewness becomes more negative, kurtosis increases simultaneously. These periods often occur when market volatility is low, i.e. in low volatile periods risk "hides in the tails". Similarly, Ghysels et al. (2016) find that skewness is typically hidden in the tails and that skewness in the tails has a high impact on portfolio allocations. Gormsen and Jensen (2017) also show that volatility targeting strategies still exhibit high tail risk, i.e. managing volatility does not mean managing extreme negative returns. Further, the authors find that times of negative skewness and/or high kurtosis are typically followed by low future returns. Similarly, Liu et al. (2003) and Das and Uppal (2004) find that in times of huge price jumps, like the financial crisis, skewness and kurtosis are higher than in normal times, which is again not captured by managing volatility. Jarrow and Zhao (2006) show that the portfolio allocation between volatility and downside risk managed strategies can be vastly different when returns exhibit price jumps. These rare tail events are not predictable and can not completely be avoided by dynamically managing risk (Bollerslev and Todorov, 2011).[18] However, rare events occur in the tail of the loss distribution and are often accompanied with changes in moments higher than volatility (Poon et al., 2004, p. 582). To better manage the potential event risk, combining an estimation method that reflects the current market condition, measured by a dynamic volatility model, combined with an estimation method that directly models the tail of the distribution, like EVT, should be used instead of volatility alone (Longin, 2000). Additionally, Jondeau and Rockinger (2003) show that skewness and kurtosis of risky assets comove, i.e. large (negative) returns in different risky assets tend to occur simultaneously. Simply combining several risky assets or managing volatility does not reduce the occurrence of extreme (negative) returns.

Benson et al. (2014, p. 96) state that mitigating drawdowns by an investment strategy, as done by the target volatility strategy, leads to a better absolute and risk-adjusted performance compared to the benchmark-index even in the absence of a negative return-volatility correlation.

[18] Systematic event risk like unpredictable jumps effect the allocation between the risky and the riskless asset (see Poon et al. (2004, p. 602) and Das and Uppal (2004)). Liu et al. (2003) find that investors should avoid leveraged positions to account for the potential of unpredictable price jumps.

By comparing arithmetic and geometric returns, they find that the enhanced risk-return profile of volatility targeting comes from avoiding huge negative returns and not from a negative relation between risk and future returns. Similarly, Harvey et al. (2018) find that risk targeting successfully reduces extreme negative returns. Due to the asymmetric behavior of compounded returns, mitigating high negative returns is more crucial than achieving high positive returns.[19] This is confirmed by the results of Barroso and Santa-Clara (2015) who find that the superior performance of the volatility managed momentum strategy is significantly driven by drawdown reduction (see also Moreira and Muir (2017) and Barroso and Maio (2016)). This indicates that a main driver of the superior performance of the target volatility strategy is drawdown protection and not the negative risk-return relation. Dachraoui (2018) theoretically show that a negative risk-return relation is not needed in order to provide an enhanced risk-return profile of risk targeting. Since asset returns usually are non-elliptically distributed, managing volatility underestimates the potential of extreme losses (Szegö, 2002, p. 1255). Consequently, managing downside risk instead of volatility should be more successful in mitigating drawdowns and hence should result into an even better (risk-adjusted) performance compared to both the target volatility and buy-and-hold strategies.[20]

Concluding, the demand for tail risk hedging strategies is high since these strategies fit well to the preferences of most investors and deliver an enhanced risk-return profile and drawdown protection. Main approaches to reduce tail risk of a risky portfolio are derivative based and cash based strategies (see Strub (2013, p. 1) and Happersberger et al. (2019)). Derivative based strategies manage the tail risk by buying or selling derivatives, e.g. options or futures on the risky asset, in order to achieve downside risk protection. Cash based strategies, as the here presented target risk strategies, dynamically allocate the wealth invested in the risky and riskless asset, based on the expected risk of the risky asset and are related to portfolio insurance strategies like CPPI (Happersberger et al., 2019). So far, almost all studies on cash based tail risk hedging strategies focused on allocating money based on forecasted volatility instead of

[19] As stated above, besides obtaining a higher total return, this also fits better to the loss aversion of most investors.

[20] This is confirmed by Strub (2013, p. 6) who finds that "larger than normal tail risk is partly responsible for the outsized drawdowns experienced in market downturns, thus being able to accurately measure and control it is likely to yield significant improvements in risk adjusted performance" (see also Hocquard et al. (2013)).

forecasted tail risk.[21] Similarly, there exists a huge literature on the economic value of volatility timing, whereas the economic value of *downside risk timing* is hardly examined (Basak and Shapiro, 2001). Cuoco et al. (2008) find that dynamically reallocating the amount invested in several assets based on downside risk is beneficial and superior to static approaches or approaches that do not account for downside risk. Therefore, we will assess the economic value of downside risk timing and compare it to the economic value of volatility timing.

To account for the above mentioned drawbacks of the target volatility strategy we next present the target Value at Risk (target VaR) and target Conditional Value at Risk (target CVaR) strategies, which aim to achieve a constant VaR or CVaR of the portfolio over time. VaR is a widely used tool to measure market risk (Alexander and Baptista, 2004, Bali et al., 2008, Berkowitz et al., 2011, Berkowitz and O'Brien, 2002, Cuoco et al., 2008), however, CVaR is getting more important in recent years from a regulatory and practical view (Du and Escanciano, 2016). By construction both, the target VaR and CVaR strategy, automatically manage the downside risk of the risky asset and thus correct for the drawbacks of the target volatility strategy.[22]

3.2 Target Value at Risk Strategy

We again consider a portfolio that invests w_t in a risky asset and $1 - w_t$ in a riskless asset. The goal of the target VaR strategy is to determine w_t such that the portfolio achieves a constant Value at Risk over time. By definition, the Value at Risk at a significance level α is the maximum loss defined as the negative daily return that is only exceeded with a probability of $100 \cdot \alpha\%$ (see Szegö (2002), Yamai and Yoshiba (2005) among others). In order to achieve a constant

[21] Strub (2013) and Happersberger et al. (2019) use a cash based tail risk strategy that relies on a similar weighting as in the target volatility strategy, but replaces the volatility in Equation (4) by an estimate of the risky asset's downside risk. Essentially, as we will see later, these strategies do not aim to target a constant level of portfolio risk over time, and hence do not belong to the class of risk targeting strategies. See also Basak and Shapiro (2001), Alexander and Baptista (2004), Cuoco et al. (2008) and Packham et al. (2017) for other tail risk based investment strategies.

[22] These tail risk targeting strategies are similar to the approach of Basak and Shapiro (2001, p. 376) and Cuoco et al. (2008) who incorporate downside risk measures in an asset allocation framework, but instead of targeting a constant level of tail risk the authors require the downside risk to be below some prespecified limit (see also Ang and Bekaert (2002), Wang et al. (2012) and Alexander and Baptista (2004)). Similar to our tail risk targeting strategies, this downside risk managed strategy also allocates wealth between a riskless asset and an (optimal) portfolio of risky assets (see Cuoco et al. (2008, Remark 3) for example). Basak and Shapiro (2001, p. 376) call this approach a softer form of portfolio insurance.

Value at Risk level $\text{VaR}_\alpha^{\text{target}}$, the investor specifies the desired (daily) Value at Risk level, i.e. the critical loss or loss threshold the investor is willing to accept as well as the corresponding significance level α, i.e. the exceedance probability. For example, a target VaR level $\text{VaR}_\alpha^{\text{target}}$ of 1% with a corresponding significance level α of 5% translates into a strategy, where daily returns below -1% only occur with a probability of 5%. In other words, with a probability of 95% daily returns should be higher than -1%.[23] Similar strategies are already available for retail investors.[24] The choice of α and $\text{VaR}_\alpha^{\text{target}}$ strongly depends on the investor's preferences and degree of risk aversion (Alexander and Baptista, 2004). In summary, the target VaR strategy has two advantages for an investor compared to a target volatility strategy. First, it manages extreme losses instead of loss deviations. Second, it is an easy to interpret strategy where investors can prescribe an acceptable loss limit.

As usual when working with tail risk measures we define the daily portfolio loss at day t as

$$L_t^P := -R_t^P. \tag{9}$$

Similarly, the day t loss of the risky asset is defined as $L_t := -R_t$. Thus, the portfolio loss can be written as

$$L_t^P = w_t \cdot L_t - (1 - w_t) \cdot R_t^f. \tag{10}$$

The day t VaR of the portfolio for a significance level α, denoted by $\text{VaR}_\alpha^{P,t}$, is defined through the relation[25]

$$\mathrm{P}\big(L_t^P \leqslant \text{VaR}_\alpha^{P,t} \mid \mathcal{F}_{t-1}\big) = 1 - \alpha. \tag{11}$$

[23] The target VaR strategy has the advantage of being better interpretable for investors than the target volatility strategy. Moreover, by choosing low values of $\text{VaR}_\alpha^{\text{target}}$ and α, this strategy can also be used by hedge fund managers as an alternative to absolute return strategies. These strategies typically have absolute return targets which are independent of the current market environment whereas most mutual fund managers have relative return targets that are compared to a benchmark asset (see Fung and Hsieh (1997) and Agarwal and Naik (2004)). For example, assuming 250 trading days per year and by choosing $\text{VaR}_\alpha^{\text{target}} = 0.5\%$ and $\alpha = 0.4\%$, a daily return below -0.5% should only occur once a year (see Figure III in Appendix D for a performance chart of this strategy). Thus, regardless of if the underlying asset is in a bear or bull market the target VaR strategy aims to constantly produce returns with limited downside risk. This is even advantageous to some hedge funds strategies since some hedge funds strategies exhibit huge losses during market downturns and bear significant tail risk (Agarwal and Naik, 2004). Similarly, the authors find low correlations between hedge funds and the market in times the market moves upwards but higher positive correlations during market downturn periods and that hedge funds often resemble a short Put payoff profile. Investors who are interested in an absolute return strategy should therefore better use a risk targeting strategy.

[24] See for example the strategies offered by Scalable Capital (http://www.scalable.capital).

[25] Throughout the paper, we assume that the loss variables L_t^P and L_t are continuously distributed.

Therefore, the portfolio VaR is given by the $(1-\alpha)$-quantile of the (conditional) portfolio loss distribution, denoted by $F^{-1}_{L^P_t|\mathcal{F}_{t-1}}(1-\alpha)$, i.e. $\mathrm{VaR}^{P,t}_\alpha = F^{-1}_{L^P_t|\mathcal{F}_{t-1}}(1-\alpha)$. In Appendix B we show that the portfolio VaR is given by

$$\mathrm{VaR}^{P,t}_\alpha = w_t \cdot \mathrm{VaR}^t_\alpha - (1-w_t) \cdot R^f_t, \tag{12}$$

where $\mathrm{VaR}^t_\alpha := F^{-1}_{L_t|\mathcal{F}_{t-1}}(1-\alpha)$ denotes the day t VaR of the risky asset.[26] In order to achieve a constant portfolio VaR level $\mathrm{VaR}^{\mathrm{target}}_\alpha$ over time, i.e. $\mathrm{VaR}^{P,t}_\alpha = \mathrm{VaR}^{\mathrm{target}}_\alpha$ for all t, the weight of the risky asset has to be chosen as

$$w_t = \frac{\mathrm{VaR}^{\mathrm{target}}_\alpha + R^f_t}{\mathrm{VaR}^t_\alpha + R^f_t}. \tag{13}$$

By construction, since VaR^t_α and R^f_t are $\mathcal{F}_{t-1}$-measurable the weight w_t is known at time $t-1$. Furthermore, the weight of the risky asset is increased, if the downside risk of the risky asset, measured by VaR, is expected to be low and vice versa. By doing this, the tail risk of the portfolio is managed by allocating money between the risky and the riskless asset.[27] If a market crash becomes more likely the amount invested in the risky asset is reduced. When market risk declines the amount invested in the risky asset is subsequently increased.[28]

Similar to the volatility the VaR of the risky asset is not observable, and hence a forecast of the risky asset's VaR is needed.[29] As first method we estimate VaR by Historical Simulation (HS) using a rolling window of n days, i.e. we estimate VaR^t_α by the empirical $(1-\alpha)$-quantile of the past n daily losses (see Kuester et al. (2006, p. 56-57) or Halbleib and Pohlmeier (2012)).

[26]The representation in Equation (12) can directly be seen by positive homogeneity and translation invariance of VaR (Szegö, 2002, p. 1259-1260).

[27]Many tail hedging strategies, that aim to reduce the tail risk, only work well when markets exhibit huge drawdowns. In times the markets go up, the tail hedging strategy usually performs worse than a simple buy and hold strategy, translating in a worse overall performance (Hocquard et al., 2013). The target VaR strategy has the advantage that this strategy increases the weight of the risky asset as downside shrinks, and hence captures the upside potential while downside risk is managed (Wang et al., 2012, p. 38). Dopfel and Ramkumar (2013) show that the periods following high risk periods are the most attractive ones (see also Muir (2017)). Hence, risk targeting delivers an option-like return profile similar to portfolio insurance strategies (see also Fung and Hsieh (1997) who found a similar behavior of dynamic trading strategies used by hedge fund managers).

[28]Similarly, Chabi-Yo et al. (2018) show in a cross-sectional setting that assets with lower crash sensitivity outperform during times of market distress but underperform when markets are calm (see also van Oordt and Zhou (2016)). Moreover, assets with a high crash sensitivity exhibit higher returns after huge market declines. Thus, during a crash period the amount invested in crash-sensitive assets should be decreased and then subsequently increased when crash risk declines.

[29]See Taylor (2005, Sec. 3) and Kuester et al. (2006) for a survey of VaR estimation models.

More formally, for a sample $l_{t-n}, \dots, l_{t-1}$ of n realized losses the day t VaR is given by

$$\widehat{\mathrm{VaR}}_{\alpha}^{t} = l_{(\lceil n(1-\alpha) \rceil), t-1}, \tag{14}$$

where $l_{(1),t-1} \leqslant \dots \leqslant l_{(n),t-1}$ denotes the order statistics of the sample $l_{t-n}, \dots, l_{t-1}$.

Historical Simulation relies on the assumption that the loss distribution can be estimated by the empirical distribution of past losses (McNeil and Frey, 2000, p. 273). Hence, Historical Simulation assumes that losses are iid, an assumption that does not hold for losses of most risky assets, since asset returns (or losses respectively) are known to exhibit a time-varying volatility and volatility clustering (Pritsker, 2006, p. 563). Further, Pritsker (2006) shows that Historical Simulation does not respond to the 1987 crash. Most VaR estimation models frequently used in the financial industry, like Historical Simulation, work well in calm periods but fail to produce accurate risk forecast in times of high downside risk just in that time when reliable forecasts are most needed (Berkowitz et al., 2011, Halbleib and Pohlmeier, 2012). Using a static model that does not account for the current market environment to manage portfolio risk can translate in high probabilities of extreme losses (Cuoco et al., 2008). Hence, using Historical Simulation in the context of a target VaR strategy can translate in a high exposure to the risky asset in times when financial markets are very risky, although a good risk-managed investment strategy should exhibit a low weight in the risky asset during times of high market risk. Thus, a fast adapting estimation model is crucial for the quality of the target VaR strategy (see also Taylor (2014) and Bollerslev et al. (2018) who find a similar result for volatility managed portfolios). However, estimating VaR by Historical Simulation is easy, straightforward and is the current industry standard for estimating VaR (see Berkowitz et al. (2011) and references therein). Thus, this approach is in particular interesting for index providers and practitioners, who are interested in a simple target VaR strategy. Consequently, the target VaR strategy based on Historical Simulation deals as a benchmark strategy for more complex target VaR strategies.

Additionally, we use three VaR forecasting models based on a volatility forecast of the EWMA or GARCH(1,1) model given in Equation (7) or (8), respectively. McNeil and Frey (2000, p. 273-274) propose to reflect the current volatility background, estimated by a dynamic volatility model, and account for heavy tails in the conditional loss distribution when estimating quantile risk measures (see also Longin (2000)). Christoffersen and Diebold (2000) find that volatility is highly forecastable for short horizons of less than 10 days and thus is highly relevant and should be incorporated when short-term risk is managed. Under the assumption that the daily return can be described by Equation (6), the day t VaR of the risky asset is given by

$$\mathrm{VaR}_\alpha^t = \sigma_t \cdot F_{L*}^{-1}(1-\alpha), \tag{15}$$

where $L_t^* := -Z_t$ is a random variable representing a standardized loss with expectation zero, variance one and $F_{L*}^{-1}(1-\alpha)$ denotes the $(1-\alpha)$-quantile of L_t^*.[30] We also denote dynamic risk models that account for the current volatility as conditional models and static models like Historical Simulation or HSD that are based on the assumption that returns are iid as unconditional models (Longin, 2000). The forecast $\widehat{\mathrm{VaR}}_\alpha^t$ for the day t VaR based on the information at time $t-1$ is then given by

$$\widehat{\mathrm{VaR}}_\alpha^t = \hat{\sigma}_t \cdot \hat{F}_{L*,t}^{-1}(1-\alpha), \tag{16}$$

where $\hat{F}_{L*,t}^{-1}(1-\alpha)$ denotes the estimator of $F_{L*}^{-1}(1-\alpha)$ given the available information at day $t-1$. $\widehat{\mathrm{VaR}}_\alpha^t$ is then estimated in the following way, using a two-stage approach as described by McNeil and Frey (2000, p. 277). In the first stage, we estimate the volatility $\hat{\sigma}_t$ using the EWMA or the GARCH(1,1) model given in Equation (7) or (8), respectively. The parameters of the GARCH(1,1) model are estimated using a Quasi Maximum Likelihood (QML) approach, i.e. assuming a standard normal distribution for the innovations Z_t. In the second stage, the standardized losses, i.e. $l_t^* = -R_t/\hat{\sigma}_t$ are calculated and used to calculate $\hat{F}_{L*,t}^{-1}(1-\alpha)$.[31] In this context, VaR is often estimated by assuming a standard normal distribution for Z_t, and hence $\hat{F}_{L*,t}^{-1}(1-\alpha)$ is given by the $(1-\alpha)$-quantile of the standard normal distribution. However, even after standardizing returns or losses by a time-varying volatility, these observations exhibit a non-zero skewness and fatter tails than a normal distribution (see Campbell and Hentschel (1992), Bollerslev et al. (1992), Glosten et al. (1993), Harvey and Siddique (1999), Ghysels

[30] By assumption the quantile $F_{L*}^{-1}(1-\alpha)$ of the standardized loss $L_t^* := -Z_t = -R_t/\sigma_t$ does not depend on t (McNeil and Frey, 2000, p. 276). We further follow Jondeau and Rockinger (2003) and Bali et al. (2008) and use a more sophisticated approach below that does not assume that Z_t is iid.

[31] The volatility σ_t as described in Section 2 is calculated using daily returns and thus represents a forecast for the return volatility. However, since $\mathrm{var}(R_t|\mathcal{F}_{t-1}) = \mathrm{var}(-R_t|\mathcal{F}_{t-1})$, the volatility forecast $\hat{\sigma}_t$ can directly be used as a forecast for the volatility of the losses.

et al. (2016), Jondeau and Rockinger (2003) and Bali et al. (2008)).[32] Similarly, Kelly and Jiang (2014) find that volatility standardized returns still exhibit significant tail risk. Therefore, we estimate $\hat{F}^{-1}_{L^*,t}(1-\alpha)$ based on a sample of n past standardized losses, denoted by $l^*_{t-n}, \ldots, l^*_{t-1}$, using three different methods that account for that stylized fact. First, we use the Filtered Historical Simulation (FHS) approach (Barone-Adesi et al., 2008, 1999), i.e. we estimate $\hat{F}^{-1}_{L^*,t}(1-\alpha)$ by the empirical $(1-\alpha)$-quantile of the standardized losses $l^*_{t-n}, \ldots, l^*_{t-1}$ (Kuester et al., 2006, p. 57). The estimator for the day t VaR of the risky asset is then given by

$$\widehat{\mathrm{VaR}}^{t}_{\alpha} = \hat{\sigma}_t \cdot l^*_{(\lceil n(1-\alpha)\rceil),t-1}, \tag{17}$$

where $l^*_{(1),t-1} \leqslant \ldots \leqslant l^*_{(n),t-1}$ denotes the order statistics of the sample $l^*_{t-n}, \ldots, l^*_{t-1}$.[33] The FHS approach easily combines the conditional heteroscedasticity and the non-normality of asset returns in a simple estimation method without any distributional assumption of the losses (Giannopoulos and Tunaru, 2005, p. 983). As second estimation method, we use the Extreme Value Theory (EVT) approach of McNeil and Frey (2000).[34] The EVT approach is based on the assumption that the tail of the distribution of the standardized losses can be described by a Generalized Pareto Distribution (GPD).[35] The tail of this distribution is defined in terms of a threshold u.[36] Then, the standardized losses above the threshold u follow a GPD, that is defined

[32] This stylized fact holds for standardized equity returns but does not hold in the foreign exchange rate market (Bollerslev et al., 1992, p. 38).

[33] Pritsker (2006) states that the choice of n is not straightforward. However, $n = 1000$ is frequently used in applications (see Kuester et al. (2006) or Christoffersen (2012)). See also Halbleib and Pohlmeier (2012) on how the window size impacts estimation results of VaR forecasts.

[34] See also McNeil et al. (2015, Section 5.2.6) for a survey of estimating quantile risk measures, when using GARCH volatility models in the first stage and Kuester et al. (2006, Section 1.4) for a good survey of how to estimate VaR using EVT. EVT is also used by Poon et al. (2004), Longin and Solnik (2001), Kelly and Jiang (2014), Longin (2000) and van Oordt and Zhou (2016) in other related financial topics.

[35] The EVT approach assumes that the losses are iid. Therefore, when working with short horizons – in this paper we work with daily losses – standardizing losses by a time-varying volatility is crucial for this approach. It is a well-known stylized fact, that daily losses are far away from iid, whereas the iid assumption fits quite well to standardized losses (Kuester et al., 2006, p. 62). McNeil and Frey (2000) state that when working with longer time horizons, the EVT approach can be applied to the non-standardized losses directly.

[36] One drawback of EVT is the choice of the threshold u (Kellner and Rösch, 2016). If u is chosen too high, the estimation of the parameters is based on only few exceedance observations, making the estimation less precise. Choosing u too low contradicts to the approximation in Equation (18), since this approximation only holds for the tails of the distribution (see Longin and Solnik (2001, Sec. II.A), Kuester et al. (2006, p. 62) and Yamai and Yoshiba (2005, p. 1008)). Longin and Solnik (2001, Appendix 1) show how u can be optimally chosen based on Monte Carlo Simulations. Packham et al. (2017, p. 740) find that their VaR-based tail risk protection strategy is robust against changes in u. As in Kellner and Rösch (2016) we choose the threshold as the 90%-quantile.

as

$$G_{\xi,\beta}(y) = \begin{cases} 1 - (1 + \xi y/\beta)^{-1/\xi}, & \text{if } \xi \neq 0 \\ 1 - \exp(-y/\beta), & \text{if } \xi = 0, \end{cases} \tag{18}$$

with $\beta > 0$. The support of this distribution is given by $y \geqslant 0$ if $\xi \geqslant 0$ and $0 \leqslant -\beta/\xi$ if $\xi < 0$ (McNeil and Frey, 2000, p. 280). The parameter ξ is usually called the *shape* parameter and β is called the *scale* parameter (McNeil et al., 2015, p. 147). For the estimation of $\hat{F}^{-1}_{L*,t}(1-\alpha)$ we again assume that the sample contains n standardized losses. Then, the estimator $\hat{F}^{-1}_{L*,t}(1-\alpha)$ is given by

$$\hat{F}^{-1}_{L*,t}(1-\alpha) = u + \frac{\hat{\beta}}{\hat{\xi}}\left(\left(\frac{\alpha n}{N_u}\right)^{-\hat{\xi}} - 1\right), \tag{19}$$

where $\hat{\beta}$ and $\hat{\xi}$ are the Maximum Likelihood estimates and N_u denotes the number of standardized losses that exceed the threshold u (McNeil et al., 2015, p. 154 and 349). The VaR forecast is then given by Equations (16) and (19).

We will next use a further extension of the models presented above. The Historical Simulation approach assumes that returns are iid, which is not realistic in practice. The EVT and FHS approaches defined above are more realistic by assuming that only volatility standardized returns are iid. However, several studies show that even this assumption is too restrictive, since even volatility standardized returns exhibit autoregressive patterns in conditional skewness and kurtosis (Bali et al., 2008, Harvey and Siddique, 1999, Jondeau and Rockinger, 2003). Thus, we follow Jondeau and Rockinger (2003) and Bali et al. (2008) and use the EWMA and GARCH based approach combined with the skewed t distribution of Hansen (1994), where conditional skewness and kurtosis are modeled autoregressively. Similar to Equation (6), we assume that the daily return can be described by

$$R_t = \sigma_t \cdot Z_t,\ Z_t \sim stsk(\eta_t, \lambda_t), \tag{20}$$

where $Z_t \sim stsk(\eta_t, \lambda_t)$ means that Z_t is skewed t distributed with mean zero, variance one and time-varying parameters η_t and λ_t. The skewed t distribution of Hansen (1994) is characterized by the pdf

$$f_{stsk}(z \mid \eta, \lambda) = \begin{cases} bc\left(1 + \frac{1}{\eta-2}\left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-(\eta+1)/2} & \text{if } z < -\frac{a}{b} \\ bc\left(1 + \frac{1}{\eta-2}\left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-(\eta+1)/2} & \text{if } z \geqslant -\frac{a}{b} \end{cases} \tag{21}$$

where

$$a := 4\lambda c\frac{\eta-2}{\eta-1}, \quad b^2 := 1+3\lambda^2-a^2, \quad c := \frac{\Gamma\left(\frac{\eta+1}{2}\right)}{\sqrt{\pi(\eta-2)}\Gamma\left(\frac{\eta}{2}\right)}.$$

The parameters of this distribution are restricted to $\eta > 2$ and $-1 < \lambda < 1$ (see Hansen (1994, p. 710) and Jondeau and Rockinger (2003, p. 1702)). Further, for $\lambda = 0$ this distribution is symmetric and equals the standardized t distribution. For $\lambda > 0$ ($\lambda < 0$) the distribution is positively (negatively) skewed (Hansen, 1994). Moreover, skewness exists for $\eta > 3$ and kurtosis exists for $\eta > 4$ (Jondeau and Rockinger, 2003). Jondeau and Rockinger (2003) show that although η is often referred as the parameter that determines kurtosis and λ determines skewness, both parameters, η and λ, affect both, skewness and kurtosis. In particular, the relation between the parameters and higher moments is highly non-linear. The parameters η_t and λ_t of the skewed t distribution are then modeled autoregressively by

$$\tilde{\eta}_t = a_1 + b_1 R_{t-1} + c_1\tilde{\eta}_{t-1}, \tag{22}$$

$$\tilde{\lambda}_t = a_2 + b_2 R_{t-1} + c_2\tilde{\lambda}_{t-1}. \tag{23}$$

To guarantee that the standardized skewed t distribution is well defined, the parameters have to be restricted to fulfill the conditions $\eta_t > 2$ and $-1 < \lambda_t < 1$. We follow Jondeau and Rockinger (2003) and Bali et al. (2008) and use a logistic transformation to guarantee that these restrictions hold. The parameters η_t and λ_t are then given by

$$\eta_t = 2 + \exp\left(\tilde{\eta}_t\right) \tag{24}$$

$$\lambda_t = \frac{2}{1+\exp\left(-\tilde{\lambda}_t\right)} - 1. \tag{25}$$

The α-quantile of the skewed t distribution is given by

$$F_{stsk}^{-1}\left(\alpha \mid \eta, \lambda\right) = \begin{cases} \frac{1}{b}\left((1-\lambda)\sqrt{\frac{\eta-2}{\eta}}F_t^{-1}(\frac{\alpha}{1-\lambda}|\eta) - a\right) & \text{if } \alpha < \frac{1-\lambda}{2} \\ \frac{1}{b}\left((1+\lambda)\sqrt{\frac{\eta-2}{\eta}}F_t^{-1}(\frac{\alpha+\lambda}{1+\lambda}|\eta) - a\right) & \text{if } \alpha \geqslant \frac{1-\lambda}{2}, \end{cases} \tag{26}$$

where $F_t^{-1}(z|\eta)$ is the inverse of the t distribution's cdf $F_t(z|\eta) = \int_{-\infty}^{z} f_t(u|\eta)du$ (Jondeau and Rockinger, 2003). The t distribution's pdf with η degrees of freedom is given by

$$f_t(z|\eta) = \frac{\Gamma\left(\frac{\eta+1}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)\sqrt{\pi\eta}}\left(1+\frac{z^2}{\eta}\right)^{-(\eta+1)/2}, \tag{27}$$

where $\Gamma(\cdot)$ denotes the Gamma function. The VaR forecast for day t is then again given by Equation (16), where

$$\hat{F}^{-1}_{L*,t}(1-\alpha) = F^{-1}_{stsk}\left(1-\alpha \mid \hat{\eta}_t, \hat{\lambda}_t\right), \tag{28}$$

and $\hat{\eta}_t$ and $\hat{\lambda}_t$ denote the Maximum Likelihood estimates of η_t and λ_t.

Kuester et al. (2006) compare several VaR forecasting approaches and find that the GARCH-based EVT, FHS and skewed t distribution approaches always belong to the best conditional models, where the authors only use the skewed t distribution with constant parameters instead of time-varying parameters. The authors state that unconditional models, like Historical Simulation, fail to produce adequate VaR forecast and that only conditionally heteroskedastic models deliver acceptable VaR forecasts. Therefore, the authors prefer a conditional approach that accounts for the volatility dynamics. Furthermore, the authors find that the VaR violations of dynamic models are reasonably independent over time, which usually does not hold for Historical Simulation. This indicates the importance of a fast adapting model, which is crucial for the target risk strategies, since wrong risk timing translates into a high exposure of the risky asset when the market's downside risk is high and vice versa.[37] Moreover, McNeil and Frey (2000, p. 283) state that using a symmetric distribution, like the normal or the t-distribution, underestimates the loss potential (see also Szegö (2002)). Kellner and Rösch (2016) find that only models that account for fat tails and/or skewness deliver accurate VaR forecasts. Since standardized losses typically follow an asymmetric distribution, using an approach like EVT, FHS or skewed t is a better choice for modeling the right tail of the standardized losses (Xiong and Idzorek, 2011). In particular, modeling the right tail of the loss distribution directly instead of modeling the whole distribution usually gives a better fit for tail risk forecasting (Longin, 2000). McNeil and Frey (2000, p. 290-291) compare the GARCH-EVT with the unconditional EVT, the GARCH-normal and the GARCH-t models and find that in most cases the GARCH-EVT is superior to the benchmark models and that the GARCH-EVT model is the only model that is not rejected in all 15 cases. Packham et al. (2017) use several alternative generalized distributions for Z_t and find that only the GPD distribution works well for tail risk management.

[37] We will come back to this point in Section 4.2, when we show how to backtest the accuracy of the target VaR strategy.

Halbleib and Pohlmeier (2012) also find convincing results of combining a dynamic volatility model like GARCH(1,1) with an estimate of $F_{L*}^{-1}(1-\alpha)$ that accounts for skewness like EVT or the skewed t-distribution. Bali et al. (2008) find convincing results of combining a dynamic volatility model with the skewed t-distribution. As a consequence, combining a dynamic volatility model with the EVT, FHS or skewed t approach should result in portfolio VaRs that are closer to the desired target VaR level than the portfolio VaRs of the simple Historical Simulation approach. Moreover, Halbleib and Pohlmeier (2012) present an easy and straightforward framework of combining several VaR forecasting models, which could also be an interesting approach to improve the accuracy of VaR targeting in a simple manner. We answer the question, which model delivers the most accurate portfolio VaR later, when we assess the accuracy of several target VaR strategies using the backtesting method described in Section 4.2.

Since R_t^f is (typically) small compared to $\mathrm{VaR}_\alpha^{\mathrm{target}}$ and VaR_α^t, the weight w_t of the target VaR strategy can be approximated by[38]

$$w_t \approx \frac{\mathrm{VaR}_\alpha^{\mathrm{target}}}{\mathrm{VaR}_\alpha^t}. \tag{29}$$

The structure in Equation (29) is similar to the weight of the target volatility strategy given in Equation (3), but the volatility of the risky asset is replaced by the VaR of the risky asset. The weighting in Equation (29) was also used by Happersberger et al. (2019). Using the approximation in (29) and the decomposition of the VaR in (15), the weight of the risky asset of the target VaR strategy can be approximated by

$$w_t \approx \frac{\mathrm{VaR}_\alpha^{\mathrm{target}}}{\mathrm{VaR}_\alpha^t} = \frac{F_{L*}^{-1}(1-\alpha) \cdot \dfrac{\mathrm{VaR}_\alpha^{\mathrm{target}}}{F_{L*}^{-1}(1-\alpha)}}{\sigma_t \cdot F_{L*}^{-1}(1-\alpha)} = \frac{\sigma^{\mathrm{target}}}{\sigma_t}, \tag{30}$$

with $\sigma^{\mathrm{target}} := \mathrm{VaR}_\alpha^{\mathrm{target}}/F_{L*}^{-1}(1-\alpha)$. Therefore, the weight of a target VaR strategy can be approximated by the weight of a target volatility strategy, where the target volatility level is determined by the target VaR level and the $(1-\alpha)$-quantile of the standardized losses. Hence, one could argue that every target VaR strategy based on the decomposition (15) can be ap-

[38]For our sample the mean of R_t^f and VaR_α^t is 0.007% and 4.0766%, respectively. Thus, for our $\mathrm{VaR}_\alpha^{\mathrm{target}}$ of 1.947% and these averages the weight based on Equation (13) would be 0.4785 whereas the approximated weight would be 0.4776.

proximated by a target volatility strategy.[39] Further, a constant portfolio volatility can also be achieved by controlling VaR instead of volatility. Taylor (2005) shows that incorporating higher moments in volatility forecasts is beneficial, since the shape of the conditional return distribution is not fix over time as shown by Jondeau and Rockinger (2003) and Bali et al. (2008). However, the approximation in Equation (30) is not straightforward. First, Equation (29) is just an approximation, which is only exact for $R_t^f = 0$. Second, to transform a target VaR strategy into the corresponding target volatility strategy the distribution of the standardized losses, or at least the quantile $F_{L*}^{-1}(1-\alpha)$, has to be known. In practice both are unknown, and hence this transformation is not directly feasible. As a rough approximation quantiles of the standard normal distribution can be used. Then, the target VaR strategy can be approximated by the target volatility strategy, using a target volatility level of

$$\sigma^{\text{target}} = \frac{\text{VaR}_\alpha^{\text{target}}}{N_{1-\alpha}}, \tag{31}$$

where $N_{1-\alpha}$ denotes the $(1-\alpha)$-quantile of the standard normal distribution.[40]

3.3 Target CVaR Strategy

The target VaR strategy presented in the previous section has the advantage that the weight of the risky asset is a function of the expected downside risk instead of expected volatility. As stated before, focusing on downside risk management instead of volatility management has several advantages (see Lee and Rao (1988), Szegö (2002), Basak and Shapiro (2001), Bali et al. (2009), Bollerslev et al. (2006), Ang et al. (2006a) among others). Although, VaR has become the industry standard when downside risk is measured and managed in recent years (Bali et al., 2008, Berkowitz et al., 2011, Berkowitz and O'Brien, 2002), Conditional Value at Risk (CVaR) is becoming more important and is establishing as the more relevant risk measure for managing market risk and from a regulatory perspective (Du and Escanciano, 2016, Kellner and Rösch, 2016). The reason for this development is that the CVaR corrects for several drawbacks of

[39] This is a contrarian approach to Taylor (2005) who uses VaR forecasts based on the CAViaR model and Historical Simulation to obtain estimates of conditional volatility.

[40] Remind that the volatility target is usually denoted as an annualized volatility, whereas the VaR target is chosen as a daily loss. For example, a target VaR strategy with $\alpha = 5\%$ and $\text{VaR}_\alpha^{\text{target}} = 1\%$ can be approximated by a target volatility strategy with an (annualized) volatility target of $\sigma^{\text{target}} = (0.01/1.645) \cdot \sqrt{252} = 9.6\%$.

VaR. VaR has been criticized in the academic literature due to its lack of subadditivity and the disregarding of extreme losses, which are of main interest in risk-management (see Artzner et al. (1999), Giannopoulos and Tunaru (2005, p. 980), McNeil and Frey (2000, p. 291-292) or Yamai and Yoshiba (2005, p. 998)). VaR only contains information on a certain quantile whereas CVaR contains information on the whole right tail of the loss distribution (Du and Escanciano, 2016, p. 942). Moreover, VaR may underestimate risk in times of market stress, i.e. times of high asset price fluctuations (see Yamai and Yoshiba (2005, p. 998) or Du and Escanciano (2016)).[41] The CVaR corrects these drawbacks of VaR, and thus is often claimed as a better risk measure than VaR (see Szegö (2002) and Cuoco et al. (2008) for example).[42] Another disadvantage of managing VaR instead of CVaR is that only the exceedance probability is managed instead of the expected loss magnitude (see Basak and Shapiro (2001, p. 385) and Aït-Sahalia and Brandt (2001)). By managing CVaR, both the exceedance probability and the size of extreme losses are manged. Berkowitz and O'Brien (2002), examining the VaR models of six commercial banks, demonstrate that the size of a VaR violation can be surprisingly large. This is confirmed by the study of Du and Escanciano (2016) who find that VaR responds less to extreme losses such as those experienced during the recent financial crisis. Basak and Shapiro (2001) and Alexander and Baptista (2004) find that managing CVaR is superior to managing VaR in an asset allocation context, especially if a risk free asset is available. Therefore, in this section we extend the target VaR strategy to the target CVaR strategy, that aims to have a constant portfolio CVaR over time. As before, the weight of the risky asset is then a function of the expected CVaR of the risky asset, i.e. if the CVaR of the risky asset is expected to be low, the weight of the risky asset is increased and vice versa.

The CVaR is defined as the average loss in the worst $100 \cdot \alpha\%$ cases, i.e. the cases where the loss exceeds the VaR (see Acerbi and Tasche (2002, p. 1488) or Yamai and Yoshiba (2005,

[41] See also Basak and Shapiro (2001) who demonstrate that in the context of asset allocation decisions VaR can lead to portfolios exhibiting losses that exceed the desired VaR extremely (see also Alexander and Baptista (2004), Berkowitz et al. (2011), Cuoco et al. (2008) and references therein). However, Cuoco et al. (2008) show that this observation does no longer hold once risk is managed dynamically, i.e. taking the actual information into account and reevaluating the risk level dynamically, instead of managing risk by a static model (see also Berkowitz et al. (2011)). This highlights the importance of managing portfolio risk dynamically as done by risk targeting.

[42] See Yamai and Yoshiba (2005) for a good comparison of VaR and CVaR. Moreover, see Szegö (2002, p. 1261) for a list of drawbacks of VaR. See Du and Escanciano (2016) for a good motivation of why CVaR is becoming the more relevant risk measure for managing downside risk.

p. 999)). More formally, we define the portfolio CVaR, denoted by $\mathrm{CVaR}_\alpha^{P,t}$, as

$$\mathrm{CVaR}_\alpha^{P,t} = \mathbb{E}\big(L_t^P \,|\, L_t^P \geqslant \mathrm{VaR}_\alpha^{P,t}, \mathcal{F}_{t-1}\big)\,. \tag{32}$$

In Appendix B.2 we show that the portfolio CVaR is given by

$$\mathrm{CVaR}_\alpha^{P,t} = w_t \cdot \mathrm{CVaR}_\alpha^t - (1 - w_t) \cdot R_t^f\,, \tag{33}$$

where $\mathrm{CVaR}_\alpha^t := \mathbb{E}\big(L_t \,|\, L_t \geqslant \mathrm{VaR}_\alpha^t, \mathcal{F}_{t-1}\big)$ denotes the day t CVaR of the risky asset.[43] In order to achieve a constant portfolio CVaR of $\mathrm{CVaR}_\alpha^{\text{target}}$ over time, i.e. $\mathrm{CVaR}_\alpha^{P,t} = \mathrm{CVaR}_\alpha^{\text{target}}$ for all t, the weight of the risky asset has to be chosen as

$$w_t = \frac{\mathrm{CVaR}_\alpha^{\text{target}} + R_t^f}{\mathrm{CVaR}_\alpha^t + R_t^f}. \tag{34}$$

Due to the definition of CVaR the target CVaR strategy manages expected losses, where the acceptable loss magnitude can be governed by the investor by choosing the values α and $\mathrm{CVaR}_\alpha^{\text{target}}$. For example, a target CVaR level $\mathrm{CVaR}_\alpha^{\text{target}}$ of 2% with a corresponding significance level α of 5% translates into a strategy with an average loss of 2% on the 5% worst days. In other words, the target CVaR strategy's average return on the worst 5 out of 100 days will be -2%.[44] The choices of α and $\mathrm{CVaR}_\alpha^{\text{target}}$ again strongly depend on the individual investor's preferences and risk aversion (Alexander and Baptista, 2004).

Again, since the CVaR of the risky asset is not observable, a forecast of CVaR_α^t is needed. We use the same estimation methods as we used for estimating the VaR of the risky asset. First, and especially interesting for practical implementations, we use Historical Simulation. For this method we again assume that a data set of n realized losses $l_{t-1}, .., l_{t-n}$ with order statistics $l_{(1),t-1} \leqslant l_{(2),t-1} \leqslant ... \leqslant l_{(n),t-1}$ exists. Based on the ordered losses we estimate CVaR_α^t by [45]

$$\widehat{\mathrm{CVaR}}_\alpha^t = \frac{1}{n - \lceil n(1-\alpha) \rceil + 1} \cdot \sum_{j=\lceil n(1-\alpha) \rceil}^{n} l_{(j),t-1}. \tag{35}$$

[43] The representation in Equation (33) again follows by positive homogeneity and translation invariance of CVaR.

[44] By choosing the value $\mathrm{CVaR}_\alpha^{\text{target}}$ and α adequately, this strategy can also be an alternative to absolute return or hedge fund strategies examined in Fung and Hsieh (1997) and Agarwal and Naik (2004).

[45] See Ko et al. (2009, p. 719) or Giannopoulos and Tunaru (2005, p. 985-986).

This estimator is motivated by Acerbi and Tasche (2002, Proposition 4.1),[46] which demonstrates that the estimator in (35) is only unbiased for n converging to infinity. For small n the estimator in Equation (35) is biased. Methods that account for this estimation bias are presented in Ko et al. (2009) among others. As before, the quality of the target CVaR strategy strongly depends on the accuracy of the CVaR estimation.[47] However, since Historical Simulation deals as a benchmark model, which is interesting for practitioners in particular, we keep the estimation as simple as possible.

For the return decomposition, given in Equation (6), the CVaR of the risky asset is given by

$$\mathrm{CVaR}_\alpha^t = \sigma_t \cdot \mathrm{CVaR}_\alpha^*, \tag{36}$$

where we define $\mathrm{CVaR}_\alpha^* := \mathbb{E}\big(L^* \mid L^* \geqslant F_{L^*}^{-1}(1-\alpha)\big)$ and L^* is again a continuously distributed random variable – representing a standardized loss – with expectation zero, variance one and $F_{L^*}^{-1}(1-\alpha)$ denotes the $(1-\alpha)$-quantile of L^* (see McNeil and Frey (2000, p. 276) for example). As for the VaR, we next present the estimation of CVaR based on a two-stage approach, where in the first stage the volatility is estimated by one of the volatility models presented in Equation (7) or (8). In the second stage, we again consider a sample $l_{t-n}^*, \dots, l_{t-1}^*$ of n standardized losses with order statistics $l_{(1),t-1}^* \leqslant \dots \leqslant l_{(n),t-1}^*$. We denote the estimator for $\mathrm{CVaR}_\alpha^* = \mathbb{E}\big(L^* \mid L^* \geqslant F_{L^*}^{-1}(1-\alpha)\big)$ based on the available information at day $t-1$, by $\widehat{\mathrm{CVaR}}_\alpha^{t,*}$. Hence, the estimator for the CVaR of the risky asset on day t, denoted by $\widehat{\mathrm{CVaR}}_\alpha^t$, is given by

$$\widehat{\mathrm{CVaR}}_\alpha^t = \hat{\sigma}_t \cdot \widehat{\mathrm{CVaR}}_\alpha^{t,*}. \tag{37}$$

By using the FHS approach the estimator $\widehat{\mathrm{CVaR}}_\alpha^{t,*}$ is given by Equation (35), where the j-th order statistic $l_{(j),t-1}$ of the loss variables is replaced by the j-th order statistic $l_{(j),t-1}^*$ of the standardized losses (see Giannopoulos and Tunaru (2005) for example). By using the EVT

[46] In this paper we assume that losses are continuously distributed. This is confirmed by Giannopoulos and Tunaru (2005, p. 982) who state that only continuous probability distributions are used in practice. In this case the CVaR, defined as Tail Conditional Expectations (TCE) in Acerbi and Tasche (2002, Definition 2.3), is equal to the Expected Shortfall (ES) (Acerbi and Tasche, 2002, Corollary 5.3).

[47] This is in line with Yamai and Yoshiba (2005, p. 999) who find that the "effectiveness of expected shortfall, however, depends on the accuracy of estimation."

approach the estimator $\widehat{\text{CVaR}}_\alpha^{t,*}$ is given by

$$\widehat{\text{CVaR}}_\alpha^{t,*} = \frac{\hat{F}_{L^*,t}^{-1}(1-\alpha)}{1-\hat{\xi}} + \frac{\hat{\beta}-\hat{\xi}u}{1-\hat{\xi}}, \tag{38}$$

where the estimator $\hat{F}_{L^*,t}^{-1}(1-\alpha)$ is given in Equation (19), u denotes the predetermined threshold and the parameters $\hat{\xi}$ and $\hat{\beta}$ are the QML estimators (see McNeil and Frey (2000, p. 293) or McNeil et al. (2015, p. 154)).

We next also use the skewed t distribution with time-varying parameters to forecast next day's CVaR. Christoffersen (2012) and Rickenberg (2019, Appendix A) show that for $Z_t \sim stsk(\eta, \lambda)$ it holds that

$$\begin{aligned}
&\mathbb{E}\big(Z_t | Z_t < F_{stsk}^{-1}(\alpha|\,\eta,\ \lambda)\big) \\
&= \begin{cases} \frac{1}{\alpha}\frac{(1-\lambda)^2}{b}\left(f_{st}\left(z^{(-)}|\eta\right)\cdot\frac{\eta-2+\left(z^{(-)}\right)^2}{1-\eta} - \frac{a\cdot F_{st}\left(z^{(-)}|\eta\right)}{1-\lambda}\right) & \text{for } F_{stsk}^{-1}(\alpha|\,\eta,\ \lambda) < -\frac{a}{b} \\ \frac{1}{\alpha}\frac{(1+\lambda)^2}{b}\left(f_{st}\left(z^{(+)}|\eta\right)\cdot\frac{\eta-2+\left(z^{(+)}\right)^2}{1-\eta} + \frac{a\cdot\left(1-F_{st}\left(z^{(+)}|\eta\right)\right)}{1+\lambda}\right) & \text{for } F_{stsk}^{-1}(\alpha|\,\eta,\ \lambda) \geqslant -\frac{a}{b}, \end{cases}
\end{aligned} \tag{39}$$

where $z^{(-)}$ and $z^{(+)}$ are given by

$$z^{(-)} = \frac{b\cdot F_{stsk}^{-1}(\alpha|\,\eta,\ \lambda)+a}{1-\lambda}, \qquad z^{(+)} = \frac{b\cdot F_{stsk}^{-1}(\alpha|\,\eta,\ \lambda)+a}{1+\lambda}.$$

Further, $f_{st}(z|\eta)$ and $F_{st}(z|\eta) = \int_{-\infty}^{z} f_{st}(u|\eta)du$ correspond to the pdf and cdf of the *standardized t distribution* with mean zero and variance one. The pdf of the standardized t distribution is given by (see Bollerslev (1987, p. 543) and Hansen (1994, p. 709))

$$f_{st}(z|\eta) = \frac{\Gamma\left(\frac{\eta+1}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)\sqrt{\pi(\eta-2)}}\left(1+\frac{z^2}{\eta-2}\right)^{-(\eta+1)/2}. \tag{40}$$

For the cdf of the t and standardized t distributions it holds $F_{st}(z|\eta) = F_t\left(\sqrt{\frac{\eta}{\eta-2}}z|\eta\right)$.[48] Bollerslev (1987) uses the standardized t distribution in the context of the GARCH(1,1) model and finds that the GARCH(1,1)-t model is superior to both, the GARCH(1,1)-normal and the unconditional t distribution. The forecast of day t's CVaR is then given by Equation (37). In this case, $\widehat{\text{CVaR}}_\alpha^{t,*}$ is given by

$$\widehat{\text{CVaR}}_\alpha^{t,*} = \mathbb{E}\left(Z_t | Z_t < F_{stsk}^{-1}\left(\alpha|\,\hat{\eta}_t,\ \hat{\lambda}_t\right)\right), \tag{41}$$

[48] This relation is advantageous since the cdf of the t distribution is often available in most software packages whereas the cdf of the standardized t distribution is not available (Jondeau and Rockinger, 2003).

where $\hat{\eta}_t$ and $\hat{\lambda}_t$ are again the Maximum Likelihood estimates of η_t and λ_t.

McNeil and Frey (2000, p. 292) find that the quality of the CVaR estimation strongly depends on the model used for the tail of the loss distribution. Correctly modeling the tails, which is crucial for the estimation of VaR, becomes even more important when CVaR is estimated (Yamai and Yoshiba, 2005). Therefore, correctly modeling the tails of the loss distribution is a central issue in achieving a constant portfolio CVaR over time. Again, Kellner and Rösch (2016) find that only models that account for fat tails and/or skewness are able to produce accurate CVaR forecasts.

As in Equation (29), since R_t^f is (typically) small compared to the CVaR values, the weight of the risky asset can be approximated by

$$w_t \approx \frac{\mathrm{CVaR}_\alpha^{\mathrm{target}}}{\mathrm{CVaR}_\alpha^t}. \tag{42}$$

By using Equation (42), similarly to Equation (30), we can approximate a target CVaR strategy by a target volatility strategy. The weight of the risky asset for this target volatility strategy is then given by

$$w_t \approx \frac{\mathrm{CVaR}_\alpha^{\mathrm{target}}}{\mathrm{CVaR}_\alpha^t} = \frac{\mathbb{E}\big(L^* \mid L^* \geqslant F_{L^*}^{-1}(1-\alpha)\big) \cdot \dfrac{\mathrm{CVaR}_\alpha^{\mathrm{target}}}{\mathbb{E}\big(L^* \mid L^* \geqslant F_{L^*}^{-1}(1-\alpha)\big)}}{\mathbb{E}\big(L^* \mid L^* \geqslant F_{L^*}^{-1}(1-\alpha)\big) \cdot \sigma_t} = \frac{\sigma^{\mathrm{target}}}{\sigma_t}, \tag{43}$$

with $\sigma^{\mathrm{target}} = \mathrm{CVaR}_\alpha^{\mathrm{target}} / \mathbb{E}\big(L^* \mid L^* \geqslant F_{L^*}^{-1}(1-\alpha)\big)$. Again, since the distribution of the standardized losses, or at least $\mathbb{E}\big(L^* \mid L^* \geqslant F_{L^*}^{-1}(1-\alpha)\big)$, is not known in practice, the volatility target σ^{target} can not directly be calculated. An approximation can be done by using a standard normal distribution for L^*. Then, $\mathbb{E}\big(L^* \mid L^* \geqslant F_{L^*}^{-1}(1-\alpha)\big)$ is given by $\frac{\varphi(N_{1-\alpha})}{\alpha}$, where φ denotes the density function and $N_{1-\alpha}$ the $(1-\alpha)$-quantile of the standard normal distribution. The volatility target is then given by

$$\sigma^{\mathrm{target}} = \frac{\mathrm{CVaR}_\alpha^{\mathrm{target}}}{\varphi(N_{1-\alpha})/\alpha}. \tag{44}$$

For example, a target CVaR strategy with a significance level of $\alpha = 5\%$ and desired CVaR target $\mathrm{CVaR}_\alpha^{\mathrm{target}} = 2\%$ can be approximated by a target volatility strategy with an annualized volatility target of $\sigma^{\mathrm{target}} = \frac{0.02}{2.063} \cdot \sqrt{252} = 15.4\%$. Further, a constant volatility of 15.4% can be achieved by managing CVaR and thus incorporating skewness and kurtosis. Moreover, by Equations (42) and (15) we obtain

$$w_t \approx \frac{\mathrm{CVaR}_\alpha^{\mathrm{target}}}{\mathrm{CVaR}_\alpha^t} = \frac{\mathrm{CVaR}_\alpha^{\mathrm{target}}}{\sigma_t \cdot F_{L^*}^{-1}(1-\alpha) \cdot \dfrac{\mathbb{E}\big(L^* \mid L^* \geqslant F_{L^*}^{-1}(1-\alpha)\big)}{F_{L^*}^{-1}(1-\alpha)}} = \frac{\mathrm{VaR}_\alpha^{\mathrm{target}}}{\mathrm{VaR}_\alpha^t}, \tag{45}$$

with $\mathrm{VaR}_\alpha^{\mathrm{target}} = \frac{F_{L^*}^{-1}(1-\alpha)}{\mathbb{E}\left(L^* \mid L^* \geqslant F_{L^*}^{-1}(1-\alpha)\right)} \cdot \mathrm{CVaR}_\alpha^{\mathrm{target}}$. Therefore, a target CVaR strategy can be approximated by a target VaR strategy with an adjusted target VaR level.[49] However, the approximation of the target CVaR strategy by a target volatility strategy given in Equation (43) is appealing, since forecasting volatility is much easier than forecasting CVaR, but this argument is only partly valid for the approximation by a target VaR strategy given in Equation (45).[50] Nevertheless, Equation (45) is helpful for comparing target VaR and target CVaR strategies. By assuming a standard normal distribution for L^*, we obtain the comparable target VaR level

$$\mathrm{VaR}_\alpha^{\mathrm{target}} = \frac{N_{1-\alpha}}{\varphi(N_{1-\alpha})/\alpha} \cdot \mathrm{CVaR}_\alpha^{\mathrm{target}}. \tag{46}$$

A target CVaR strategy with $\alpha = 5\%$ and $\mathrm{CVaR}_\alpha^{\mathrm{target}} = 2\%$ should then be compared to a target VaR strategy with $\alpha = 5\%$ and $\mathrm{VaR}_\alpha^{\mathrm{target}} = \frac{1.645}{2.063} \cdot 0.02 = 1.6\%$ which is again approximated by a target volatility strategy with $\sigma^{\mathrm{target}} = \frac{0.016}{1.645} \cdot \sqrt{252} = 15.4\%$.

This approach of comparing a target CVaR strategy with a target volatility or target VaR strategy is similar to the approach of Strub (2013). The author starts with a predefined volatility target and transforms this volatility target to a CVaR target by using a normality assumption. The weight of the risky asset is then obtained as the ratio of the transformed CVaR target and the forecast of the risky asset's CVaR. Strub (2013, p. 16) finds that the volatility and CVaR managed strategies offer a substantial drawdown protection, especially in the years when the underlying index suffers the most. By comparing a volatility managed strategy with a CVaR managed strategy, Strub (2013, p. 17) finds that managing CVaR translates into a better risk-adjusted performance and lower drawdowns. Moreover, he finds that even after transaction

[49] This approach is similar to Cuoco et al. (2008) who show how a VaR limit can be transformed in a CVaR limit and vice versa.

[50] Yamai and Yoshiba (2005, p. 1012) state that the estimation error for CVaR is larger than for VaR, especially when the return distribution exhibits fat tails. Similarly, Kellner and Rösch (2016) find that the model risk for CVaR is higher than for VaR, which is mainly driven by fat tails in the return distribution. Especially in times of financial market turmoils like the recent financial crisis, CVaR forecasts among different models are more volatile than VaR forecasts.

costs the risk managed strategies still deliver convincing performances, which makes the risk managed strategies interesting for practical applications and an interesting alternative to hedge fund strategies as examined in Fung and Hsieh (1997) and Agarwal and Naik (2004). Similarly, the CVaR-managed strategy of Wang et al. (2012) reduces drawdowns without sacrificing returns and hence captures the upside potential, while downside risk is limited (Wang et al., 2012, p. 38).[51] This is in line with Basak and Shapiro (2001) who also find convincing results by managing expected losses, as done by managing CVaR, and conclude that managing expected losses is superior to managing exceedance probabilities, as done by managing VaR (see also Aït-Sahalia and Brandt (2001, p. 1316)).

3.4 VaR and CVaR Targeting as Optimal Trading Strategies under Risk Limits

In this section we motivate the VaR and CVaR targeting strategies from another perspective as optimal trading strategies where a trader faces a risk limit as done by Basak and Shapiro (2001), Wang et al. (2012), Cuoco et al. (2008) and Alexander and Baptista (2004).[52] We again consider a trader who invests w_t in the risky and $1 - w_t$ in the riskless asset and define the trader's portfolio value by $W_t := W_{t-1} \cdot (1 + R_t^P)$, $W_0 > 0$. Further, we define the absolute loss in t by $L_t^{abs} := W_{t-1} - W_t = -W_{t-1} \cdot R_t^P$. We now consider a portfolio optimization problem under an (absolute) risk limit $\overline{\mathrm{VaR}}_t$ given by

$$\max_{w_t} \mathbb{E}\big(R_t^P \mid \mathcal{F}_{t-1}\big) \quad \text{s.t.} \quad \mathrm{VaR}_\alpha^{t,abs} \leqslant \overline{\mathrm{VaR}}_t, \tag{47}$$

where $\mathrm{VaR}_\alpha^{t,abs} = W_{t-1} \cdot \mathrm{VaR}_\alpha^{P,t}$ denotes the VaR of the absolute loss L_t^{abs}. The risk limit $\mathrm{VaR}_\alpha^{t,abs} \leqslant \overline{\mathrm{VaR}}_t$ can then be rewritten as $\mathrm{VaR}_\alpha^{P,t} \leqslant \overline{\mathrm{VaR}}_t / W_{t-1}$. From Equation (12) it follows that the risk limit holds if

$$w_t \leqslant \frac{\overline{\mathrm{VaR}}_t / W_{t-1} + R_t^f}{\mathrm{VaR}_\alpha^t + R_t^f}. \tag{48}$$

Under the assumption $\mathbb{E}(R_t \mid \mathcal{F}_{t-1}) > R_f^f$, which is typically fulfilled in practice (Benartzi and Thaler, 1995), the expected portfolio return $\mathbb{E}\big(R_t^P \mid \mathcal{F}_{t-1}\big)$ is increasing in w_t. Hence, the

[51] Wang et al. (2012) call their strategy a target CVaR strategy. However, the authors do not target a constant level of risk over time, but allow a maximum level of risk (see also Basak and Shapiro (2001) and Alexander and Baptista (2004)).

[52] I thank Peter Albrecht and Markus Huggenberger for this helpful comment.

investor chooses the highest possible equity exposure that still fulfills the risk limit $\mathrm{VaR}_{\alpha}^{t,abs} \leqslant \overline{\mathrm{VaR}}_t$. Thus, w_t is given by

$$w_t = \frac{\overline{\mathrm{VaR}}_t / W_{t-1} + R_t^f}{\mathrm{VaR}_{\alpha}^t + R_t^f}. \tag{49}$$

Consequently, by choosing a constant relative risk limit $\overline{\mathrm{VaR}}_t / W_{t-1} = \mathrm{VaR}_{\alpha}^{\mathrm{target}}$, the target VaR strategy follows as optimal dynamic trading strategy under a risk limit. By the same arguments as above, the weighting for the target CVaR strategy can be obtained if an investor faces a CVaR limit.

4 Assessing the Accuracy of Target Risk Strategies

In this section we present methods to test the accuracy of the target risk strategies, i.e. we test if the risk models are successful in targeting a constant level of portfolio risk over time. A constant portfolio risk is important for several reasons. First, a constant risk of the strategies should be achieved by definition of risk targeting. Second, an investor who chose a fund that targets a volatility level that fits to his risk preferences would divest if it achieves a significantly higher volatility as expected. Similarly, an investor who expects only a limited number of days where the portfolio return is smaller than $-\mathrm{VaR}_{\alpha}^{\mathrm{target}}$ would also divest if the fund exhibits too many extremely negative returns. Third, risk-averse investors are willing to pay for hedges against changing portfolio volatility (Adrian and Rosenberg, 2008, Ang et al., 2006b, Bollerslev and Todorov, 2011). These investors are willing to pay higher fees for strategies with a more constant portfolio risk. Fourth, several studies show that a higher forecast accuracy, and hence a more constant portfolio risk, coincides with higher (risk-adjusted) performance and utility gains (Bollerslev et al., 2018, Fleming et al., 2003, Marquering and Verbeek, 2004, Moreira and Muir, 2017, Taylor, 2014). Consequently, a fund that fails to target a constant risk over time typically achieves a sub optimal risk-return profile. For example, Bollerslev et al. (2018, p. 2732) write:

> “the investor achieves the maximum utility by successfully targeting a constant risk level, while the utility decreases with the volatility-of-volatility. Hence, risk models that help the investor achieve more accurate volatility forecasts are associated with higher levels of utility”

Bollerslev et al. (2018) find that an investor who uses volatility targeting is willing to pay a fee of 0.48% per year to switch from an inaccurate to a more accurate volatility model. The authors find that there exists a positive, non-linear relation between forecasting accuracy of volatility models and utility benefits. Further, they find that a model with perfect foresight, i.e. a model that produces a totally constant portfolio volatility over time, exhibits the highest utility benefit (see also Benson et al. (2014)). Similarly, in a cross-sectional setting, Baltussen et al. (2018) find that assets with a high volatility of volatility (vol-of-vol) underperform assets with a more constant volatility. Further, higher vol-of-vol assets also exhibit higher downside risk. This especially holds during down markets when high vol-of-vol assets underperform low vol-of-vol assets by 0.83% per month. Further, targeting a constant level of portfolio risk is also frequently used by practitioners (Barroso and Santa-Clara, 2015, p. 112). Consequently, forecasting accuracy is an important driver of the investor's benefit of risk targeting and should therefore be tested. Besides backtesting the accuracy of volatility targeting we additionally show how the accuracy of VaR and CVaR targeting can be tested. To assess the accuracy of several volatility models, Bollerslev et al. (2018) use the R^2 as well as the DM-test of Diebold and Mariano (1995) which tests for equal predictive ability. However, both methods have several disadvantages. Therefore, we use more powerful tools to assess the accuracy of volatility targeting as presented in the next section.

4.1 Assessing the Accuracy of Volatility Targeting

Although several studies on volatility targeting have been made, only a few studies statistically assess if it is possible to achieve the desired volatility target over time. To assess the accuracy of volatility targeting for some set of models $\mathcal{M}$, we measure the portfolio variance of model k on day t by $RV_{k,t}^2 := w_{k,t}^2 \cdot RV_t^2$, where $w_{k,t}$ is the weight of strategy k, $k \in \mathcal{M}$, on day t and RV_t denotes the Realized Volatility on day t of the risky asset (see Andersen et al. (2001), Patton (2011) or Bollerslev et al. (2018) for a definition of RV_t).[53] Motivated by Hansen and

[53] The Realized Volatility data are downloaded from the Oxford Man Realized Library (https://realized.oxford-man.ox.ac.uk/). As in Hansen and Lunde (2005) we scale the Realized Volatility to a measure of the close-to-close volatility of day t. Bollerslev et al. (2018) simply add the squared overnight return to the Realized Volatility to obtain a measure for the whole day's variance. However, both methods deliver similar results.

Lunde (2005) and Patton (2011) we define the QLIKE loss function of model k on day t by

$$L_{k,t} := L\left(RV_{k,t}^2, \sigma_{\text{target,d}}^2\right) := \frac{RV_{k,t}^2}{\sigma_{\text{target,d}}^2} - \ln\left(\frac{RV_{k,t}^2}{\sigma_{\text{target,d}}^2}\right) - 1, \tag{50}$$

where $\sigma_{\text{target,d}} = \sigma^{\text{target}}/\sqrt{252}$ (see Christoffersen (2012, p. 85) and Taylor (2014, p. 475)). Patton (2011) shows that the QLIKE and the MSE loss functions are robust against noise in the volatility proxy. The MSE relies on the absolute forecast error whereas the QLIKE relies on the relative forecast error. We choose the QLIKE instead of the MSE since the QLIKE penalizes models that underestimate risk and hence produce a portfolio volatility that is too high. We use a slightly different representation than that used by Hansen and Lunde (2005) and Patton (2011), which is also used by Christoffersen (2012). This representation has the advantage that $L\left(RV_{k,t}^2, \sigma_{\text{target,d}}^2\right) = L\left(RV_t^2, \sigma_t^2\right)$ holds, which is the usual choice that is made when volatility forecasts are evaluated. Moreover, our loss function is normalized in the sense that $L_{k,t} = 0$ holds if the portfolio volatility on day t equals the desired volatility target, whereas the representation of Hansen and Lunde (2005) and Patton (2011) is not normalized. In particular, our representation still fulfills Proposition 1 of Patton (2011) by choosing $C(z) = \frac{1}{z}, \tilde{C}(z) = \log(z)$ and $B(z) = -\log(z)$. Thus, our representation is a robust loss function in the sense of Patton (2011, Definition 1), and hence is robust against noise in the volatility proxy. Further, Patton (2011) shows that using squared daily returns instead of Realized Volatility leads to quite similar conclusions. We also used squared daily returns instead of the Realized Volatility in the empirical part and found similar results for both methods.

We next define the relative loss between model i and j as $X_{ij,t} = L_{i,t} - L_{j,t}$ and the average relative loss as

$$\overline{X}_{ij} = \frac{1}{T}\sum_{t=1}^{T} X_{ij,t}. \tag{51}$$

The basic idea of testing for predictive accuracy is that a positive value of $\overline{X}_{ij}$ indicates that model j is more accurate than model i, i.e. model j is more successful in targeting a constant level of volatility.

To test for the accuracy of the different target volatility strategies we apply the test for equal predictive ability (DM-test) of Diebold and Mariano (1995) which was also used by Patton

(2011) and Bollerslev et al. (2018). Further, we use the Reality Check (RC-test) of White (2000) and Sullivan et al. (1999) and its extension, the test for Superior Predictive Ability (SPA-test) of Hansen (2005) and Hansen and Lunde (2005). Contrary to the DM-test both, the RC- and SPA-test, test for superior predictive ability and can also be applied to more than two models simultaneously. Both tests test the null-hypothesis that a chosen benchmark model is more accurate than all the remaining models. Moreover, we use the stepwise extensions of the RC-test and SPA-test presented in Romano and Wolf (2005) and Hsu et al. (2010), which we denote by Step-RC and Step-SPA, respectively. These approaches can be used to construct sets of models that are superior to a chosen benchmark model. Similarly, we also use the algorithm based on the False Discovery Rate (FDR) presented in Barras et al. (2010) and Bajgrowicz and Scaillet (2012). The authors show how the FDR can be used to identify models that are superior to a chosen benchmark model. Finally, we use the Model Confidence Set (MCS) of Hansen et al. (2011) and Hansen et al. (2003), where we mainly follow Hansen et al. (2003) who also applied this algorithm to assess the accuracy of volatility models. The MCS also identifies a set of superior models and has the advantage that no benchmark model is needed. A short summary of the tests can be found in Rickenberg (2019, Appendix C).

For the DM-test, the Step-RC, Step-SPA and the FDR approach a certain benchmark model has to be chosen to which the alternative models are compared. As benchmark model we choose the easiest one which is the HSD. This model is similar to the one used by Barroso and Santa-Clara (2015), Barroso and Maio (2016) and Moreira and Muir (2017). When applying the RC- and SPA-test we choose each model once as the benchmark and test if this benchmark is outperformed by any other model. Romano and Wolf (2005) and Bajgrowicz and Scaillet (2012) argue that doing this has several disadvantages that are corrected by the Step-RC, Step-SPA, the MCS and the FDR approaches.

4.2 Assessing the Accuracy of VaR Targeting

In Section 3 we have presented several VaR forecasting methods and we have shown how these VaR forecasting methods can be used to derive the weight w_t of the risky asset in a target VaR strategy. Moreover, we have shown how a target VaR strategy can be approximated by a target volatility strategy with an adjusted target volatility level given in Equation (30). Next we want to asses the quality of the forecasting methods and compare the "true" target VaR strategies based on a proper VaR forecast for the risky asset with the approximated target VaR strategies, that are based on the risky asset's volatility solely. In other words, we want to assess if the different target VaR strategies succeed to produce a constant portfolio VaR over time and if controlling volatility is sufficient for this task. Similarly, Christoffersen and Diebold (2000) show how VaR backtesting methods can be used to backtest the accuracy of volatility models.

To assess the quality of the target VaR strategies we define the hit variables

$$H_t^P = \begin{cases} 1, & \text{if } L_t^P > \mathrm{VaR}_\alpha^{\text{target}} \\ 0, & \text{if } L_t^P \leqslant \mathrm{VaR}_\alpha^{\text{target}} \end{cases} \tag{52}$$

i.e. H_t^P is equal to one if the portfolio loss is higher than the VaR target $\mathrm{VaR}_\alpha^{\text{target}}$, called a hit, and zero else. An accurate target VaR strategy should exhibit two abilities. First, the percentage of days when the portfolio loss is higher than the predefined VaR target, i.e. the proportion of hits in the hit-series $\{H_t^P\}_{t=1}^T$, should be equal to the desired significance level α. Second, the days when the portfolio loss is higher than the VaR target should occur randomly over time and should not be clustered (see Berkowitz and O'Brien (2002, p. 1101) and Berkowitz et al. (2011, p. 2217)). Assume the hits of a target VaR strategy occur clustered on many subsequent days, i.e. the portfolio losses are higher than the predefined VaR target on every day in a certain period. As a consequence, investors would remove money from a fund using this strategy, since this strategy seems to fail the aim of having a constant VaR over time.[54]

To test these two abilities we resort to the VaR backtesting method of Christoffersen (1998), which is one of the most widely used VaR backtests in the academic literature (Du and Escanciano, 2016).[55] In Appendix C.1 we show that the variable H_t^P is equivalent to

$$H_t = \begin{cases} 1, & \text{if } L_t > \mathrm{VaR}_\alpha^t \\ 0, & \text{if } L_t \leqslant \mathrm{VaR}_\alpha^t, \end{cases} \tag{53}$$

[54] Besides this economic importance of independent hits, this ability should also hold by definition of VaR. See for example McNeil et al. (2015, Lemma 9.5) who show that the process of hit variables is a process of iid Bernoulli random variables with probability α (see also Christoffersen (1998), Berkowitz and O'Brien (2002) and Berkowitz et al. (2011)).

[55] See also Berkowitz and O'Brien (2002), Berkowitz et al. (2011, p. 2217) and Kuester et al. (2006, Sec. 2) for a short overview of this backtesting procedure.

i.e. the hit variable based on the losses and VaRs of the risky asset solely, which are used in the backtest of Christoffersen (1998). Consequently, the backtesting approach of Christoffersen (1998) can directly be adopted for the variables H_t^P. Moreover, this result directly provides critical values which allows us to draw conclusions on the accuracy of the target VaR strategies.[56] The backtest of the target VaR strategy is then formed with the variables

$$\hat{H}_t^P = \begin{cases} 1, & \text{if } l_t^P > \mathrm{VaR}_\alpha^{\text{target}} \\ 0, & \text{if } l_t^P \leqslant \mathrm{VaR}_\alpha^{\text{target}}, \end{cases} \tag{54}$$

where l_t^P is the realized portfolio loss on day t. The first above mentioned ability, i.e. the correct hit proportion, is then tested with the unconditional coverage test. The second ability, i.e. independence of the hits, is tested with the test of independence and both abilities are simultaneously tested by the conditional coverage test (Christoffersen, 1998).

4.3 Assessing the Accuracy CVaR Targeting

For backtesting the target CVaR strategy we again use backtesting methods developed in the context of CVaR forecasting. For backtesting CVaR there does not exist a common backtesting procedure (Du and Escanciano, 2016). Further, backtesting CVaR is more challenging than backtesting VaR. Therefore, we will use two different CVaR backtesting procedures that help us to draw more sound conclusions on the accuracy of the target CVaR strategies.[57] For this purpose, as first CVaR backtest, we use the CVaR backtesting procedure described in McNeil and Frey (2000, Section 4.3). This backtesting method compares the loss of the risky asset with the CVaR of the risky asset and is based on the result that the variables

$$X_t = \frac{L_t - \mathrm{CVaR}_\alpha^t}{\sigma_t} = L_t^* - \mathrm{CVaR}_\alpha^{t,*} \tag{55}$$

are iid with expectation zero, given the loss L_t exceeds VaR_α^t. Based on this result a backtest procedure using a distribution free bootstrap is derived. However, in this paper we are interested in the (normalized) difference between the portfolio loss and target CVaR level $\mathrm{CVaR}_\alpha^{\text{target}}$, i.e.

[56] Christoffersen (1998) shows that, under the null hypothesis, the test statistic asymptotically follows a χ^2 distribution.

[57] Both backtests used in this paper are unconditional backtests which are less powerful than conditional backtests (Du and Escanciano, 2016). However, opposed to the VaR backtesting literature, there does not exist a widely used conditional CVaR backtesting method.

we are interested in the ratio

$$X_t^P = \frac{L_t^P - \mathrm{CVaR}_\alpha^{\mathrm{target}}}{\sqrt{\mathrm{var}(R_t^P \mid \mathcal{F}_{t-1})}}, \tag{56}$$

where we normalize these differences by the portfolio volatility. In Appendix C.2 we show that X_t^P equals X_t and thus, given $L_t^P - \mathrm{VaR}_\alpha^{t,P} > 0$, X_t^P should be iid with expectation zero as well. Hence, we can adopt the backtest procedure of McNeil and Frey (2000, Section 4.3) for the variables X_t^P. The backtest is then formed using the realizations

$$x_t^P = \frac{l_t^P - \mathrm{CVaR}_\alpha^{\mathrm{target}}}{w_t \cdot \hat{\sigma}_t}, \tag{57}$$

where l_t^P denotes the day t realized portfolio loss based on the weight w_t. If the weight w_t of the risky asset is estimated correctly, the sample

$$\left\{ x_t^P : t = 1, ..., T, l_t^P > \widehat{\mathrm{VaR}}_\alpha^{t,P} \right\} \tag{58}$$

should behave like an iid sample with mean zero.[58]

As second target CVaR backtesting procedure we use the backtest derived in Embrechts et al. (2005). We again consider the days, where the portfolio loss is higher than the portfolio VaR, i.e. we consider the days where $L_t^P > \mathrm{VaR}_\alpha^{t,P}$ holds. In these cases, stemming from the definition of CVaR, the mean between the portfolio loss and the portfolio CVaR should be zero. Since the portfolio CVaR should be equal to $\mathrm{CVaR}_\alpha^{\mathrm{target}}$ over time, the measure

$$V_1 = \frac{\sum_{t=1}^T \left(L_t^P - \mathrm{CVaR}_\alpha^{\mathrm{target}} \right) \cdot \mathbb{1}_{\{L_t^P > \mathrm{VaR}_\alpha^{t,P}\}}}{\sum_{t=1}^T \mathbb{1}_{\{L_t^P > \mathrm{VaR}_\alpha^{t,P}\}}} \tag{59}$$

should exhibit a low absolute value (Embrechts et al., 2005, p. 72).[59] Nevertheless, Embrechts et al. (2005) argue that the measure V_1 has the drawback that it relies on an estimate of the

[58]We standardize the strategies that rely on Historical Simulation by the HSD volatility. Moreover, backtesting the target CVaR strategies for which a proper VaR forecast, and hence portfolio VaR, exists is straightforward. For the strategies that are only based on a volatility forecast, the time series of portfolio VaR is not available. Since in these cases the target volatility level is derived by assuming a normal distribution for Z_t, we solve this problem in the following way. If a target CVaR strategy relies on a volatility forecast $\hat{\sigma}_t$ solely, we estimate the corresponding VaR by $\widehat{\mathrm{VaR}}_\alpha^t = \hat{\sigma}_t \cdot N_{1-\alpha}$, i.e. again assuming that Z_t follows a standard normal distribution. Then, the portfolio VaR is given by $\widehat{\mathrm{VaR}}_\alpha^{t,P} = w_t \cdot \hat{\sigma}_t \cdot N_{1-\alpha} - (1 - w_t) \cdot R_t^f$. An alternative would be to use the VaR target $\mathrm{VaR}_\alpha^{\mathrm{target}}$ as proxy for the portfolio VaR, i.e. $\widehat{\mathrm{VaR}}_\alpha^{t,P} = \mathrm{VaR}_\alpha^{\mathrm{target}}$.

[59]For the volatility based strategies we again use $\widehat{\mathrm{VaR}}_\alpha^{t,P} = w_t \cdot \hat{\sigma}_t \cdot N_{1-\alpha} - (1 - w_t) \cdot R_t^f$ as forecast for the portfolio VaR in this backtest.

portfolio VaR. In the definition of the measure V_1 the worst cases are defined as the days when the portfolio loss exceeds the estimated portfolio VaR. If the risky asset's VaR forecast, and hence by Equation (12) the portfolio VaR, is not credible, the validity of the measure V_1 is doubtful. To account for this, the authors propose a second measure V_2 that does not rely on a proper forecast of VaR. The motivation of this measure stems from the interpretation, that the CVaR is the expected loss in the α "worst" cases. Therefore, we denote the difference between the portfolio loss and the CVaR target by $D_t := L_t^P - \mathrm{CVaR}_\alpha^{\mathrm{target}}$. Then, we define the worst α cases as the $100 \cdot \alpha\%$ highest differences D_t, i.e. we define the worst cases as the cases when the target CVaR level is exceeded the most. This has the advantage that the worst cases do not depend on an estimate of the VaR, where we do not know if this estimate is credible. We denote the $(1-\alpha)$-quantile of $\{D_t\}_{t=1}^T$ by $D^{1-\alpha}$ and calculate V_2 by

$$V_2 = \frac{\sum_{t=1}^T D_t \cdot \mathbb{1}_{\{D_t > D^{1-\alpha}\}}}{\sum_{t=1}^T \mathbb{1}_{\{D_t > D^{1-\alpha}\}}}. \tag{60}$$

Again, for a successful target CVaR strategy the absolute value of V_2 should be low. As a third measure, denoted by V, Embrechts et al. (2005, p. 72) combine the measures V_1 and V_2 and define

$$V = \frac{|V_1| + |V_2|}{2}, \tag{61}$$

which again should be low for a good target CVaR strategy.

5 Empirical results

5.1 Data

To evaluate the performance of the different target risk strategies and to backtest the ability of achieving a constant level of portfolio risk over time we use data for the DAX Performance Index as risky asset. As a proxy for the risk free rate we use the three month Euribor.[60] The data range from 01.01.2000 to 31.12.2018 and are obtained from Datastream. Although many

[60]This is similar to Marquering and Verbeek (2004) who use the S&P 500 as risky asset and the three month US T-bill rate as risk free asset to examine the economic value of volatility timing in the US market. For risk targeting it is important to frequently reallocate the weight of the risky asset. Hence, it is crucial to use a highly liquid asset as underlying risky asset since times of increasing volatility, which induce a portfolio reallocation, typically coincide with times of lower market liquidity (Ang et al., 2006b).

studies on investment or fund strategies use monthly data, we use daily data, since daily data better capture the dynamics of the financial markets and are more close to the manner how funds are managed (see Busse (1999, p. 1015) and Karolyi and Stulz (1996, p. 952)). Further, even long-term investors typically reevaluate their portfolio frequently on short horizons (Benartzi and Thaler, 1995) and should also time short-term volatility (Moreira and Muir, 2019). Since extreme price changes can occur during short time intervals focusing on daily return data is also beneficial to better manage potential extreme events (Longin, 2000, p. 1104). Most studies on risk targeting – or more precisely volatility targeting – use data for the S&P 500, whereas risk targeting for German stocks is not examined so far.[61] Some additional results for US data and small caps, proxied by the S&P 500 and the German small cap index SDAX, are given in Appendix D. The chosen period is marked by changing periods of low and high risk containing the collapse of the tech bubble, the global financial crisis and the European debt crisis, but also times of continuously up-trending markets. This illustrates how risk targeting works in different market environments and whether the models are successful in adapting to changing market regimes. Dopfel and Ramkumar (2013) demonstrate how important portfolio risk management was in the financial crisis, where volatility managing delivers higher returns with lower volatility. A well performing strategy should limit the downside, while the upside potential is captured as found for many hedge fund strategies (Fung and Hsieh, 1997). We also show in Appendix D how risk targeting works for a longer data set that covers about 88 years. As in Kellner and Rösch (2016) we use an estimation window of $n = 1000$ days for Historical Simulation, FHS, EVT, the skewed t distribution and for estimating the GARCH(1,1) parameters. The HSD is estimated with an estimation window of $m = 30$ days. As benchmark portfolios for the risk targeting strategies we use two buy-and-hold investment strategies. The first benchmark strategy is fully invested in the risky asset, i.e. $w_t = 1$ for all t and the second strategy initially invests $w_0 = 60\%$ of wealth in the risky asset and the remaining $1 - w_0 = 40\%$ in the risk-free asset without rebalancing the weights over time. Benartzi and Thaler (1995) state that portfolios that contain approximately 50% stocks and 50% bonds are optimal for loss-averse investors. Simi-

[61] Packham et al. (2017) examine data for German stocks as well but in a slightly different setting. Barroso and Santa-Clara (2015) examine volatility targeting for a momentum portfolio consisting of German stocks. Ang et al. (2009) examine the low volatility anomaly internationally including Germany.

larly, Ang et al. (2005, Fig. 3) find that such portfolios are also held by moderately risk-averse investors. Further, 60/40 portfolios are frequently used by pension funds (Benartzi and Thaler, 1995, p. 87) for which risk targeting can be an interesting alternative.

To better manage extreme losses and to better mitigate drawdowns we choose a low significance level of $\alpha = 0.5\%$ for the target VaR and CVaR strategies. Low significance levels are frequently used in practice and are important from a regulatory perspective. For example, the Bank of Internal Settlements has set the significance level to 1% for measuring market risk and only 0.1% for credit risk. Further, a significance level of $\alpha = 0.5\%$ is also set to calculate the Solvency Capital Requirement under Solvency II. Bali et al. (2008) also use a significance level of 0.5% in VaR forecasting. Happersberger et al. (2019) find better result for downside risk managed strategies when a lower α is chosen. Further, Ghysels et al. (2016) find that skewness information is hidden in the distribution's tails and that this "tail skewness" is important to determine the portfolio allocation. Thus, lower significance levels should be chosen to better capture skewness risk. Additional results for significance levels of $\alpha = 1\%, 2.5\%$ and 5% are given in Appendix D. As in Barroso and Santa-Clara (2015) and Barroso and Maio (2016) we choose an annualized volatility target of $\sigma^{\text{target}} = 12\%$. By using Equations (31) and (46) we obtain VaR and CVaR target levels of $\text{VaR}_{\alpha}^{\text{target}} = 1.9471\%$ and $\text{CVaR}_{\alpha}^{\text{target}} = 2.1861\%$ for a significance level of $\alpha = 0.5\%$.[62]

5.2 Testing the Accuracy of Target Risk Strategies

We start by assessing the accuracy of the different target risk strategies. By definition the aim of the target risk strategies is to achieve a predefined level of portfolio risk constantly over time. In particular, we are interested in the question if more advanced models produce a more constant portfolio risk over time and what kind of risk – volatility, VaR or CVaR – an investor should manage if the investor targets a predefined level of volatility, VaR or CVaR, respectively.

[62] We have chosen the same α for both, the target VaR and target CVaR strategies, but different target risk levels. Another possibility would be to choose the same target level, but different significance levels as in Alexander and Baptista (2004, p. 1262), i.e. $\text{VaR}_{\alpha}^{\text{target}} = \text{CVaR}_{\tilde{\alpha}}^{\text{target}}$ with $\alpha < \tilde{\alpha}$. Du and Escanciano (2016) suggest that the significance level for CVaR should be about twice the significant level of VaR, i.e. $2\alpha \approx \tilde{\alpha}$. For example, a significance level of $\tilde{\alpha} = 5\%$ for the target CVaR strategy requires a significance level of about $\alpha = 1.96\%$ for the target VaR strategy to guarantee that both strategies have the same target risk level when Equations (31) and (46) are used.

Further, we are interested in the question if managing volatility is sufficient or if incorporating higher moments, as done by managing VaR and CVaR, leads to a higher accuracy as found by Taylor (2005) in a different setting. Testing the accuracy of the different risk models is also important from an economical perspective, since previous studies have shown that a higher forecasting accuracy coincides with a higher risk-adjusted performance and economic value in terms of utility gains (Bollerslev et al., 2018, Fleming et al., 2003, Marquering and Verbeek, 2004, Moreira and Muir, 2017, Taylor, 2014). Consequently, a high forecasting accuracy, and hence a more constant portfolio risk, is beneficial for risk targeting. We first test the accuracy of the strategies when the investor's aim is to target a certain level of portfolio volatility over time. Whenever a benchmark model is needed we choose the HSD as benchmark, which we denote by model 0, to assess if more advanced models are more successful in volatility targeting than the model used in Barroso and Santa-Clara (2015), Moreira and Muir (2017) and Barroso and Maio (2016). This model is then tested against the remaining models $k = 1, ..., 16$. Bollerslev et al. (2018) find that more advanced models produce more accurate forecasts and higher utility gains than static forecasting models like HSD.

Tables I and II give the results for the tests presented in Section 4.1, where Table I shows the results for the DM-, RC- and SPA-test. The first column of Table I contains the average loss of all models, normalized by the average loss of the HSD model, i.e. $\overline{L}_k^{norm} = \frac{\frac{1}{n}\sum_{t=1}^{T} L_{k,t}}{\frac{1}{n}\sum_{t=1}^{T} L_{0,t}}$, $k = 1, ..., 16$. A normalized loss smaller than 100% indicates that model k is more accurate on average, whereas values greater than 100% indicate that the HSD model is more successful in achieving a constant portfolio volatility. Table I clearly shows that the dynamic volatility models, i.e. the EWMA and GARCH based target volatility strategies, are the most accurate models, whereas managing VaR or CVaR typically leads to a less accurate portfolio volatility. In particular, the Historical Simulation managed strategies (VaR-HS and CVaR-HS) are the least accurate models when the aim is to target volatility. Further, the DM-test indicates that most of the CVaR models are significantly less accurate in targeting a constant level of volatility with values of less than -1.64. When using the RC test only three strategies – EWMA, GARCH and VaR-GARCH-FHS – can not be rejected. The RC-test tests if a chosen benchmark model

Table I. Testing the accuracy of volatility targeting

This table contains the results of the tests of predictive accuracy presented in Section 4.1. $\overline{L}_k^{norm} = \frac{\frac{1}{n}\sum_{t=1}^{n} L_{k,t}}{\frac{1}{n}\sum_{t=1}^{n} L_{0,t}}$ defines the average loss of model k normalized by the loss of model 0 and is given in percent. DM-test stands for the test statistic of the Diebold and Mariano (1995) test. The null-hypothesis of equal predictive ability is rejected for $|\text{DM-test}| > 1.64$, where positive values indicate that model k is more accurate than the HSD model. Bold numbers of DM-test indicate that the model is significantly superior to the HSD model. $p^{RC,n}$ and p^{RC} stand for the naive p-value and p-value of the RC-test of White (2000) and Sullivan et al. (1999). $p^{SPA,n}$ and $p^{SPA,c}$ stand for the naive p-value and p-value for the SPA-test of Hansen (2005) and Hansen and Lunde (2005). $p^{SPA,l}$ and $p^{SPA,u}$ give lower and upper bounds for the p-value of the SPA-test. Bold numbers of these tests indicate that the null-hypothesis that model k is the best model cannot be rejected at a test level of 10%. All p-values are given in per cent.

Model	$\overline{L}_k^{norm}$	DM-test	$p^{RC,n}$	p^{RC}	$p^{SPA,n}$	$p^{SPA,l}$	$p^{SPA,c}$	$p^{SPA,u}$
Vola Hist	100.00	-	0.00	0.45	0.00	0.00	0.00	0.00
Vola EWMA	83.47	**5.49**	3.07	**64.48**	3.07	3.07	3.07	**13.26**
Vola GARCH	80.88	**4.04**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**	**100.00**
VaR Hist	278.33	-7.43	0.00	0.00	0.00	0.00	0.00	0.00
VaR EWMA FHS	98.44	0.31	0.00	0.75	0.00	0.00	0.00	0.00
VaR EWMA EVT	108.13	-1.35	0.00	0.00	0.00	0.00	0.00	0.00
VaR EWMA Stsk	111.72	-2.41	0.00	0.00	0.00	0.00	0.00	0.00
VaR GARCH FHS	89.25	**1.97**	0.00	**11.94**	0.00	0.00	0.00	0.00
VaR GARCH EVT	98.76	0.20	0.00	0.28	0.00	0.00	0.00	0.00
VaR GARCH Stsk	119.79	-2.79	0.00	0.00	0.00	0.00	0.00	0.00
CVaR Hist	273.18	-7.72	0.00	0.00	0.00	0.00	0.00	0.00
CVaR EWMA FHS	119.44	-2.74	0.00	0.00	0.00	0.00	0.00	0.00
CVaR EWMA EVT	124.49	-3.32	0.00	0.00	0.00	0.00	0.00	0.00
CVaR EWMA Stsk	142.35	-5.76	0.00	0.00	0.00	0.00	0.00	0.00
CVaR GARCH FHS	100.38	-0.06	0.00	0.11	0.00	0.00	0.00	0.00
CVaR GARCH EVT	106.77	-1.03	0.00	0.00	0.00	0.00	0.00	0.00
CVaR GARCH Stsk	149.67	-5.88	0.00	0.00	0.00	0.00	0.00	0.00

is at least as accurate as all the remaining models. If for a model the null-hypothesis cannot be rejected, i.e. the p-value is higher than the chosen test level of 10%, there is no indication that any other model is more successful in targeting a constant level of portfolio volatility over time than this model. The SPA-test, which extends the RC-test by using a studentized test statistic and a sample dependent null distribution, is typically more powerful in determining inferior models (Hansen, 2005, Hansen and Lunde, 2005). This is confirmed by our results, since more null-hypotheses are rejected. The SPA-test rejects all null-hypotheses of superior predictive ability except for the null-hypothesis when the GARCH model is used as benchmark model. Concluding, Table I shows that the dynamic volatility models produce the most accurate portfolio volatility.

Table II shows the sets of superior models identified by the stepwise RC-test, the stepwise SPA-test, the MCS and the FDR approach. Whenever a benchmark model is needed we choose the HSD model as benchmark strategy. The MCS has the advantage that no benchmark model has to be chosen. The MCS, which is an extension of the SPA-test, produces similar results

Table II. Sets of accurate volatility targeting models

This table contains the results of the stepwise RC-test of Romano and Wolf (2005), the stepwise SPA-test of Hsu et al. (2010), the MCS of Hansen et al. (2003) and Hansen et al. (2011) as well as the FDR method of Barras et al. (2010) and Bajgrowicz and Scaillet (2012). p^R and p^{SQ} stand for the p-values of the MCS of Hansen et al. (2003) and Hansen et al. (2011) and are given in per cent. Bold values indicate that the model is contained in the MCS for a test level of 10%. Step-RC and Step-RCst contain the step in which the model is added to the set of superior models using the stepwise multiple testing of Romano and Wolf (2005), where Step-RCst uses a studentized test statistic. Step-SPA and Step-SPAst contain the step in which the model is added to the set of superior models using the stepwise multiple testing of Hsu et al. (2010), where Step-SPAst uses a studentized test statistic. A value of zero means that the model is not added to the set of superior models. The tests are performed for a test-level of 10%. The last column contains the step in which the model is added to the set of superior models targeting an FDR^+ of 10%. A value of zero indicates that the model is not contained in the superior set.

Model	p^R	p^{SQ}	Step-RC	Step-RCst	Step-SPA	Step-SPAst	$FDR^+ = 10\%$
Vola Hist	0.00	0.00	-	-	-	-	-
Vola EWMA	4.90	4.90	**1**	**1**	**1**	**1**	**1**
Vola GARCH	**100.00**	**100.00**	**1**	**1**	**1**	**1**	**2**
VaR Hist	0.00	0.00	0	0	0	0	0
VaR EWMA FHS	0.00	0.00	0	0	0	0	**4**
VaR EWMA EVT	0.00	0.00	0	0	0	0	0
VaR EWMA Stsk	0.00	0.00	0	0	0	0	0
VaR GARCH FHS	0.00	0.00	**1**	**1**	**1**	**1**	**3**
VaR GARCH EVT	0.00	0.00	0	0	0	0	0
VaR GARCH Stsk	0.00	0.00	0	0	0	0	0
CVaR Hist	0.00	0.00	0	0	0	0	0
CVaR EWMA FHS	0.00	0.00	0	0	0	0	0
CVaR EWMA EVT	0.00	0.00	0	0	0	0	0
CVaR EWMA Stsk	0.00	0.00	0	0	0	0	0
CVaR GARCH FHS	0.00	0.00	0	0	0	0	0
CVaR GARCH EVT	0.00	0.00	0	0	0	0	0
CVaR GARCH Stsk	0.00	0.00	0	0	0	0	0

to the SPA-test for all reasonable test levels, since only the GARCH model is contained in the MCS for a test level of 10%. The Step-RC and Step-SPA produce larger sets than the MCS and contain the EWMA, GARCH and VaR-GARCH-FHS models. This result is similar to the result of the RC-test. There are no differences between the sets of the Step-RC and Step-SPA test. Further, studentizing does not lead to different results. The FDR approach, which is known to typically produce sets of superior models that are at least as large as the sets of the Step-RC and Step-SPA approaches, chooses one additional model. Besides the three models that are identified by stepwise multiple testing, the set that targets an FDR^+ of 10% also contains the VaR-GARCH-EVT. However, the volatility models (EWMA and GARCH) are chosen in the first two steps and the VaR-GARCH-EVT is chosen the last.

To summarize the results of Tables I and II we find convincing results of the EWMA, GARCH and VaR-GARCH-FHS model, where the GARCH model delivers the best results.

Hence, an investor who wants to achieve a constant portfolio volatility over time should manage volatility directly by a dynamic risk model. Managing downside risk typically fails to target a constant level of volatility. Further, unconditional models, i.e. HSD or Historical Simulation, produce a portfolio volatility that significantly deviates from the desired volatility target. Since a higher forecast accuracy typically coincides with a higher risk-adjusted performance and utility gains we expect higher risk-adjusted performance and utility gains for conditional models, i.e. models that are based on a dynamic volatility model. Bollerslev et al. (2018) also find that static models like HSD are inaccurate and, due to their inaccuracy, produce lower utility gains for an investor who targets a constant level of volatility.

Table III reports results for the VaR backtest of Christoffersen (1998) presented in Section 4.2, where we report p-values for the unconditional and conditional coverage test for significance levels of $\alpha = 0.5\%, 1\%, 1.5\%$ and 5%. These significance levels are also frequently used in the literature on VaR forecasting (see Bali et al. (2008) for example). The VaR backtesting results demonstrate that for all significance levels controlling volatility is not sufficient when an investor's aim is to target a constant portfolio VaR over time. Contrary, managing CVaR is feasible for an investor who targets a constant VaR over time. However, VaR-based strategies are more successful in targeting a constant portfolio VaR over time than strategies that manage CVaR. Only two of the VaR-based strategies that rely on a conditional volatility model can be rejected for a significance level of $\alpha = 0.5\%$ and a test level of 10%. Further, for higher significance levels of α unconditional models based on Historical Simulation also fail to target a constant VaR, whereas these models cannot be rejected for low significance levels. A possible explanation for this result is that low significance levels produce only a limited number of hits. Historical Simulation is known for typically producing adequate hit ratios, but these hits are usually clustered over time. Hence, when testing for unconditional coverage, Historical Simulation usually delivers convincing results. However, Historical Simulation is often rejected once the independence or conditional coverage test is applied, due to a failure of producing independent hits (Kuester et al., 2006). Since the independence test of Christoffersen (1998) only regards successive hits, low significance levels, and hence only very few hits over the whole sample, imply that the independence test fails to detect the lack of independence. This explains why Historical Simulation seems to perform well for low levels of α. Pritsker (2006) also finds that VaR backtests fail to identify inferior models when only few exceedances occurred over the sample.

Table III. VaR backtesting results

The table reports the backtesting results of the Christoffersen (1998) VaR backtest for significance levels of $\alpha = 0.5\%, 1\%, 2.5\%$ and 5%. p_{uc} and p_{cc} are the p-values for the unconditional coverage and conditional coverage test and are given in percent. Bold numbers mark the models that are not rejected at a test level of 10%.

	$\alpha = 0.5\%$		$\alpha = 1\%$		$\alpha = 2.5\%$		$\alpha = 5\%$	
Model	p_{uc}	p_{cc}	p_{uc}	p_{cc}	p_{uc}	p_{cc}	p_{uc}	p_{cc}
Vola Hist	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.00
Vola EWMA	0.01	0.03	0.00	0.00	0.00	0.01	0.02	0.04
Vola GARCH	1.46	2.90	2.09	6.87	0.10	0.40	5.98	**18.22**
VaR Hist	**82.09**	**25.17**	**68.80**	**27.71**	**55.22**	0.00	**51.14**	0.00
VaR EWMA FHS	**43.75**	**27.08**	**41.01**	**63.71**	**49.33**	**25.13**	**64.51**	**59.47**
VaR EWMA EVT	**27.94**	**10.06**	**22.13**	**30.36**	**67.98**	**71.83**	**15.16**	8.21
VaR EWMA Stsk	**66.34**	**21.64**	**79.56**	**80.76**	3.53	5.07	0.17	0.34
VaR GARCH FHS	**56.13**	**29.04**	**58.70**	**74.75**	**61.45**	**86.73**	**64.51**	**90.32**
VaR GARCH EVT	**19.21**	6.95	**28.47**	**37.41**	**96.32**	**99.90**	**39.42**	**72.83**
VaR GARCH Stsk	**19.21**	6.95	**12.52**	**18.43**	**74.79**	**85.83**	**26.64**	**50.57**
CVaR Hist	**19.21**	**39.97**	4.45	1.13	0.61	0.08	0.00	0.00
CVaR EWMA FHS	**27.94**	**10.06**	**12.52**	**18.43**	6.43	**12.41**	2.57	4.24
CVaR EWMA EVT	**12.57**	4.51	1.96	3.17	0.32	1.12	0.99	2.10
CVaR EWMA Stsk	**19.21**	6.95	**12.52**	**18.43**	**28.10**	**52.96**	**69.05**	**14.61**
CVaR GARCH FHS	**27.94**	**10.06**	**35.88**	**44.91**	**11.92**	**26.42**	8.03	**19.06**
CVaR GARCH EVT	4.57	1.55	3.00	4.80	1.46	5.08	5.06	**12.92**
CVaR GARCH Stsk	4.57	1.55	0.15	0.24	0.16	0.68	0.99	2.55

Table IV shows the backtesting results for the two CVaR backtests presented in Section 4.3. We again choose the four significance levels $\alpha = 0.5\%, 1\%, 1.5\%$ and 5%. The p-value of the backtest of McNeil and Frey (2000) is denoted by p_{CVaR}. All target volatility and nearly all target VaR strategies fail to accurately target the portfolio CVaR and are rejected at a test level of 10%. In contrast, only one of the target CVaR strategies for a significance level of $\alpha = 0.5\%$ can be rejected, indicating that by controlling the CVaR of the risky asset it is possible to achieve a constant portfolio CVaR over time. This finding is also supported by the results of the CVaR backtest of Embrechts et al. (2005), which exhibits the lowest values for CVaR managed strategies. Further, the values V of the dynamically managed target CVaR strategies are systematically lower than the values of the remaining models, indicating that when the objective is to achieve a constant portfolio CVaR over time, the CVaR of the risky asset should be managed dynamically. Interestingly, the CVaR-HS cannot be rejected by the backtest of

McNeil and Frey (2000). However, this backtest is an unconditional backtest which only tests if the produced CVaR is correct on average (Du and Escanciano, 2016). As mentioned above, Historical Simulation is typically rejected once a conditional backtest is applied. The lower values of V of the conditionally managed strategies indicate a higher accuracy of conditional models.

Table IV. CVaR backtesting results

This table reports the backtesting results of the McNeil and Frey (2000) and Embrechts et al. (2005) CVaR backtests for significance levels of $\alpha = 0.5\%, 1\%, 2.5\%$ and 5%. V denotes the measure given in Equation (61) of the backtest of Embrechts et al. (2005). Bold numbers mark the lowest value of V. p_{CVaR} denotes the p-value of the backtest of McNeil and Frey (2000) and is given in percent. Bold numbers mark the models that are not rejected at a test level of 10%.

	$\alpha = 0.5\%$		$\alpha = 1\%$		$\alpha = 2.5\%$		$\alpha = 5\%$	
Model	V	p_{CVaR}	V	p_{CVaR}	V	p_{CVaR}	V	p_{CVaR}
Vola Hist	0.5825	1.11	0.4257	0.39	0.2840	0.04	0.2097	0.00
Vola EWMA	0.4934	1.85	0.3164	1.93	0.2068	0.42	0.1527	0.09
Vola GARCH	0.3337	**10.03**	0.2150	7.84	0.1163	9.48	0.0917	1.66
VaR Hist	0.5699	3.30	0.4179	3.69	0.3849	0.20	0.3169	0.00
VaR EWMA FHS	0.3621	2.30	0.2282	2.62	0.0843	**11.46**	0.0891	1.05
VaR EWMA EVT	0.3145	3.94	0.1842	3.94	0.0898	8.12	0.0957	1.16
VaR EWMA Stsk	0.3356	5.53	0.2182	3.53	0.1506	1.37	0.1472	0.06
VaR GARCH FHS	0.2825	**10.42**	0.1369	**13.93**	0.0681	**16.24**	0.0688	3.30
VaR GARCH EVT	0.2318	**14.41**	0.1094	**16.87**	0.0637	**14.92**	0.0738	2.50
VaR GARCH Stsk	0.1955	**26.53**	0.1219	**13.24**	0.0732	**12.96**	0.0867	1.60
CVaR Hist	0.3052	**12.97**	0.1376	**51.87**	0.0848	**77.77**	0.0526	**15.62**
CVaR EWMA FHS	0.1814	**28.01**	0.0772	**63.80**	0.0200	**93.16**	0.0085	**90.92**
CVaR EWMA EVT	0.2305	5.81	0.0857	**17.86**	0.0209	**35.76**	0.0186	**14.09**
CVaR EWMA Stsk	0.1913	**27.28**	0.0894	**34.94**	0.0390	**54.87**	0.0384	**47.23**
CVaR GARCH FHS	0.1602	**44.38**	0.0691	**57.07**	**0.0133**	**99.67**	**0.0070**	**94.12**
CVaR GARCH EVT	0.1936	**17.85**	0.0508	**41.94**	0.0211	**49.32**	0.0139	**32.10**
CVaR GARCH Stsk	**0.0634**	**70.90**	**0.0387**	**73.45**	0.0455	**14.36**	0.0315	4.93

The backtesting results presented in this section demonstrate two important issues. First, if an investor is interested in targeting portfolio risk in terms of volatility, VaR or CVaR the investor should directly manage volatility, VaR or CVaR, respectively. In particular, when the aim is to target a certain level of tail risk, it is not sufficient to manage volatility. Second, when portfolio risk is managed, the investor should use a fast-adapting dynamic risk model instead of simpler empirical models like HSD or Historical Simulation as done by Barroso and Santa-Clara (2015), Barroso and Maio (2016) and Moreira and Muir (2017).

5.3 Performance of Target Risk Strategies

We next assess the performance of the different risk targeting strategies and the two benchmark portfolios. Results of the performance analysis are given in Table V. All target risk strategies except for the Historical Simulation managed target VaR strategy deliver higher returns than the 60/40 portfolio with a risk, measured by volatility, drawdown, VaR or CVaR, that is lower – or comparable in the case of the volatility managed strategies – than the risk of the 60/40 portfolio. Further, the risk targeting strategies deliver higher returns with lower risk compared to the DAX. Therefore, dynamically managing the portfolio risk can significantly reduce the portfolio risk measured by volatility, drawdown, VaR and CVaR without simultaneously sacrificing returns (see Fung and Hsieh (1997) who found a similar behavior for dynamic trading strategies used by hedge funds). This is also reflected in higher Sharpe Ratios for the dynamically managed target risk strategies compared to the two benchmark portfolios. Moreover, within the (dynamically) managed target risk strategies, returns are quite similar, however, the downside risk managed strategies exhibit a significantly lower risk than the volatility managed strategies. The highest Sharpe Ratio is found for the CVaR-EWMA-Stsk strategy, which is about 287.5% higher than the Sharpe Ratio of the DAX and 51.96% higher that the Sharpe Ratio of the HSD model. The Sharpe Ratio of the best volatility managed strategy is still 205% higher than the Sharpe Ratio of the DAX. That is, the Sharpe Ratio of the best CVaR managed strategy is $0.155/0.122 - 1 = 27.05\%$ higher than the Sharpe Ratio of the best volatility managed strategy. In particular, Sharpe Ratios of the dynamically managed CVaR strategies are all higher than the Sharpe Ratios of the volatility managed strategies. This can also be seen by the modified Sharpe Ratio which measures the risk-adjusted annualized excess return (see Jondeau and Rockinger (2012) for a definition of the modified Sharpe Ratio). Best results in terms of Sharpe Ratios are found for the strategies based on the skewed t distribution of Jondeau and Rockinger (2003) and Bali et al. (2008). Generally, managing risk dynamically instead of statically by HSD or Historical Simulation is crucial in order to increase the risk-adjusted performance. Sharpe Ratios of the statically managed strategies are significantly lower than the Sharpe Ratios of the dynamically managed strategies. This finding is also in line with Bollerslev et al. (2018) since models that produce a more constant portfolio risk over time, as shown in Section 5.2, also yield a higher risk-adjusted return. Although the differences in the Sharpe Ratio seem small, the results indicate significant performs gains of portfolio risk management, especially when downside risk is managed. This is because our strategies are highly correlated, which results in very small standard errors for the relative Sharpe Ratios as highlighted in Kirby and Ostdiek (2012). For example, the average correlation between all risk targeting models is 97.34% and the maximum correlation between two strategies is 99.98%. This high correlation between the risk targeting strategies demonstrates that even small differences in the Sharpe Ratios indicate a striking improvement in performance. For example, Kirby and Ostdiek (2012) find Sharpe Ratios for their strategies in the range of 0.47 to 0.49, compared to the benchmark's Sharpe Ratio of 0.46, and they conclude that, due to the high correlation of the strategies, "[t]hese differences translate into significant performance gains". The performance gains of CVaR targeting compared to volatility targeting are even higher in magnitude than the gains found by Kirby and Ostdiek (2012), demonstrating the vast performance gains of managing CVaR instead of volatility. To test if any model produces a statistically higher Sharpe Ratio than the HSD managed model, we use the corrected version of the Sharpe Ratio test of Jobson and Korkie (1981) which is also used by DeMiguel et al. (2009, p. 1928). This test could also be applied to more strategies simultaneously as shown by Jobson and Korkie (1981, II.C). However, we apply more sophisticated approaches to test for higher performance gains in Section 5.4 to all portfolios simultaneously and only test each strategy with respect to the HSD model here. The test of Jobson and Korkie (1981) indicates that only the VaR-EWMA-Stsk model exhibits a Sharpe Ratio that is significantly higher than the Sharpe Ratio of HSD model when using a test level of 10%.

We now turn to the drawdown protection ability of risk targeting. Several studies demonstrate that volatility targeting is an easy but successful method to significantly reduce drawdowns (see Benson et al. (2014), Barroso and Santa-Clara (2015), Harvey et al. (2018) and Moreira and Muir (2017) among others). As expected, all risk targeting strategies and the 60/40 portfolio are successful in mitigating portfolio drawdowns. The maximum drawdown (MDD) of the risk targeting strategies and the 60/40 portfolio is about half of the maximum drawdown of the DAX, whereby the risk targeting strategies exhibit a slightly higher drawdown reduction than the 60/40 portfolio. This can be seen by ΔMDD which measures the percentage drawdown reduction compared to the drawdown of the DAX. Managing downside risk, espe-

Table V. Performance results of risk targeting

This table shows the performance results of all target risk strategies and the two benchmark portfolios over the whole period. Return and Vola stand for the annualized return and volatility, respectively. SR stands for the annualized Sharpe Ratio, z_{JK} stands for the test statistic of the corrected version of the test of Jobson and Korkie (1981) and mSR is the modified Sharpe Ratio defined in Jondeau and Rockinger (2012). MDD and ΔMDD stand for the maximum drawdown and the reduction of the maximum drawdown in contrast to the maximum drawdown of the DAX. Calmar Ratio stands for the drawdown adjusted return and is defined in Farinelli et al. (2008) and Eling and Schuhmacher (2007). VaR and CVaR are the in-sample VaR and CVaR, which are estimated with Historical Simulation using all data. Min and Max stand for the minimum and maximum daily return, respectively. Return, Vola, MDD, ΔMDD, Min. and Max. are given in percent. Bold numbers of z_{JK} show significance at a level of 10%, i.e. $z_{JK} \geqslant 1.6449$.

Model	Return	Vola	SR	z_{JK}	mSR	MDD	ΔMDD	Calmar	VaR	CVaR	Min	Max
Vola Hist	3.12	12.84	0.102	-	1.49	41.24	41.45	0.032	1.38	1.82	-5.64	5.14
Vola EWMA	3.33	12.50	0.122	0.95	1.97	40.55	42.42	0.038	1.34	1.76	-5.28	4.98
Vola GARCH	3.19	11.97	0.116	0.38	1.81	39.28	44.23	0.035	1.29	1.67	-5.29	4.03
VaR Hist	2.07	9.69	0.029	-0.68	-0.25	31.62	55.10	0.009	0.97	1.44	-4.99	5.29
VaR EWMA FHS	3.23	11.09	0.128	0.87	2.12	37.09	47.33	0.038	1.19	1.56	-4.33	4.24
VaR EWMA EVT	3.23	10.43	0.136	1.29	2.30	35.28	49.91	0.040	1.11	1.47	-4.24	4.01
VaR EWMA Stsk	3.43	10.83	0.150	**1.68**	2.63	35.07	50.20	0.046	1.16	1.53	-4.76	4.11
VaR GARCH FHS	3.11	11.33	0.115	0.34	1.81	38.32	45.59	0.034	1.22	1.59	-4.67	3.52
VaR GARCH EVT	3.13	10.63	0.125	0.62	2.04	36.55	48.10	0.036	1.15	1.49	-4.63	3.37
VaR GARCH Stsk	3.34	10.39	0.147	1.10	2.58	34.68	50.75	0.044	1.11	1.46	-4.84	3.32
CVaR Hist	2.43	9.26	0.069	-0.33	0.70	28.51	59.52	0.022	0.95	1.37	-5.12	4.17
CVaR EWMA FHS	3.22	10.16	0.139	1.13	2.38	34.35	51.22	0.041	1.07	1.43	-4.01	3.79
CVaR EWMA EVT	3.23	9.93	0.143	1.32	2.48	33.80	52.00	0.042	1.05	1.40	-3.90	3.74
CVaR EWMA Stsk	3.38	10.14	0.155	1.53	2.77	33.12	52.97	0.048	1.09	1.44	-4.67	3.70
CVaR GARCH FHS	3.23	10.68	0.134	0.79	2.25	37.00	47.47	0.039	1.14	1.49	-4.58	3.26
CVaR GARCH EVT	3.26	10.33	0.141	0.99	2.42	35.51	49.58	0.041	1.10	1.44	-4.41	3.19
CVaR GARCH Stsk	3.21	9.74	0.145	0.94	2.51	33.02	53.11	0.043	1.04	1.38	-4.75	2.98
DAX	2.73	23.46	0.040	-0.57	-	70.42	-	0.013	2.36	3.46	-8.49	11.40
60/40	2.37	11.94	0.048	-0.59	0.21	40.07	43.10	0.014	1.24	1.74	-4.32	5.10

cially managing CVaR, instead of volatility results in a higher drawdown reduction without simultaneously sacrificing returns. Interestingly, the Historical Simulation managed strategies exhibit the highest drawdown reduction. However, this superior drawdown protection comes along with significantly lower returns. This is confirmed by the Calmar Ratio which measures the drawdown-adjusted return and takes the highest values for the dynamically managed target CVaR strategies, whereas the Calmar Ratios of the Historical Simulation managed strategies are significantly lower.[63]

[63] Since asset returns are usually non-normally distributed, performance measurement based on the Sharpe Ratio solely can lead to wrong conclusions (see Farinelli et al. (2008) or Eling and Schuhmacher (2007) for example). Only for elliptical distributions, a class of distributions that contains the normal distribution, Sharpe Ratio is an adequate risk-adjusted performance measure (Eling and Schuhmacher, 2007, p. 2633). Therefore, besides the Sharpe Ratio as risk-adjusted performance measure we additionally use the Calmar Ratio. This measure replaces the volatility in the Sharpe Ratio by the Maximum Drawdown. For a motivation of enhanced risk-adjusted performance measures and definitions see Eling and Schuhmacher (2007). Further, Jobson and Korkie (1981, Sec. I) gives an overview on several performance measures. We also used other performance measures which are not reported here, since results were quite similar to the Sharpe Ratio and Calmar Ratio. Similarly, Eling and Schuhmacher (2007) find a similar ranking order across hedge funds when several risk-adjusted performance measures are used.

The highest (least negative) minimum daily return is achieved by the CVaR managed strategies, which is in line with the results of the maximum drawdown. The minimum return of the VaR managed strategies is also comparable to the minimum return of the CVaR managed strategies but slightly lower. The minimum return of the volatility managed strategies is significantly more negative than the minimum return of the downside risk managed strategies. Further, the minimum daily return of the Historical Simulation based strategies is significantly more negative than the minimum return of the dynamically managed strategies. This is somewhat surprising since the Historical Simulation based strategies exhibit the lowest drawdowns which indicates that these models are the most conservative. Further, the Historical Simulation based strategies also have the lowest average equity weights which are not shown here. The lower average equity exposure of the Historical Simulation managed strategies is consistent with Berkowitz and O'Brien (2002) who find that banks – who often use Historical Simulation as risk measurement tool – typically exhibit too conservative, i.e. too high, risk estimates that translates in lower equity weights of strategies that are managed by Historical Simulation. Further, Berkowitz and O'Brien (2002) find that although commercial banks' internal risk models produce more conservative risk estimates than a GARCH-based VaR model, the banks' models deliver comparable – or even more – VaR violations. This explains the somewhat surprising result of lower equity exposure and drawdown but a more negative minimum return of the Historical Simulation managed strategies. The Historical Simulation based strategies are more conservative on average but fail to correctly manage downside risk, just when downside risk protection is most needed. This result again highlights the need of a fast adapting risk model when portfolio risk is managed. Again, a higher forecast accuracy, as examined in Section 5.2 coincides with a better performance. However, even the dynamically managed strategies exhibit minimum daily returns of the size of the 60/40 portfolio. This highlights the fact, that unpredictable negative price jumps cannot be completely avoided by risk targeting, but as supposed by Longin (2000), these jumps are best managed by models using EVT. By comparing the minimum return of the EVT, FHS and skewed t distribution based approaches that use the same volatility model, the minimum return of the EVT based model is always higher (less negative) than the minimum return of the other models. Results found for the minimum daily return reverse when the maximum daily return is compared. Now, the volatility managed strategies exhibit higher maximum returns than the downside risk managed strategies. This indicates that downside risk timing seems superior in crash periods but volatility timing is superior in bull markets, motivating a strategy that switches between downside risk targeting in down periods and volatility targeting in up periods as examined later.

The performance evaluation in Table V does not consider transaction costs. However, many studies demonstrated that volatility managing is also beneficial after realistic transaction costs were considered (see Moreira and Muir (2017), Kirby and Ostdiek (2012), Fleming et al. (2003), Fleming et al. (2001), Marquering and Verbeek (2004), Harvey et al. (2018) and Bollerslev et al. (2018)). In unreported results, we find that most downside risk managed strategies exhibit lower turnovers than the volatility managed strategies (see Kirby and Ostdiek (2012, p. 442) for a definition of the turnover). Hence, the superiority of downside risk managed strategies, especially CVaR managed strategies, in contrast to volatility managed strategies would be even more striking if realistic transaction costs were considered. However, the skewed t distribution based strategy produces a higher turnover compared to the FHS and EVT based approaches. Hence, the outperformance of the skewed t distribution based strategy over the FHS and EVT based strategies will be lowered after transaction costs.

To better assess the mitigation of extreme negative returns and how risk targeting works in different market environments we next consider the days when the underlying risky asset, i.e. the DAX, suffers the highest losses or obtains the highest gains. Table VI reports the five lowest and five highest daily DAX returns in conjunction with the corresponding returns of the target risk strategies and the 60/40 portfolio. On the days with the worst DAX returns, both the target risk strategies and the 60/40 portfolio deliver significantly higher, i.e. less negative, daily returns. The returns of the target risk strategies are usually higher than the returns of the DAX, with the exception of the day with the third lowest DAX return. This day indicates a day with an unpredictable negative price jump in the DAX as examined in Liu et al. (2003) and Das and Uppal (2004). On this day, returns of the Historical Simulation based strategies are higher than the returns of the remaining models, which is in line with our earlier finding. However, for the days with the two most negative DAX returns, the dynamic risk models produce higher returns, which shows that these models are better in timing (partly) predictable losses. Further, the returns of the downside risk managed strategies, in particular the CVaR managed strategies, are significantly higher than the returns of the volatility managed strategies. Hence, for mitigating extreme negative returns, an investor should manage CVaR instead of volatility.

Table VI. Lowest and highest DAX returns

Panel A shows the five days with the lowest DAX returns and the corresponding returns of the target risk strategies and the 60/40 portfolio on these days. Panel B shows the five days with the highest DAX returns and the corresponding returns of the target risk strategies and the 60/40 portfolio on these days. All entries correspond to daily returns and are given in percent.

Model	Panel A: Low DAX Return (%)					Panel B: High DAX Return (%)				
Vola Hist	-5.638	-4.125	-4.142	-2.829	-2.736	3.092	3.204	2.517	3.157	5.141
Vola EWMA	-5.279	-3.466	-3.910	-2.813	-2.659	2.928	2.679	2.600	2.857	4.978
Vola GARCH	-5.292	-3.114	-3.333	-2.408	-2.164	2.979	2.301	2.493	2.451	4.030
VaR Hist	-4.989	-3.211	-1.692	-1.187	-0.875	3.679	5.291	1.438	1.757	1.843
VaR EWMA FHS	-4.331	-3.157	-3.318	-2.857	-2.496	2.412	1.985	2.370	2.604	4.238
VaR EWMA EVT	-4.235	-3.038	-3.358	-2.649	-2.341	2.203	1.976	2.234	2.214	4.010
VaR EWMA Stsk	-4.764	-3.072	-3.233	-2.697	-2.637	2.416	2.276	2.437	2.738	4.106
VaR GARCH FHS	-4.668	-2.998	-2.943	-2.630	-2.054	2.760	1.956	2.423	2.448	3.515
VaR GARCH EVT	-4.633	-2.945	-2.984	-2.340	-2.012	2.418	1.855	2.303	2.047	3.370
VaR GARCH Stsk	-4.838	-2.799	-2.767	-2.366	-2.333	2.315	1.919	2.345	2.467	3.319
CVaR Hist	-5.123	-3.037	-1.706	-1.185	-0.874	2.834	4.165	1.453	1.721	1.653
CVaR EWMA FHS	-4.006	-2.987	-3.260	-2.734	-2.224	1.958	1.770	2.296	2.015	3.791
CVaR EWMA EVT	-3.895	-2.998	-3.111	-2.588	-2.219	1.958	1.765	2.271	2.001	3.742
CVaR EWMA Stsk	-4.666	-2.969	-2.934	-2.656	-2.609	2.187	2.239	2.344	2.698	3.697
CVaR GARCH FHS	-4.583	-2.964	-2.988	-2.464	-1.998	2.248	1.718	2.409	1.943	3.260
CVaR GARCH EVT	-4.407	-2.907	-2.905	-2.327	-1.969	2.183	1.684	2.358	1.884	3.192
CVaR GARCH Stsk	-4.749	-2.715	-2.532	-2.358	-2.316	2.041	1.885	2.253	2.438	2.979
DAX	-7.164	-8.492	-2.959	-3.455	-2.552	5.562	11.402	4.331	5.299	3.373
60/40	-4.098	-4.167	-1.688	-1.553	-1.180	2.848	5.097	2.208	2.636	2.260

The aforementioned results reverse when the highest daily DAX returns are regarded. Now, the DAX delivers higher returns than all the remaining models.[64] However, in order to achieve a high long-term performance avoiding high negative returns is more crucial than achieving high positive returns.[65] This also fits better to the preferences of most investors who treat losses and gains asymmetrically by weighting losses higher than gains (Aït-Sahalia and Brandt, 2001, Ang et al., 2005, Benartzi and Thaler, 1995, Chabi-Yo et al., 2018)). Aït-Sahalia and Brandt (2001) conjecture that loss aversion is highly related to downside risk managed portfolio strategies.

[64]Remind that we have chosen low risk targets. Risk-seeking or less risk-averse investors should use a higher risk target to better capture the upside potential of high DAX returns and to lower the underperformance on days with high DAX returns.

[65]For example, a return of -5% has to be compensated by a return of 5.26% to achieve a compounded return of zero.

This is confirmed by our results since managing CVaR delivers the most convincing mitigation of extreme losses.

Table VII. Performance during and after the financial crisis

Panel A shows the performance of all strategies from 15.07.2008 to 15.07.2009, i.e. during the height of the financial crisis. Panel B shows the performance of all strategies from 16.07.2009 to 15.07.2011, i.e. the time following the financial crisis and before the European debt crisis. See Table V for a description of Return, Volatility, SR, MDD, Min, Max. - marks a negative Sharpe Ratio.

	Panel A: 15.07.2008 - 15.07.2009						Panel B: 16.07.2009 - 15.07.2011					
Model	Return	Volatility	SR	MDD	Min	Max	Return	Volatility	SR	MDD	Min	Max
Vola Hist	-3.51	13.16	-	16.94	-2.75	3.20	9.84	12.47	0.708	12.35	-3.03	3.16
Vola EWMA	-3.40	12.83	-	16.91	-2.66	2.93	10.49	12.21	0.776	11.91	-2.85	2.86
Vola GARCH	-4.07	13.56	-	17.71	-2.62	2.98	10.22	11.46	0.803	9.97	-2.35	2.45
VaR Hist	-8.53	17.20	-	22.38	-4.11	5.29	7.12	6.11	1.003	4.17	-1.10	1.76
VaR EWMA FHS	-1.22	10.29	-	12.94	-2.19	2.41	9.09	11.12	0.727	11.52	-2.78	2.60
VaR EWMA EVT	-1.53	9.63	-	12.36	-2.00	2.20	8.20	9.49	0.759	9.66	-2.29	2.21
VaR EWMA Stsk	-0.16	10.81	-	12.02	-1.96	2.42	8.48	10.10	0.740	10.68	-2.69	2.74
VaR GARCH FHS	-2.93	12.23	-	15.71	-2.45	2.76	9.46	10.98	0.770	9.95	-2.39	2.45
VaR GARCH EVT	-2.55	11.08	-	14.26	-2.15	2.42	8.70	9.59	0.803	8.48	-1.99	2.05
VaR GARCH Stsk	-0.08	11.49	-	12.52	-2.02	3.26	8.70	9.51	0.809	8.96	-2.28	2.47
CVaR Hist	-5.92	14.99	-	19.01	-3.42	4.17	6.99	5.98	1.003	4.09	-1.08	1.72
CVaR EWMA FHS	-0.96	8.56	-	10.79	-1.77	1.96	7.28	8.76	0.718	9.55	-2.28	2.02
CVaR EWMA EVT	-0.93	8.55	-	10.76	-1.77	1.96	7.46	8.62	0.751	8.95	-2.12	2.00
CVaR EWMA Stsk	0.67	9.72	-	10.73	-1.56	2.24	7.49	9.23	0.704	10.08	-2.60	2.70
CVaR GARCH FHS	-2.13	10.24	-	13.09	-1.99	2.25	8.09	9.12	0.778	8.52	-2.00	1.94
CVaR GARCH EVT	-1.95	10.01	-	12.74	-1.92	2.18	8.04	8.84	0.797	7.97	-1.87	1.88
CVaR GARCH Stsk	0.05	10.13	-	11.52	-1.75	2.10	8.00	8.70	0.805	8.45	-2.20	2.44
DAX	-18.36	41.48	-	44.53	-7.07	11.40	19.41	18.43	0.993	12.29	-3.33	5.30
60/40	-8.15	18.80	-	23.13	-3.58	5.10	9.89	9.48	0.936	6.86	-1.72	2.64

Table VI demonstrates that volatility and downside risk targeting behave differently in different market environments. In up-trending markets, volatility targeting delivers higher returns whereas in bear markets downside risk targeting is more convincing. This again motivates a strategy that switches between CVaR and volatility targeting as examined later. To strengthen this observation we next assess risk targeting in two sub samples, one high risk and one low risk period. Table VII shows the performance of the strategies in the period from 15.07.2008 to 15.07.2011, i.e. during the height of the financial crisis and the time following the financial crisis, which is split into two sub-periods. The first sub period, given in Panel A, covers the financial crisis and ranges from 15.07.2008 to 15.07.2009. This period is marked by high negative returns and high risk. During the financial crisis, the dynamically managed risk targeting strategies have significantly higher (less negative) returns and significantly lower risk measured by volatility, drawdown and minimum return than the two benchmark portfolios. For example, the DAX and the 60/40 portfolio exhibit a return of -18.36% and -8.15% as well as a volatility of 41.48% and 18.80%, respectively. In contrast, return and volatility of the EWMA managed target volatility strategy are -3.40% and 12.83%, respectively. However, the CVaR-EWMA-Stsk strategy is even more convincing and achieves a positive return of 0.67% with a volatility of only 9.72%. Hence, managing CVaR performs significantly outperform the two benchmark portfolios and the volatility managed strategies by producing higher returns with lower risk. The statically managed target VaR and target CVaR strategies (VaR-HS and CVaR-HS) exhibit significantly lower returns with higher risk than the dynamically managed strategies. Furthermore, there are significant differences between the EWMA and GARCH managed strategies. The EWMA managed strategies achieve higher returns with lower risk than the GARCH managed strategies. A possible explanation for the better results of the EWMA model could be the higher estimation risk of the GARCH model during highly volatile periods.

The second sub-period, ranging from 16.07.2009 to 15.07.2011, covers the time following the financial crisis, but excludes the European financial debt crisis. This period is marked by a continuously uptrending market with high returns and low risk. Results for the second sub-period are given in Panel B. For this period the DAX clearly outperforms the remaining strategies. Further, in this period, the different target risk strategies perform significantly diverse. The unconditional models, VaR-HS and CVaR-HS, perform very well in this calm market by taking less risk. Moreover, the volatility targeting strategies produce higher returns than the downside risk targeting strategies, again motivating a strategy that switches between volatility and CVaR targeting. However, volatility targeting has also much higher risk measured by volatility, drawdown and minimum return. Consequently, the Sharpe Ratios of the dynamically managed target volatility, VaR and CVaR strategies are only slightly different. The Sharpe Ratios of the two benchmark portfolios are slightly higher than the Sharpe Ratios of the target risk strategies. However, the differences in risk-adjusted performance are only small compared to the differences in Panel A. This demonstrates that risk targeting strategies are able to extremely lower the downside risk but still capture the upside potential of the DAX. In particular, the downside risk managed strategies significantly outperform the remaining strategies in bear markets but exhibit an only slightly worse risk-adjusted performance in uptrending markets compared to the volatility managed strategies.

5.4 Economic Value of Risk Targeting

In the previous section we examined the (risk-adjusted) performance and drawdown protection ability of risk targeting and found that CVaR targeting is superior to volatility targeting. However, conclusions solely based on *unconditional* risk-adjusted performance measures like the Sharpe Ratio can be misleading, since these measures do not account for a time-varying volatility (Han, 2005, Marquering and Verbeek, 2004).[66] Therefore, we next assess the economic value of volatility, VaR and CVaR timing, where we define the economic value as the annualized fee an investor is willing to pay to switch from a static portfolio allocation to a risk-managed portfolio. The economic value of volatility timing has already been examined by Fleming et al. (2001), Fleming et al. (2003), Kirby and Ostdiek (2012), Marquering and Verbeek (2004), Taylor (2014), Han (2005), Moreira and Muir (2017) and Bollerslev et al. (2018).[67] All authors find huge improvements of volatility timing in terms of high utility gains. Fleming et al. (2001), Fleming et al. (2003), Kirby and Ostdiek (2012), Han (2005) and Taylor (2014) examine utility gains in a multivariate framework using several asset classes, whereas Marquering and Verbeek (2004), Moreira and Muir (2017), Moreira and Muir (2019) and Bollerslev et al. (2018) work with only one risky asset. Further, Jondeau and Rockinger (2006) examine the economic value of portfolio strategies that incorporate higher moments and find that the opportunity costs of ignoring higher moments can become very large when asset returns are non-normally distributed or investors are highly risk-averse. Similarly, Jondeau and Rockinger (2012) assess the economic value of dynamic timing strategies that also incorporate higher moments like skewness and kurtosis and find a higher economic value of these strategies compared to strategies that only time volatility. Ghysels et al. (2016) find that investors are willing to pay high fees to switch

[66] Marquering and Verbeek (2004, p. 419-421) write: "It is important to realize that the Sharpe ratio does not appropriately take into account time-varying volatility. The risk of the dynamic strategies is typically overestimated by the sample standard deviation, particularly in the presence of volatility timing, because the ex post (unconditional) standard deviation is an inappropriate measure for the (conditional) risk an investor was facing at each point in time. This indicates a potentially severe disadvantage of the use of Sharpe ratios to evaluate dynamic strategies."

[67] Calculating the economic value, defined as fee an investor is willing to pay to switch from one strategy to another strategy, is similar to calculating the certainty equivalent as done by Ang and Bekaert (2002), Ghysels et al. (2016), Das and Uppal (2004), Guidolin and Timmermann (2008), DeMiguel et al. (2009) and Moreira and Muir (2019).

from mean-variance optimization to mean-variance-skewness optimization. In most studies, the economic value is defined as the maximum fee (in percent) a mean-variance investor is willing to pay to switch from one strategy to another strategy. We further follow Jondeau and Rockinger (2012) and also calculate the economic value for an investor with constant relative risk aversion (CRRA). CRRA utility is frequently used in portfolio selection problems (see Ang and Bekaert (2002), Liu et al. (2003), Das and Uppal (2004), Aït-Sahalia and Brandt (2001), Guidolin and Timmermann (2008), Ghysels et al. (2016) among others). Guidolin and Timmermann (2008), Jondeau and Rockinger (2012) and Bali et al. (2009) show that, for reasonable levels of risk aversion, CRRA utility implies that investors prefer higher skewness and lower kurtosis which is in line with Scott and Horvath (1980). That is, by calculating the economic value of risk targeting for an investor with CRRA utility, we explicitly take preferences for higher moments into account. Guidolin and Timmermann (2008) compare the asset allocation under CRRA preferences with portfolio allocations under four moment preferences and find only minor differences. Hence, portfolio selection under CRRA utility is mainly driven by preferences for the first four moments. A similar result also holds for investors with constant absolute risk aversion (CARA) as shown by Bali et al. (2009) and Jondeau and Rockinger (2006). Jondeau and Rockinger (2006) show that portfolio allocation under CARA utility is mainly driven by preferences for the first four moments and that portfolio allocations under CARA and CRRA utility produce similar results. For that reason, we do not calculate the economic value for CARA utility. Finally, we calculate the economic value for loss-averse investors. Portfolio selection for loss-averse investors has been examined by Benartzi and Thaler (1995), Aït-Sahalia and Brandt (2001) and Ang et al. (2005). Aït-Sahalia and Brandt (2001) compare portfolio selection for mean-variance investors, CRRA investors and loss-averse investors and find that mean-variance and CRRA preferences produce only slightly different optimal portfolio selections (see also Guidolin and Timmermann (2008)), but loss-aversion leads to a significantly different portfolio selection. Ang et al. (2005) examine portfolio selection under disappointment aversion preferences, which also treat gains and losses asymmetrically, and CRRA preferences. The authors find more realistic asset allocations for disappointment aversion preferences and that disappointment aversion preferences can resemble portfolio allocations under CRRA preferences, whereas the opposite does not hold. Generally, portfolio allocations under loss-aversion are more realistic than portfolio allocation under mean-variance or CRRA preferences, since equity holdings of investors are typically much lower than predicted for mean-variance or CRRA investors (see Benartzi and Thaler (1995) and Ang et al. (2005)).

As first method to calculate the economic value of risk targeting we follow Fleming et al. (2001), Fleming et al. (2003) and Kirby and Ostdiek (2012) and assume that the investor's true utility function can be approximated by quadratic utility. For this investor the realized day t utility is given by

$$U_{MV}(R_{t,a}) = W_{t-1}(1 + R_{t,a}) - \frac{1}{2}\gamma_{abs}W_{t-1}^2(1 + R_{t,a})^2, \tag{62}$$

where γ_{abs} is the investor's absolute risk aversion, W_{t-1} denotes the investor's wealth on day $t-1$ and $R_{t,a}$ denotes the day t return of strategy a. We call an investor with preferences given in Equation (62) a mean-variance investor since this approach is highly related to the mean-variance theory (Fleming et al., 2001, p. 334). By assuming that this investor has a constant relative risk aversion γ, Equation (62) can be rewritten as

$$U_{MV}(R_{t,a}) = W_0\left((1 + R_{t,a}) - \frac{\gamma}{2(1+\gamma)}(1 + R_{t,a})^2\right). \tag{63}$$

The economic value of a strategy a is then given by the percentage fee Δ_{MV} the investor with utility in Equation (62) is willing to pay to switch from the 60/40 portfolio to the strategy a. The fee Δ_{MV} is defined by equating the expected utilities

$$\mathbb{E}(U_{MV}(R_{t,a} - \Delta_{MV})) = \mathbb{E}(U_{MV}(R_{t,b})), \tag{64}$$

where $R_{t,b}$ denotes the return of the 60/40 portfolio. The expected utility in (64) is then estimated by the average realized utility. Hence, the fee Δ_{MV} is calculated by solving

$$\overline{U}_{MV}\left(R_{1,a} - \Delta_{MV}, ..., R_{T,a} - \Delta_{MV}\right) = \overline{U}_{MV}\left(R_{1,b}, ..., R_{T,b}\right). \tag{65}$$

where $\overline{U}_{MV}(R_1, ..., R_T) = \sum_{t=1}^{T}(1 + R_t) - \frac{\gamma}{2(1+\gamma)}(1 + R_t)^2$. We calculate the fee Δ_{MV} for levels of risk aversion given by $\gamma = 2, 5, 10$ and 15 which are in line with previous studies using this approach (see Marquering and Verbeek (2004), Aït-Sahalia and Brandt (2001), Jondeau and Rockinger (2006) and Jondeau and Rockinger (2012) for example).

In the case of the CRRA investor, the realized day t utility is given by

$$U_{CRRA}(R_{t,a}) = \begin{cases} \frac{(1+R_{t,a})^{(1-\gamma)}}{1-\gamma}, & \text{if } \gamma > 1 \\ \ln(1+R_{t,a}), & \text{if } \gamma = 1. \end{cases} \tag{66}$$

Since we choose the same levels of γ as stated above, the investor's utility simplifies to the case $U_{CRRA}(R_{t,a}) = \frac{(1+R_{t,a})^{(1-\gamma)}}{1-\gamma}$. Following Jondeau and Rockinger (2012) the economic value for an investor with CRRA utility is defined by equating the expected utilities

$$\mathbb{E}(U_{CRRA}(R_{t,a} - \Delta_{CRRA})) = \mathbb{E}(U_{CRRA}(R_{t,b})), \tag{67}$$

which is again estimated by the average realized utility. The percentage fee Δ_{CRRA} is then calculated by solving

$$\overline{U}_{CRRA}(R_{1,a} - \Delta_{CRRA}, ..., R_{T,a} - \Delta_{CRRA}) = \overline{U}_{CRRA}(R_{1,b}, ..., R_{T,b}), \tag{68}$$

where $\overline{U}_{CRRA}(R_1, ..., R_T) = \sum_{t=1}^{T} \frac{(1+R_t)^{(1-\gamma)}}{1-\gamma}$.

Lastly, to account for the loss-aversion of investors, we use a utility function that gives more weight on negative returns. Following Aït-Sahalia and Brandt (2001) and Benartzi and Thaler (1995) we define the investor's day t utility by

$$U_{LA}(R_{t,a}) = \begin{cases} (R_{t,a})^b, & \text{if } R_{t,a} \geqslant 0 \\ -l(-R_{t,a})^b, & \text{if } R_{t,a} < 0, \end{cases} \tag{69}$$

where $l > 1$ determines the loss aversion and b measures the degree of risk seeking for negative returns and risk aversion for positive returns (see Aït-Sahalia and Brandt (2001, p. 1314) or Benartzi and Thaler (1995, p. 79)).[68] Typical values of l and b are in the range of $l = 2.25$ and $b = 0.88$, which are motivated empirically. Similar to Aït-Sahalia and Brandt (2001) we choose the four combinations of $l = 2.0, 3.0$ and $b = 0.8, 1$. A loss aversion of $l = 2$ implies that the disutility of a loss is twice as great as the utility of a positive return of the same magnitude (Benartzi and Thaler, 1995, p. 74). Another possibility to assess the economic value of an investor with unexpected utility would be to use preferences of an ambiguity-averse investor

[68] We also used the risk-free rate instead of a zero return to define the cut off point which determines a loss or a gain. Results for the economic value were nearly identical for both choices and are not reported here.

as in Aït-Sahalia and Brandt (2001) and Jondeau and Rockinger (2012) or preferences of a disappointment averse investor as in Ang et al. (2005). See also Jondeau and Rockinger (2012, Footnote 17) for a list of studies that incorporate ambiguity aversion in asset allocation. The economic value for a loss-averse investor is then given by equating the expected utilities

$$\mathbb{E}(U_{LA}(R_{t,a} - \Delta_{LA})) = \mathbb{E}(U_{LA}(R_{t,b}))\,. \tag{70}$$

As above, we calculate Δ_{LA} by solving

$$\overline{U}_{LA}\left(R_{1,a} - \Delta_{LA}, ..., R_{T,a} - \Delta_{LA}\right) = \overline{U}_{LA}\left(R_{1,b}, ..., R_{T,b}\right), \tag{71}$$

where $\overline{U}_{LA}(R_1, ..., R_T) = \sum_{t=1}^{T} R_t^b \cdot \mathbb{1}_{\{R_t \geqslant 0\}} - l(-R_t)^b \cdot \mathbb{1}_{\{R_t < 0\}}$.

Table VIII shows the values Δ_i for the three utility functions, i.e. Δ_i gives the annualized percentage fee a mean-variance, CRRA or loss-averse investor is willing to pay to switch from the 60/40 strategy to one of the risk timing strategies. In addition, we examine the economic value of risk timing during and after the financial crisis by choosing the same sub-periods as in Table VII. For these sub-periods we only report the results for a risk aversion of $\gamma = 5$ and $\gamma = 10$ in the case of the mean-variance and CRRA investor as well as $l = 2$ and $l = 3$ combined with $b = 0.8$ for the loss-averse investor.

Panel A of Table VIII shows the economic value for a mean-variance investor. The economic value over the whole sample is positive for almost all risk targeting strategies and levels of risk aversion. Further, we find that downside risk timing delivers a significantly higher economic value than volatility timing and that managing CVaR delivers the highest economic value. In other words, a mean-variance investor should manage CVaR instead of volatility. Interestingly, as in Marquering and Verbeek (2004) we find that the economic value of volatility timing is decreasing in the level of risk aversion γ. This result reverses when the economic value of downside risk timing is assessed. Now, the economic value is increasing in the risk aversion, that is, for a highly risk-averse mean-variance investor, timing CVaR instead of volatility or using a static portfolio allocation becomes more important. For example, a mean-variance investor with risk-aversion of $\gamma = 15$ would pay an annualized fee of 0.756% to switch from the 60/40 portfolio to the GARCH managed strategy, but is not willing to pay a positive fee to

Table VIII. Economic value of risk targeting

This table shows the economic value given as annualized percentage fee Δ_i an investor is willing to pay to switch from the 60/40 portfolio to a risk timing strategy for a given utility function U_i, $i \in \{\text{MV}, \text{CRRA}, \text{LA}\}$. Panel A shows results for a mean-variance investor. Panel B shows results for an investor with CRRA utility. Panel C shows results for a loss-averse investor. γ indicates the investor's risk aversion. l determines the investor's loss aversion and b measures the investor's degree of risk seeking for negative returns and risk aversion for positive returns.

	Whole Sample				Crash		Recovery	
Panel A: Δ_{MV}	$\gamma = 2$	$\gamma = 5$	$\gamma = 10$	$\gamma = 15$	$\gamma = 5$	$\gamma = 10$	$\gamma = 5$	$\gamma = 10$
Vola Hist	0.618	0.280	-0.283	-0.843	8.881	13.836	-1.343	-2.953
Vola EWMA	0.876	0.669	0.323	-0.021	9.184	14.394	-0.637	-2.100
Vola GARCH	0.799	0.789	0.772	0.756	8.018	12.641	-0.527	-1.556
VaR Hist	-0.049	0.682	1.912	3.158	0.731	2.172	-1.493	-0.186
VaR EWMA FHS	0.940	1.237	1.735	2.236	12.968	20.081	-1.396	-2.227
VaR EWMA EVT	1.009	1.521	2.381	3.248	12.906	20.407	-1.540	-1.542
VaR EWMA Stsk	1.168	1.552	2.194	2.841	13.930	20.780	-1.527	-1.828
VaR GARCH FHS	0.799	1.015	1.375	1.737	10.057	15.743	-0.995	-1.752
VaR GARCH EVT	0.898	1.346	2.098	2.856	11.070	17.580	-1.121	-1.171
VaR GARCH Stsk	1.126	1.653	2.537	3.430	13.672	20.057	-1.099	-1.114
CVaR Hist	0.347	1.206	2.653	4.121	5.073	8.469	-1.581	-0.236
CVaR EWMA FHS	1.031	1.629	2.634	3.648	14.001	22.157	-2.121	-1.798
CVaR EWMA EVT	1.066	1.733	2.856	3.992	14.038	22.200	-1.900	-1.513
CVaR EWMA Stsk	1.196	1.800	2.816	3.841	15.384	22.992	-2.093	-1.977
CVaR GARCH FHS	0.991	1.424	2.150	2.881	11.954	19.037	-1.503	-1.337
CVaR GARCH EVT	1.057	1.602	2.517	3.440	12.253	19.490	-1.453	-1.161
CVaR GARCH Stsk	1.068	1.794	3.016	4.253	14.491	21.799	-1.441	-1.088
Panel B: Δ_{CRRA}	$\gamma = 2$	$\gamma = 5$	$\gamma = 10$	$\gamma = 15$	$\gamma = 5$	$\gamma = 10$	$\gamma = 5$	$\gamma = 10$
Vola Hist	0.731	0.387	-0.197	-0.793	7.878	12.680	-1.022	-2.649
Vola EWMA	0.945	0.733	0.371	0.000	8.130	13.173	-0.345	-1.820
Vola GARCH	0.803	0.789	0.759	0.723	7.076	11.540	-0.322	-1.358
VaR Hist	-0.292	0.438	1.668	2.917	0.430	1.810	-1.753	-0.452
VaR EWMA FHS	0.841	1.135	1.625	2.117	11.550	18.457	-1.232	-2.074
VaR EWMA EVT	0.839	1.348	2.202	3.065	11.418	18.707	-1.542	-1.551
VaR EWMA Stsk	1.041	1.422	2.057	2.697	12.566	19.230	-1.469	-1.775
VaR GARCH FHS	0.728	0.940	1.292	1.645	8.910	14.410	-0.845	-1.610
VaR GARCH EVT	0.749	1.194	1.939	2.692	9.768	16.084	-1.112	-1.167
VaR GARCH Stsk	0.952	1.476	2.355	3.247	12.401	18.629	-1.097	-1.115
CVaR Hist	0.062	0.918	2.365	3.837	4.384	7.688	-1.849	-0.510
CVaR EWMA FHS	0.833	1.427	2.426	3.438	12.390	20.322	-2.187	-1.873
CVaR EWMA EVT	0.844	1.509	2.627	3.762	12.426	20.363	-1.979	-1.599
CVaR EWMA Stsk	0.995	1.596	2.606	3.630	13.877	21.285	-2.118	-2.006
CVaR GARCH FHS	0.847	1.277	1.996	2.721	10.543	17.423	-1.538	-1.377
CVaR GARCH EVT	0.876	1.418	2.327	3.248	10.814	17.844	-1.513	-1.226
CVaR GARCH Stsk	0.828	1.550	2.768	4.005	13.040	20.150	-1.512	-1.162
	$b = 0.8$		$b = 1$		$b = 0.8$		$b = 0.8$	
Panel C: Δ_{LA}	$l = 2$	$l = 3$	$l = 2$	$l = 3$	$l = 2$	$l = 3$	$l = 2$	$l = 3$
Vola Hist	-4.057	-6.599	-4.707	-7.490	19.566	28.406	-9.141	-13.852
Vola EWMA	-3.191	-5.377	-3.606	-5.932	20.476	29.792	-8.289	-12.770
Vola GARCH	-1.786	-3.157	-1.865	-3.224	17.193	25.036	-6.377	-9.761
VaR Hist	8.009	12.568	10.125	15.791	6.462	10.148	9.090	15.170
VaR EWMA FHS	1.423	1.749	1.978	2.617	31.116	45.090	-5.690	-8.333
VaR EWMA EVT	3.564	5.064	4.635	6.710	32.736	47.798	-1.133	-1.114
VaR EWMA Stsk	2.641	3.537	3.426	4.733	30.469	43.812	-2.489	-3.272
VaR GARCH FHS	0.251	0.013	0.613	0.591	22.955	33.377	-5.268	-7.836
VaR GARCH EVT	2.574	3.590	3.473	4.971	26.938	39.426	-1.381	-1.666
VaR GARCH Stsk	3.851	5.479	5.013	7.230	28.531	40.709	-0.666	-0.561
CVaR Hist	9.139	14.167	11.688	18.014	13.800	20.473	9.395	15.689
CVaR EWMA FHS	4.589	6.648	5.887	8.645	36.478	53.296	0.674	2.023
CVaR EWMA EVT	5.332	7.800	6.836	10.110	36.525	53.359	1.188	2.758
CVaR EWMA Stsk	5.171	7.479	6.541	9.562	34.719	50.002	-0.055	0.858
CVaR GARCH FHS	2.565	3.526	3.439	4.864	30.050	43.969	-0.239	0.328
CVaR GARCH EVT	3.752	5.352	4.918	7.129	30.919	45.186	0.620	1.669
CVaR GARCH Stsk	6.149	9.106	7.867	11.715	32.692	47.158	1.759	3.466

switch to the HSD managed strategy. However, the same investor would even pay an annualized fee of 4.253% to switch from the 60/40 portfolio to the CVaR-GARCH-Stsk strategy. The differences between volatility and downside risk timing become even more striking during the financial crisis. An investor with risk-aversion $\gamma = 5$ would pay an annualized fee of 9.184% to switch from the 60/40 portfolio to the EWMA managed strategy during the financial crisis. However, the same investor is even willing to pay an annualized fee of 15.384% to switch to the CVaR-EWMA-Stsk strategy. During crash periods mean-variance investors are willing to pay extremely high fees to switch to a risk targeting strategy, where the willingness to pay for CVaR managed strategies is significantly higher than the willingness to pay for volatility managed strategies. Hence, investors are willing to pay extremely high fees to mitigate crashes as best done by managing CVaR. This is in line with the results of Bollerslev and Todorov (2011) and Chabi-Yo et al. (2018) that investors are crash-averse. We further find that during the financial crisis the EWMA model is again superior to the GARCH model, what is in line with the results of Table VII. Further, during the financial crisis, investors are willing to pay much lower fees to switch to the static VaR-HS and CVaR-HS models. Hence, static models fail to achieve a good downside risk protection just when it is most needed. As expected, results reverse when the period following the financial crisis is assessed. Now, all risk targeting strategies exhibit a negative economic value. Panel B shows the results of risk timing for an investor with CRRA utility, i.e. incorporating preferences for higher moments like skewness and kurtosis. However, results are quite similar to the results of a mean-variance investor in Panel A. Again, managing CVaR is superior to managing volatility, especially in times of bear markets and for highly risk-averse investors. However, results reverse during the uptrending period.

Panel C shows the economic value of risk targeting for a loss-averse investor. The economic value for a loss-averse investor is significantly different to the economic value of a mean-variance or CRRA investor. This result is in line with Aït-Sahalia and Brandt (2001) who find similar results when mean-variance or CRRA preferences are used in portfolio selection problems but vastly different allocations for loss-averse investors. Interestingly, over the whole period the economic value of the target volatility strategies is negative, regardless of the level of loss-aversion and volatility model. In other words, a loss-averse investor would pay a positive fee to switch away from a target volatility strategy to the 60/40 portfolio. In contrast, the economic value of downside risk targeting is always positive and typically very high. We again find that managing CVaR produces the highest economic value, i.e. a loss-averse investor should time CVaR or at least VaR instead volatility. Somewhat surprising, we find a higher economic value for the unconditional models (VaR-HS and CVaR-HS). However, this finding can be explained by the lower equity exposure of these strategies, what makes these strategies more conservative, and hence, more appealing for loss-averse investors. The extremely high economic value for loss-averse investors that manage downside risk can partly be explained by the daily evaluation period used in calculating the economic value. Benartzi and Thaler (1995) show that loss-aversion is more pronounced for shorter evaluation periods, i.e. the shorter the evaluation period for a loss-averse investor the less attractive are investments with higher risk. Similar horizon effects have been found by Aït-Sahalia and Brandt (2001) for loss aversion, but not for mean-variance and CRRA preferences. The authors conclude that loss-aversion implies that short term investors are extremely risk-averse, whereas long-term investors become more risk-neutral.[69] During the financial crisis the economic value becomes extremely high, i.e. a loss-averse investor is willing to pay extremely high fees for downside risk protection during crash periods. This again confirms the result of Bollerslev and Todorov (2011) and Chabi-Yo et al. (2018) that investors are crash-averse. For example, a loss-averse investor with parameters $b = 0.8$ and $l = 3$ would pay an annualized fee of 29.792% to switch from the 60/40 portfolio to the EWMA managed target volatility strategy. However, the same investor would even pay a fee of 53.359% per year to switch to a CVaR managed strategy. Further, during the crash period we find a significantly higher economic value for dynamically managed strategies, what indicates that these models are more successful in managing extreme negative returns in

[69] We also calculated the economic value of a loss-averse investor by first aggregating the daily returns to monthly returns. As expected, the economic value for a loss-averse investor calculated with monthly returns is smaller than the economic value calculated with daily data. The economic value of volatility targeting is still negative for all combinations. The economic value of downside risk targeting is still positive for all combinations except for the VaR-GARCH-FHS model for $b = 0.8$. The highest economic value is again obtained by managing CVaR. The economic value for a CRRA investor is nearly unchanged when using monthly returns instead of daily returns. This result is also found by Aït-Sahalia and Brandt (2001) for the optimal portfolio choice under CRRA preferences and loss-aversion.

crash periods. Interestingly, opposed to the results of the mean-variance and CRRA investor, we even find a positive economic value of almost all CVaR targeting strategies in the uptrending market, whereas the economic value of volatility is negative and high in magnitude. That is, a loss-averse investor should time CVaR instead of volatility regardless of whether there is a bull or a bear regime.

The fees given in Table VIII are extremely high compared to the fees found by Bollerslev et al. (2018). The authors argue that even their fees, in the range of 0.5%, are extremely beneficial for investors. This highlights the advantage of risk targeting, especially CVaR targeting, found for our data set. However, there are several differences between our study and the study of Bollerslev et al. (2018) which explain the differences in the magnitude of the fees. First, Bollerslev et al. (2018) calculate utility gains of several volatility targeting strategies, relying on different volatility models, against a benchmark volatility targeting strategy. That is, the authors choose a certain target volatility strategy as benchmark model, whereas we choose the 60/40 portfolio as benchmark similar to Marquering and Verbeek (2004). Second, the authors only compare the differences between several forecasting models, whereas we also compare the differences between volatility and downside risk targeting. In line with the results of Bollerslev et al. (2018) differences within the volatility targeting strategies are only small, whereas the differences between volatility and CVaR targeting are significantly higher. For example, a mean-variance investor with a risk-aversion of $\gamma = 5$ would pay a fee of $0.789 - 0.280 = 0.509\%$ per year to switch from the HSD managed strategy to the GARCH managed strategy. This result is comparable to the finding of Bollerslev et al. (2018) and again demonstrates the positive relation between accuracy – or equivalently constant portfolio volatility – and utility gains. However, the same investor would even pay $1.800 - 0.280 = 1.52\%$ per year to switch from the HSD managed strategy to the CVaR-EWMA-Stsk strategy. Third, Bollerslev et al. (2018) rebalance the weight of the volatility targeting strategy monthly, whereas we use daily rebalancing. Since the authors show that a higher accuracy typically coincides with higher utility gains, daily rebalancing should also produce a higher economic value. However, daily rebalancing also induces higher transaction costs, which dampens the extremely high fees of Table VIII. Rickenberg

(2019) examines volatility and downside risk targeting with monthly rebalancing. Fourth, additionally to mean-variance preferences we also calculate the economic value for loss-averse investors, who are willing to pay extremely high fees to mitigate extreme losses, as best done by the target CVaR strategy.

We next assess if the economic value found in Table VIII is also statistically significant. Bollerslev et al. (2018) use the DM-test to statistically compare the utility benefit of several volatility models used in a volatility targeting strategy. Taylor (2014, Sec. 2.2) presents a conditional test, that extends the DM-test, to asses if advanced forecasting models produce higher utility gains than simple forecasting models. Kirby and Ostdiek (2012) also use a bootstrap based test to assess the significance of the utility gain. A similar approach is also used by DeMiguel et al. (2009) to test differences in the certainty-equivalent return for mean-variance investors. We follow these approaches and also apply the tests presented in Section 4.1 to test if a strategy produces a significantly higher utility. These tests are also frequently used to test for a superior (risk-adjusted) performance of technical trading rules or mutual funds (see Sullivan et al. (1999), Hsu et al. (2010), Barras et al. (2010), Bajgrowicz and Scaillet (2012) among others). Results of these tests are shown in Table IX, where only results for $\gamma = 10$ for the mean-variance and CRRA investor as well as $l = 2$ and $b = 1$ for the loss-averse investor are shown. Whenever a benchmark model is needed, we choose the 60/40 portfolio as benchmark. Panel A shows results for the mean-variance investor. The DM-test indicates that almost all downside risk targeting strategies produce higher utility gains than the 60/40 portfolio whereas all target volatility strategies do not produce statistically higher utilities. The RC-test fails to reject any null hypotheses which again demonstrates the weaknesses of the RC-test. In contrast, the SPA-test rejects the null-hypotheses of all target volatility and target VaR strategies, whereas for most target CVaR strategies, the null-hypotheses can not bet rejected. Thus, the SPA-test indicates that the target CVaR strategies produce significantly higher utilities. Results of the MCS are quite similar to the results of the DM-test. All target volatility strategies are not contained in the MCS, which is also confirmed by the stepwise approaches, where we only show results for the studentized versions. The Step-SPA approach identifies all downside risk

Table IX. Testing the utility gain of risk targeting

This table shows the results of the tests presented in Section 4.1 used to test the significance of the utility gains. Panel A shows results for a mean-variance investor with $\gamma = 10$. Panel B shows results for a CRRA investor with $\gamma = 10$. Panel C shows results for a loss-averse investor with $b = 1$ and $l = 2$. The description of the columns is given in Tables I and II.

Panel A: MV	DM-test	p^{RC}	p^{SPA}	p^{SQ}	Step-RCst	Step-SPAst	$FDR^{+} = 10\%$
Vola Hist	-0.24	**20.35**	0.02	0.06	0	0	0
Vola EWMA	0.29	**26.86**	0.04	0.18	0	0	**16**
Vola GARCH	0.79	**31.00**	0.06	0.10	0	0	**15**
VaR Hist	**2.30**	**56.04**	1.70	**21.12**	**1**	**1**	**3**
VaR EWMA FHS	1.56	**47.29**	0.61	3.98	0	**2**	**13**
VaR EWMA EVT	**2.12**	**64.73**	0.82	**21.26**	**1**	**1**	**9**
VaR EWMA Stsk	**1.93**	**58.56**	0.93	**16.45**	**1**	**1**	**12**
VaR GARCH FHS	1.38	**39.83**	0.09	0.76	0	**3**	**14**
VaR GARCH EVT	**2.05**	**56.61**	0.39	7.23	**1**	**1**	**10**
VaR GARCH Stsk	**2.40**	**76.88**	3.40	**36.56**	**1**	**1**	**4**
CVaR Hist	**3.04**	**74.46**	**42.71**	**92.81**	**1**	**1**	**1**
CVaR EWMA FHS	**2.25**	**75.59**	1.34	**39.87**	**1**	**1**	**8**
CVaR EWMA EVT	**2.43**	**86.58**	**55.78**	**92.81**	**1**	**1**	**5**
CVaR EWMA Stsk	**2.34**	**86.24**	**48.81**	**92.81**	**1**	**1**	**7**
CVaR GARCH FHS	**2.05**	**59.26**	0.04	9.72	**1**	**1**	**11**
CVaR GARCH EVT	**2.36**	**75.53**	**20.76**	**36.56**	**1**	**1**	**6**
CVaR GARCH Stsk	**2.66**	**100.00**	**100.00**	**100.00**	**1**	**1**	**2**
Panel B: CRRA	**DM-test**	**p^{RC}**	**p^{SPA}**	**p^{SQ}**	**Step-RCst**	**Step-SPAst**	**$FDR^{+} = 10\%$**
Vola Hist	-0.27	**21.08**	0.01	0.14	0	0	0
Vola EWMA	0.27	**27.27**	0.02	0.36	0	0	**16**
Vola GARCH	0.76	**31.50**	0.07	0.21	0	0	**15**
VaR Hist	**2.29**	**56.86**	2.15	**21.21**	**1**	**1**	**4**
VaR EWMA FHS	1.55	**47.84**	0.64	4.17	0	**2**	**13**
VaR EWMA EVT	**2.11**	**65.93**	0.98	**21.21**	**1**	**1**	**9**
VaR EWMA Stsk	**1.92**	**59.03**	0.83	**16.01**	**2**	**1**	**12**
VaR GARCH FHS	1.36	**39.89**	0.11	0.88	0	0	**14**
VaR GARCH EVT	**2.04**	**57.78**	0.53	7.36	**1**	**1**	**11**
VaR GARCH Stsk	**2.40**	**77.12**	3.61	**36.32**	**1**	**1**	**6**
CVaR Hist	**3.03**	**75.18**	**42.33**	**91.85**	**1**	**1**	**1**
CVaR EWMA FHS	**2.24**	**76.41**	1.59	**39.36**	**1**	**1**	**8**
CVaR EWMA EVT	**2.43**	**86.69**	**55.11**	**91.85**	**1**	**1**	**3**
CVaR EWMA Stsk	**2.34**	**85.77**	**47.38**	**91.85**	**1**	**1**	**7**
CVaR GARCH FHS	**2.04**	**60.09**	0.05	9.79	**1**	**1**	**10**
CVaR GARCH EVT	**2.35**	**76.90**	**21.89**	**36.32**	**1**	**1**	**5**
CVaR GARCH Stsk	**2.65**	**100.00**	**100.00**	**100.00**	**1**	**1**	**2**
Panel C: LA	**DM-test**	**p^{RC}**	**p^{SPA}**	**p^{SQ}**	**Step-RCst**	**Step-SPAst**	**$FDR^{+} = 10\%$**
Vola Hist	-2.64	0.00	0.00	0.00	0	0	0
Vola EWMA	-2.21	0.00	0.00	0.00	0	0	0
Vola GARCH	-1.44	0.00	0.00	0.00	0	0	0
VaR Hist	**7.09**	**51.73**	0.00	0.20	**1**	**1**	**1**
VaR EWMA FHS	0.94	0.23	0.00	0.00	0	0	**13**
VaR EWMA EVT	**2.30**	2.51	0.00	0.00	**1**	**1**	**9**
VaR EWMA Stsk	**1.77**	1.00	0.00	0.00	**1**	**1**	**12**
VaR GARCH FHS	0.20	0.04	0.00	0.00	0	0	**14**
VaR GARCH EVT	**1.93**	0.77	0.00	0.00	**1**	**1**	**10**
VaR GARCH Stsk	**2.87**	2.74	0.00	0.00	**1**	**1**	**7**
CVaR Hist	**7.77**	**100.00**	**100.00**	**100.00**	**1**	**1**	**2**
CVaR EWMA FHS	**2.81**	6.08	0.00	0.00	**1**	**1**	**6**
CVaR EWMA EVT	**3.25**	9.83	0.02	0.20	**1**	**1**	**5**
CVaR EWMA Stsk	**3.24**	8.52	0.03	0.14	**1**	**1**	**3**
CVaR GARCH FHS	**1.87**	0.82	0.00	0.00	**1**	**1**	**11**
CVaR GARCH EVT	**2.65**	**2.66**	0.00	0.00	**1**	**1**	**8**
CVaR GARCH Stsk	**4.17**	**14.37**	0.01	0.20	**1**	**1**	**4**

targeting strategies as superior whereas the target volatility strategy strategies are not identified as superior. As expected, the FDR approach produces the largest set of superior models and picks all models except for the HSD model. Panel B shows results for the CRRA investor. As in Table VIII, these are again quite similar to the results of the mean-variance investor.

Panel C shows results for the loss-averse investor. The DM-test again indicates that almost all downside risk targeting strategies produce statistically higher utility gains. In contrast, the volatility targeting strategies produce lower utilities where the utility of the HSD and EWMA based strategies are even statistically lower with a test statistic lower than -1.64. Results for the RC- and SPA-test as well as for the MCS approach are very distinct to the findings of the DM-test. These tests indicate that the Historical Simulation based target CVaR strategy clearly outperforms the remaining models. This result is also in line with the high economic value of this strategy shown in Table VIII. In contrast, the stepwise approaches and the FDR approach produce large sets of optimal models and pick (almost) all downside risk targeting strategies whereas none of the target volatility strategies is chosen. The FDR approach further shows that the target CVaR strategies are typically picked in the first steps. The differences between the results of the MCS and the stepwise approaches can be explained by their construction. Whereas the stepwise approaches identify superior models and then test the remaining models, the MCS eliminates bad performing models and then tests the remaining models. Hence, in the MCS approach a good performing model remains in the test set until the last step and thus identifies all other models as inferior if one model clearly outperforms the remaining models. Due to the significantly higher economic value of the CVaR-HS model for a loss-averse investor, all other models are clearly eliminated in the first steps. In total, results of Panel C demonstrate that loss-averse investors should time downside risk instead of volatility where CVaR timing produces the best results. This again confirms the suggestion of Aït-Sahalia and Brandt (2001, p. 1315-1316) that loss aversion is similar to portfolio construction using CVaR as examined by Basak and Shapiro (2001).

5.5 Switching Strategies

Results so far indicate that volatility targeting produces higher returns in uptrending markets whereas CVaR targeting provides a better drawdown protection. However, in uptrending markets the CVaR targeting approach is typically too conservative. For that reason, we next examine strategies that switch between volatility and CVaR targeting, based on an estimate if the following day is an up- or down day. Combining different portfolio strategies is frequently examined in the literature (DeMiguel et al., 2009, Garlappi et al., 2006, Kan and Zhou, 2007, Tu and Zhou, 2011). Further, Wang et al. (2012) switches between different target levels where a more conservative target is chosen if a crash regime is expected. This is similar to our approach of switching to a more conservative strategy when a down-market is expected. Taylor (2014) proposes to switch between several forecasting models based on the current market environment, which is similar to switching between target risk strategies. A combined strategy, that manages portfolio risk by CVaR in times of bear markets and switches to a volatility based strategy in bull markets should be successful in mitigating drawdowns and simultaneously capturing the upside potential. Another possibility would be to buy the risky asset, i.e. $w_t = 1$, in bull markets and use a CVaR based strategy in bear markets. Following Tu and Zhou (2011) we define the weight of day t as

$$w_t^{switch} = \delta_t \cdot w_t^{CVaR} + (1 - \delta_t) \cdot w_t^{vol}, \tag{72}$$

where $\delta_t \in \mathbb{R}$ is the weight placed on the target CVaR strategy, w_t^{CVaR} is the day t weight of the CVaR targeting strategy and w_t^{vol} is the day t weight of the volatility targeting strategy. Several possibilities to define the crash indicator δ_t are possible. For example, a regime-switching process as in Ang and Bekaert (2002), Guidolin and Timmermann (2008) and Wang et al. (2012) could be used to determine bull and bear regimes. However, since risk targeting is also relevant for practical implementations we will rely on simple models to determine δ_t. For our first two switching strategies we model the parameter δ_t as a crash indicator that equals one if a negative return on day t is expected and zero else, given information up to day $t-1$. Hence, these approaches use either volatility or CVaR targeting. To determine $\delta_t \in \{0, 1\}$, we use methods from the literature on technical analysis (Bajgrowicz and Scaillet, 2012, Hsu et al., 2010, Moskowitz et al., 2012, Sullivan et al., 1999), where we use the two most prominent methods, i.e. Moving Averages (MA) and Time Series Momentum (TSMOM). Based on the

MA approach the indicator δ_t is given by

$$\delta_t = \begin{cases} 1, & \text{if } S_{t-1} \leqslant \text{MA}_{t-1,n} \\ 0, & \text{if } S_{t-1} > \text{MA}_{t-1,n}, \end{cases} \tag{73}$$

where $\text{MA}_{t-1,n} = \frac{1}{n}\sum_{i=1}^{n} S_{t-i}$ denotes the Moving Average with a length of n days. Hence, if the risky asset is in an uptrend, given if the price of day $t-1$ is higher than the average price of the days $t-1$ to $t-n$, the portfolio is manged by volatility. In contrast, if the risky asset is in a downtrend, given by $S_{t-1} \leqslant \text{MA}_{t-1,n}$, the portfolio is manged by the more conservative CVaR targeting approach.

Using the TSMOM approach of Moskowitz et al. (2012) the indicator δ_t is given by

$$\delta_t = \begin{cases} 1, & \text{if } S_{t-1} \leqslant S_{t-1-n} \\ 0, & \text{if } S_{t-1} > S_{t-1-n}. \end{cases} \tag{74}$$

Hence, the portfolio on day t is managed by volatility if the price of day $t-1$ is higher than the price of day $t-1-n$ and thus the risky asset is in an uptrend. In contrast, during a downtrend ($S_{t-1} \leqslant S_{t-1-n}$) the more conservative CVaR targeting approach is used. We show results for $n = 200$ which is the most often used length in the literature and by practitioners.[70]

The two indicators defined above are dummy variables, taking a value $\delta_t = 1$ if a negative return is likely and zero else. As a consequence, the weight of day t is either given by the volatility targeting strategy *or* the CVaR targeting strategy. We next define a third indicator, where the weight of day t is given as a combination of the volatility and CVaR targeting strategy. This is similar to combining different forecasting approaches (Halbleib and Pohlmeier, 2012, Taylor, 2014). If market risk increases, measured by expected volatility of day t, we place a higher weight on the CVaR targeting strategy, whereas CVaR targeting becomes less important when market risk decreases.[71] More formally, we define δ_t as

$$\delta_t = \frac{\hat{\sigma}_t}{\sigma^{\text{target}}}, \tag{75}$$

[70] Moskowitz et al. (2012) find good results for the TSMOM strategy for periods between one and 36 months, which corresponds to windows of approximately 21 and 756 days. We also used other lengths and found good results for other choices of n. For example, choosing $n = 150$ produces even higher risk-adjusted returns compared to $n = 200$. However, since $n = 200$ is the most relevant length, we only show results for this choice.

[71] We also used an indicator based on the two indicators defined above given by $\delta_t = (\delta_t^{MA} + \delta_t^{TSMOM})/2$. Thus, this strategy uses either volatility targeting, CVaR targeting or an equally weighted combination of both. However, results where quite similar to the previous approaches and are not reported.

Table X. Performance results of risk targeting: Switching strategies

This table shows the performance results of the strategies that switch between the GARCH volatility targeting strategy and the CVaR targeting strategies for three different indicators. Panel A shows results for the indicator δ_t based on a 200 days Time-Series-Momentum rule. Panel B shows results for the indicator δ_t based on a 200 days Moving Average rule. Panel C shows results for the indicator δ_t based on the GARCH(1,1) volatility forecast. Descriptions of the columns are given in Table V.

Panel A: TSMOM Indicator	Return	Vola	SR	z_{JK}	MDD	Calmar	VaR	CVaR	Min	Max
Vola Hist	3.12	12.84	0.102	-	41.24	0.032	1.38	1.82	-5.64	5.14
DAX	2.73	23.46	0.040	-0.57	70.42	0.013	2.36	3.46	-8.49	11.40
60/40	2.37	11.94	0.048	-0.59	40.07	0.014	1.24	1.74	-4.32	5.10
GARCH/CVaR Hist	4.09	11.74	0.193	1.57	31.17	0.073	1.23	1.66	-5.12	4.17
GARCH/CVaR EWMA FHS	3.65	11.09	0.166	**1.73**	36.59	0.050	1.18	1.56	-4.01	4.03
GARCH/CVaR EWMA EVT	3.70	11.01	0.171	**1.84**	36.17	0.052	1.17	1.55	-3.90	4.03
GARCH/CVaR EWMA Stsk	3.97	11.11	0.194	**2.48**	34.50	0.062	1.18	1.56	-4.67	4.03
GARCH/CVaR GARCH FHS	3.56	11.46	0.153	1.38	38.38	0.046	1.22	1.60	-4.58	4.03
GARCH/CVaR GARCH EVT	3.66	11.32	0.163	**1.66**	37.24	0.050	1.20	1.58	-4.41	4.03
GARCH/CVaR GARCH Stsk	3.92	11.14	0.189	**2.15**	34.92	0.060	1.18	1.56	-4.75	4.03
Panel B: MA Indicator	**Return**	**Vola**	**SR**	z_{JK}	**MDD**	**Calmar**	**VaR**	**CVaR**	**Min**	**Max**
Vola Hist	3.12	12.84	0.102	-	41.24	0.032	1.38	1.82	-5.64	5.14
DAX	2.73	23.46	0.040	-0.57	70.42	0.013	2.36	3.46	-8.49	11.40
60/40	2.37	11.94	0.048	-0.59	40.07	0.014	1.24	1.74	-4.32	5.10
GARCH/CVaR Hist	3.97	11.78	0.183	1.35	31.18	0.069	1.23	1.67	-5.12	4.17
GARCH/CVaR EWMA FHS	3.56	11.09	0.158	1.49	36.47	0.048	1.18	1.56	-4.01	4.03
GARCH/CVaR EWMA EVT	3.61	11.01	0.163	1.61	36.08	0.050	1.17	1.55	-3.90	4.03
GARCH/CVaR EWMA Stsk	3.76	11.15	0.174	**1.98**	34.96	0.056	1.18	1.57	-4.67	4.03
GARCH/CVaR GARCH FHS	3.54	11.44	0.151	1.31	38.07	0.045	1.22	1.59	-4.58	4.03
GARCH/CVaR GARCH EVT	3.63	11.29	0.161	1.56	36.92	0.049	1.20	1.57	-4.41	4.03
GARCH/CVaR GARCH Stsk	3.77	11.14	0.175	**1.80**	34.58	0.056	1.18	1.56	-4.75	4.03
Panel C: Volatility Indicator	**Return**	**Vola**	**SR**	z_{JK}	**MDD**	**Calmar**	**VaR**	**CVaR**	**Min**	**Max**
Vola Hist	3.12	12.84	0.102	-	41.24	0.032	1.38	1.82	-5.64	5.14
DAX	2.73	23.46	0.040	-0.57	70.42	0.013	2.36	3.46	-8.49	11.40
60/40	2.37	11.94	0.048	-0.59	40.07	0.014	1.24	1.74	-4.32	5.10
GARCH/CVaR Hist	2.41	12.07	0.051	-0.27	36.72	0.017	0.89	1.78	-10.17	11.61
GARCH/CVaR EWMA FHS	3.12	9.02	0.146	0.51	30.29	0.044	0.95	1.34	-3.55	3.83
GARCH/CVaR EWMA EVT	3.08	8.67	0.148	0.51	29.06	0.044	0.91	1.29	-3.40	3.79
GARCH/CVaR EWMA Stsk	3.75	9.19	0.210	1.16	26.45	0.073	0.97	1.35	-4.45	5.01
GARCH/CVaR GARCH FHS	3.27	9.94	0.148	0.70	35.29	0.042	1.06	1.42	-4.33	3.38
GARCH/CVaR GARCH EVT	3.26	9.30	0.156	0.81	32.01	0.045	1.00	1.33	-4.09	3.33
GARCH/CVaR GARCH Stsk	3.44	8.87	0.184	0.76	27.54	0.059	0.93	1.24	-4.56	11.05

where σ^{target} is again the chosen volatility target and $\hat{\sigma}_t$ is the volatility forecast of one of the volatility models. Defining δ_t with respect to the chosen volatility target is appealing since more risk-averse investors choose lower levels of σ^{target}, which implies higher values of δ_t. Further, as shown above, more risk-averse investors obtain higher utility gains from CVaR targeting compared to volatility targeting. Hence, by choosing δ_t as a function of σ^{target} more risk-averse investors place higher weights on CVaR targeting whereas risk-seeking investors place higher weights on volatility targeting.

The weight of the switching strategy under the indicator given in Equation (75) can be rewritten as

$$w_t^{switch} = w_t^{vol} + \left(\frac{w_t^{CVaR}}{w_t^{vol}} - 1 \right). \tag{76}$$

Hence, this switching strategy is similar to the volatility targeting strategy with weight w_t^{vol}, but places more (less) weight on the risky asset when the weight of the CVaR targeting strategy is higher (lower) than the weight of the volatility targeting strategy. This strategy is similar to the approach of Packham et al. (2017) who examine tail risk hedging strategies based on the difference of VaR forecasts under a normality assumption and forecasting methods that take non-normalities into account. By definition the CVaR takes non-normalities into account and w_t^{CVaR} should be higher (lower) than w_t^{vol} when the market is in an up-market (down-market) with lower (higher) left tail risk. Thus, this switching strategy should be similar to the volatility targeting strategy but reacts more sensitive to the market environment where the weight is lowered in a down-market and increased in an up-market.

Results for the three indicators are given in Table X, where we only show results for the strategies that switch between the GARCH model and the CVaR targeting strategies. For a better comparison to previous results we also show results for the HSD based target volatility strategy, the DAX and the 60/40 given in Table V. Panel A shows results for the indicator δ_t based on the TSMOM strategy. Switching between the GARCH and the target CVaR strategies successfully heightens the return while volatility is reduced compared to the HSD model. The switching strategies provide an enhanced risk-return profile indicated by a higher Sharpe Ratio and Calmar Ratio than the individual strategies given in Table V. For example, the strategy that switches between the GARCH model and the CVaR-EWMA-Stsk strategy increases the Sharpe Ratio of the HSD model by $0.194/0.102-1 = 90.2\%$. The high increase of the Sharpe Ratio can also be seen by the Sharpe Ratio test of Jobson and Korkie (1981). Most switching strategies provide a statistically higher Sharpe Ratio than the HSD model, whereas only one model in Table V was able to provide a statistically higher Sharpe Ratio. Further, the switching strategies also provide a higher drawdown protection indicated by the lower MDD and minimum return. Panel B shows results for the indicator based on the 200 day Moving Average. Results are quite similar to the TSMOM based indicator, however, results for the TSMOM based strategies

Table XI. Economic value of risk targeting: Switching strategies

This table shows the economic value given as annualized percentage fee Δ_i an investor is willing to pay to switch from the 60/40 portfolio to a strategy that switches between volatility and CVaR targeting for a given utility function U_i, $i \in \{\text{MV}, \text{CRRA}, \text{LA}\}$. Panel A shows the economic value for a mean-variance investor. Panel B shows the economic value for an investor with CRRA utility. Panel C shows the economic value for a loss-averse investor. γ indicates the investor's risk aversion. l determines the investor's loss aversion and b measures the investor's degree of risk seeking for negative returns and risk aversion for positive returns.

	Whole Sample				Crash		Recovery	
Panel A: Δ_{MV}	$\gamma = 2$	$\gamma = 5$	$\gamma = 10$	$\gamma = 15$	$\gamma = 5$	$\gamma = 10$	$\gamma = 5$	$\gamma = 10$
GARCH/CVaR Hist	0.021	-0.027	-0.107	-0.187	-17.684	-29.044	-4.000	-2.060
GARCH/CVaR EWMA FHS	1.048	1.979	3.549	5.143	18.973	28.798	-3.858	-2.974
GARCH/CVaR EWMA EVT	1.042	2.068	3.803	5.565	19.049	28.861	-3.644	-2.677
GARCH/CVaR EWMA Stsk	1.643	2.532	4.030	5.549	30.228	39.512	-3.926	-3.381
GARCH/CVaR GARCH FHS	1.108	1.775	2.897	4.031	16.608	26.172	-2.846	-2.140
GARCH/CVaR GARCH EVT	1.157	2.012	3.454	4.915	17.129	26.828	-2.929	-2.065
GARCH/CVaR GARCH Stsk	1.374	2.349	3.995	5.667	28.440	34.591	-3.081	-2.202
Panel B: Δ_{CRRA}	$\gamma = 2$	$\gamma = 5$	$\gamma = 10$	$\gamma = 15$	$\gamma = 5$	$\gamma = 10$	$\gamma = 5$	$\gamma = 10$
GARCH/CVaR Hist	0.038	0.003	-0.073	-0.253	-15.156	-27.065	-4.384	-2.457
GARCH/CVaR EWMA FHS	0.740	1.666	3.230	4.823	17.046	26.590	-4.036	-3.164
GARCH/CVaR EWMA EVT	0.702	1.724	3.452	5.217	17.124	26.655	-3.838	-2.882
GARCH/CVaR EWMA Stsk	1.349	2.235	3.734	5.263	28.434	37.634	-4.034	-3.493
GARCH/CVaR GARCH FHS	0.887	1.550	2.666	3.799	14.734	24.040	-2.988	-2.290
GARCH/CVaR GARCH EVT	0.874	1.725	3.162	4.625	15.230	24.668	-3.102	-2.246
GARCH/CVaR GARCH Stsk	1.052	2.039	3.743	5.514	27.592	35.013	-3.256	-2.381
	$b = 0.8$		$b = 1$		$b = 0.8$		$b = 0.8$	
Panel C: Δ_{LA}	$l = 2$	$l = 3$	$l = 2$	$l = 3$	$l = 2$	$l = 3$	$l = 2$	$l = 3$
GARCH/CVaR Hist	11.127	17.064	13.333	20.235	-7.410	-9.874	15.241	25.294
GARCH/CVaR EWMA FHS	9.987	14.961	12.484	18.822	49.459	71.461	4.675	9.222
GARCH/CVaR EWMA EVT	11.197	16.822	14.115	21.331	49.518	71.462	5.308	10.135
GARCH/CVaR EWMA Stsk	10.536	15.568	13.226	19.588	52.597	72.702	3.082	6.721
GARCH/CVaR GARCH FHS	5.885	8.617	7.391	10.933	45.944	67.806	3.056	6.217
GARCH/CVaR GARCH EVT	8.054	11.986	10.165	15.233	47.181	69.430	4.273	8.181
GARCH/CVaR GARCH Stsk	11.146	16.814	14.568	21.820	49.962	69.412	5.682	10.319

are slightly better. Panel C shows results for the volatility based indicator δ_t. Interestingly, although some strategies based on this indicator produce the highest Sharpe Ratios in Table X, none of these strategies produces a significantly higher Sharpe Ratio for the test of Jobson and Korkie (1981). The volatility based indicator δ_t exhibits the lowest drawdowns among the three indicators which is in line with Equation (76) that this strategy is similar to the volatility targeting strategy, but more sensitive to up- and down-markets.

Table XI shows the economic value of the switching strategies that use the volatility based indicator δ_t for the three investors. We only show results for the volatility based indicator δ_t since Table X indicates that none of the switching strategies based on this indicator produces significant performance gains for the test of Jobson and Korkie (1981). We will test in Table

XII if the same result also holds when testing for higher utilities of the volatility based indicator. The economic value is calculated with respect to the 60/40 portfolio, i.e. the numbers in this table correspond to the annual percentage fee an investor is willing to pay to switch from the 60/40 portfolio to one of the switching strategies and can be compared to results of Table VIII. Results of Panel A and B in Table XI are similar to the results of Table VIII but higher in magnitude. That is, mean-variance and CRRA investors are willing to pay higher fees for the switching strategies over the whole sample and the crash period compared to the individual strategies. However, during the calm period these investors prefer the 60/40 portfolio. Thus, a possible extension of our switching approach would be to switch between the CVaR managed strategy and a non-managed static portfolio. Panel C shows results for the loss-averse investor. Results are again similar to Table VIII but higher in magnitude for the whole period and the period capturing the financial crisis. Interestingly, during the crisis period the economic value of the strategy that switches to the Historical Simulation based target CVaR strategy is negative, although this strategy was quite convincing in Table VIII. More interestingly are the results during the calm period. Now, the economic value of all switching strategies becomes positive and high in magnitude. This holds especially for the strategy that switches to the Historical Simulation based strategy. Thus, loss-averse investors are willing to pay high fees to have access to a strategy that switches between volatility and CVaR targeting even when the market is in a calm period. In contrast, a loss-averse investor is not willing to pay a positive fee to use the GARCH based volatility strategy as shown in Table VIII.

To summarize Table XI switching between volatility and CVaR targeting heightens utility gains for all three investors compared to the static 60/40 portfolio. Further, utility gains of the switching strategies are higher in magnitude compared to the economic value of the individual risk targeting strategies shown in Table VIII. This confirms the earlier finding of Table X that the switching strategies exhibit an enhanced risk-return profile. We will next test, if these higher utilities are also statistically significant. Table XII shows results for the tests that test for higher utility gains of the switching strategies. For comparison we also include the three volatility targeting strategies. Table XII shows that the switching strategies produce statistically higher

Table XII. Testing the utility gain of risk targeting: Switching strategies
This table shows the results of the tests presented in Section 4.1 used to test the significance of the utility gains. Panel A shows results for a mean-variance investor with $\gamma = 10$. Panel B shows results for a CRRA investor with $\gamma = 10$. Panel C shows results for a loss-averse investor with $b = 1$ and $l = 2$. The description of the columns is given in Tables I and II.

Panel A: MV	DM-test	p^{RC}	p^{SPA}	p^{SQ}	Step-RCst	Step-SPAst	$FDR^{+} = 10\%$
Vola Hist	-0.24	5.16	0.33	0.73	0	0	0
Vola EWMA	0.29	7.95	0.47	1.66	0	0	0
Vola GARCH	0.79	**10.31**	0.61	4.36	0	0	**7**
GARCH/CVaR Hist	-0.05	**12.20**	**14.52**	**19.91**	0	0	0
GARCH/CVaR EWMA FHS	**2.19**	**76.03**	**11.96**	**62.96**	**1**	**1**	**6**
GARCH/CVaR EWMA EVT	**2.29**	**83.87**	**54.92**	**94.49**	**1**	**1**	**4**
GARCH/CVaR EWMA Stsk	**2.16**	**100.00**	**100.00**	**100.00**	**1**	**1**	**3**
GARCH/CVaR GARCH FHS	**2.12**	**56.80**	1.22	**23.75**	**1**	**1**	**5**
GARCH/CVaR GARCH EVT	**2.40**	**80.96**	**42.27**	**62.96**	**1**	**1**	**1**
GARCH/CVaR GARCH Stsk	**2.38**	**88.54**	**64.08**	**95.61**	**1**	**1**	**2**
Panel B: CRRA	**DM-test**	**p^{RC}**	**p^{SPA}**	**p^{SQ}**	**Step-RCst**	**Step-SPAst**	**$FDR^{+} = 10\%$**
Vola Hist	-0.27	5.33	0.38	0.66	0	0	0
Vola EWMA	0.27	8.27	0.55	1.61	0	0	0
Vola GARCH	0.76	**10.56**	0.57	3.38	0	0	**7**
GARCH/CVaR Hist	-0.05	**12.40**	**14.69**	**19.67**	0	0	0
GARCH/CVaR EWMA FHS	**2.18**	**75.36**	**12.40**	**59.93**	**1**	**1**	**5**
GARCH/CVaR EWMA EVT	**2.29**	**83.45**	**54.81**	**91.25**	**1**	**1**	**3**
GARCH/CVaR EWMA Stsk	**2.16**	**87.54**	**61.20**	**92.79**	**1**	**1**	**4**
GARCH/CVaR GARCH FHS	**2.12**	**55.68**	1.10	**22.79**	**1**	**1**	**6**
GARCH/CVaR GARCH EVT	**2.39**	**79.01**	**37.64**	**59.93**	**1**	**1**	**1**
GARCH/CVaR GARCH Stsk	**2.36**	**100.00**	**100.00**	**100.00**	**1**	**1**	**2**
Panel C: LA	**DM-test**	**p^{RC}**	**p^{SPA}**	**p^{SQ}**	**Step-RCst**	**Step-SPAst**	**$FDR^{+} = 10\%$**
Vola Hist	-2.64	0.00	0.00	0.00	0	0	0
Vola EWMA	-2.21	0.00	0.00	0.00	0	0	0
Vola GARCH	-1.44	0.00	0.00	0.00	0	0	0
GARCH/CVaR Hist	**5.52**	**100.00**	**100.00**	**87.75**	**1**	**1**	**1**
GARCH/CVaR EWMA FHS	**4.19**	**33.82**	0.00	0.08	**1**	**1**	**2**
GARCH/CVaR EWMA EVT	**4.54**	**77.05**	**64.98**	**87.75**	**1**	**1**	**3**
GARCH/CVaR EWMA Stsk	**4.39**	**53.82**	**35.97**	**53.28**	**1**	**1**	**4**
GARCH/CVaR GARCH FHS	**3.07**	0.17	0.00	0.00	**1**	**1**	**7**
GARCH/CVaR GARCH EVT	**3.92**	4.36	0.00	0.01	**1**	**1**	**5**
GARCH/CVaR GARCH Stsk	**5.14**	**81.86**	**72.42**	**100.00**	**1**	**1**	**6**

utility gains whereas the volatility targeting strategies do not. Only the GARCH managed strategy produces a significant higher utility than the 60/40 portfolio for some test procedures. For the mean-variance and CRRA investors the strategies that switch to a conditionally managed strategies clearly provide higher utility gains, whereas the volatility targeting strategies and the strategy that switches to the Historical Simulation based strategy do not. For the loss-averse investor all switching strategies clearly produce higher utility gains than the volatility targeting strategy. Thus, whereas the test of Jobson and Korkie (1981) does not indicate that the switching strategies based on the volatility indicator δ_t exhibit statistically higher performance gains, results of Table XII show that switching between volatility and CVaR targeting produces significantly higher utility gains for all three investors. This again highlights the disadvantage of the

Sharpe Ratio as performance measure for dynamic trading strategies (Han, 2005, Marquering and Verbeek, 2004).

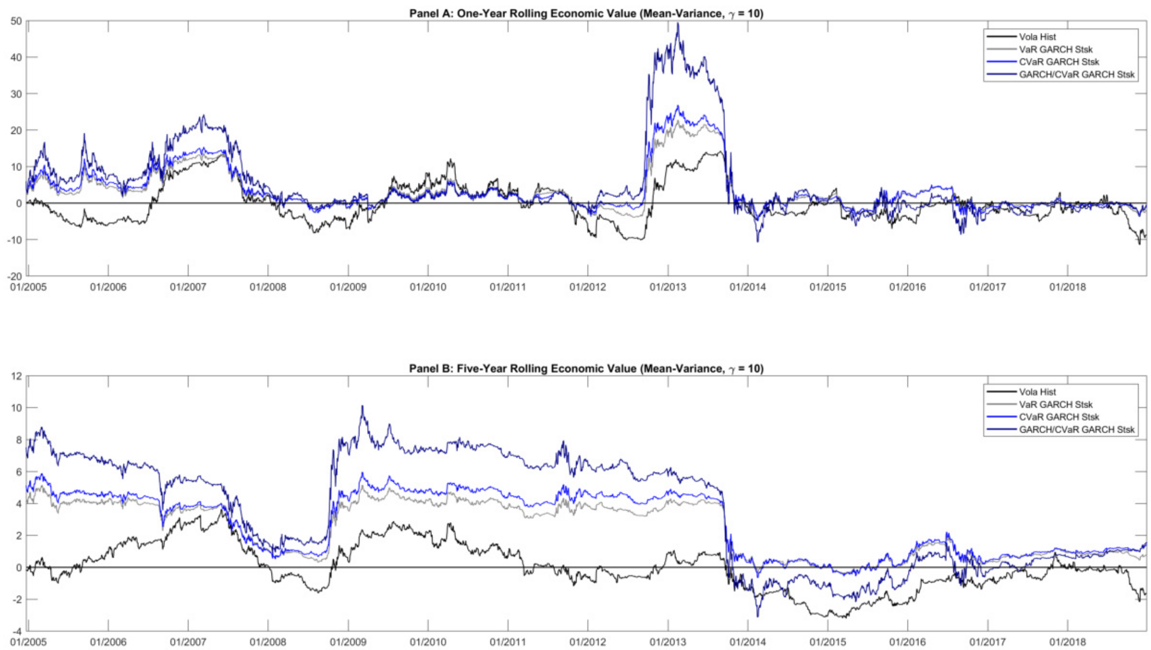

Figure I. One and five year rolling economic value for a mean-variance investor. This figure plots the one and five year rolling economic value measured by Δ_{MV} with respect to the 60/40 portfolio for a mean-variance investor with a risk aversion of $\gamma = 10$. Panel A shows the economic value for an investor with an investment horizon of one year, whereas Panel B shows the economic value for an investor with an investment horizon of five years. The dates on the x axes correspond to the end date of the one or five year investment horizon.

So far, we only calculated the economic value over the whole sample. However, most investors typically have short evaluation periods (Benartzi and Thaler, 1995). Further, timing short term risk is also beneficial for long-term investors, thus even long-term investors should be concerned about short term utility gains (Moreira and Muir, 2019). For that reason, similar to Figure 2 of Marquering and Verbeek (2004) we next plot in Figure I the rolling one and five year economic value of a mean-variance investor for our risk targeting and switching strategies. Thus, this figure plots the rolling annualized fee a mean-variance investor is willing to pay to switch from the 60/40 portfolio to the risk targeting strategies, when the investor has to be invested for one or five years. Panel A shows the rolling annualized percentage fee for an investor with a risk aversion of $\gamma = 10$ and an investment horizon of one year, whereas Panel B shows the rolling economic value for an investor with an investment horizon of five years.

Figure I shows that a mean-variance investor with an investment horizon of one year is almost always willing to pay a positive fee to have access to risk targeting. This holds especially for the strategies that target a constant level of downside risk and the switching strategy. The utility gains in the crises are substantially higher than the utility losses in the low risk periods. Hence, investors are willing to pay very high fees to avoid crash periods, whereas their utility loss of lower returns in uptrending markets is significantly smaller. In particular, during crash periods, the economic value of downside risk timing and the switching strategy is significantly higher than the economic value of volatility timing, whereas in calm periods the economic values of volatility targeting, downside risk timing and the switching strategy is comparable. Interestingly, we find that during crises the switching strategy outperforms all other strategies. Thus, even during crises switching between volatility and CVaR targeting outperforms downside risk targeting. Results in Panel B are similar to the results in Panel A but the difference between volatility targeting, downside targeting and the switching strategy becomes larger. Thus, for investors with longer investment horizons downside risk targeting becomes far more important than volatility targeting. This again holds especially for the strategy that switches between volatility and CVaR targeting. In particular, we find that on average the switching strategy produces the highest economic value, followed by CVaR targeting. In contrast, volatility targeting exhibits on average the lowest economic value, but there are also periods when volatility targeting produces the highest economic value. Results for the CRRA investor are again similar to the results of Figure I and are not shown here.

Figure II shows the rolling one and five year economic value for a loss-averse investor with parameters $b = 0.8$ and $l = 3$. Panel A shows results for an investor with an investment horizon of one year. For a loss-averse investor the economic value of downside risk targeting is always higher than the economic value of volatility targeting and the economic value of the switching strategy is always the highest. As in Figure I the economic value significantly increases for the periods that contain a crisis. Again, especially during crises, switching between volatility and CVaR targeting outperforms all the remaining strategies. Panel B shows results for a loss-averse investor with an investment horizon of five years. Interestingly, the economic value of volatility

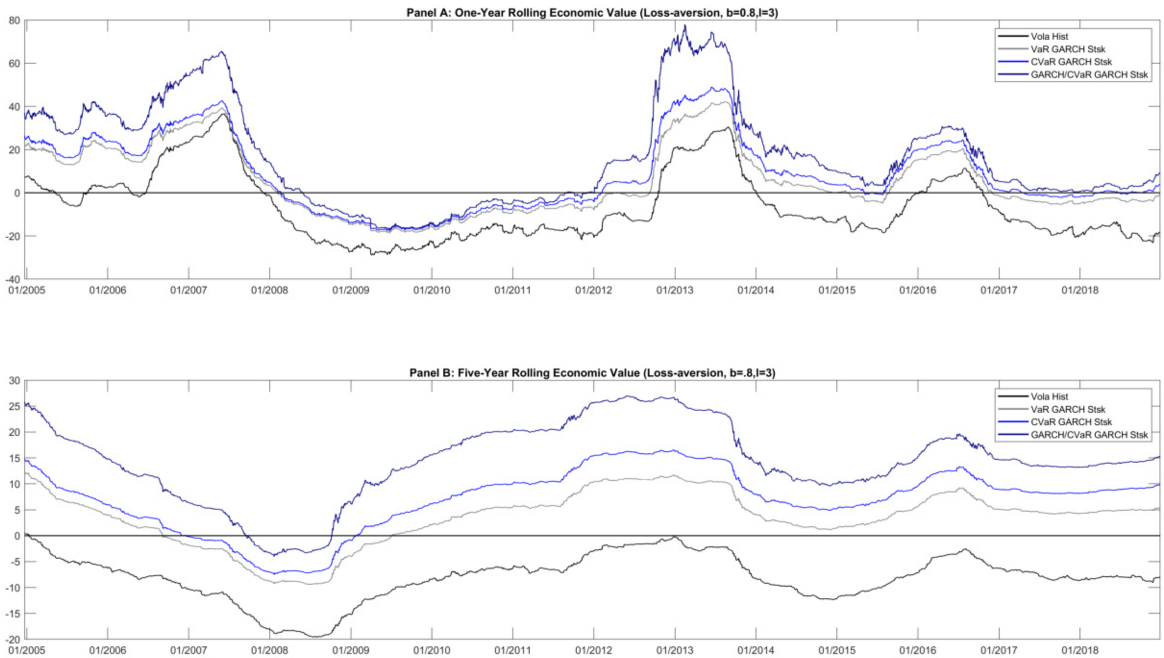

Figure II. One and five year rolling economic value for a loss-averse investor. This figure plots the one and five year rolling economic value measured by Δ_{LA} with respect to the 60/40 portfolio for a loss-averse with parameters $b = 0.8$ and $l = 3$. Panel A shows the economic value for an investor with an investment horizon of one year, whereas Panel B shows the economic value for an investor with an investment horizon of five years. The dates on the x axes correspond to the end date of the one or five year investment horizon.

targeting is always negative, i.e. a loss-averse investor with an investment horizon of five years is never willing to pay a fee to switch from the 60/40 portfolio to the HSD managed strategy. This contradicts the finding of Moreira and Muir (2019) that even long-term investors should time volatility.[72] In contrast, the economic value of downside risk targeting and the switching strategy is only negative for a short time period. Thus, a loss-averse investor with an investment horizon of five years should almost always target downside risk or, even more advantageous, should switch between volatility and CVaR targeting.

6 Conclusion

This paper studies dynamic trading strategies that target a predefined level of risk measured by volatility, Value at Risk (VaR) or Conditional Value at Risk (CVaR). We derive weights for these

[72]In contrast to our examination in Figure II, Moreira and Muir (2019) do not assess the economic value for loss-averse investors.

trading strategies and present several methods to estimate volatility, VaR and CVaR. Based on a dataset for the German stock market, we find that risk targeting offers an enhanced risk-return profile, better drawdown protection and significant utility gains compared to a buy and hold equity investment and a static portfolio consisting of equities and bonds. Most convincing results are found for strategies that target a constant level of portfolio CVaR over time. In particular, we find that mean-variance investors, CRRA investors and loss-averse investors should time downside risk, measured by CVaR, instead of volatility. This result especially holds for highly risk-averse or loss-averse investors and during crises. Generally, we find that risk should be managed by a conditional risk model instead of simple models as done by Barroso and Santa-Clara (2015), Barroso and Maio (2016) and Moreira and Muir (2017). This is in line with the result of Bollerslev et al. (2018) that a higher forecasting accuracy, and hence a more constant portfolio risk, typically coincides with higher performance benefits compared to static forecasting models.

The risk-return profile and utility gains of risk targeting can further be improved by switching between volatility and CVaR targeting, where CVaR targeting is only used when a negative market return is expected. Based on three different crash indicators we show that these switching strategies produce higher returns with lower risk compared to the volatility targeting strategies. Further, the mean-variance, CRRA and loss-averse investors are willing to pay high fees to have access to these switching strategies. When compared to the utility of a static portfolio allocation, utility gains of the switching strategies are statistically significant whereas the utility gains of volatility targeting are insignificant.

Appendices

A Advantages of Volatility Targeting

This section summarizes several reasons why investors should target a constant level of volatility. Most of these advantages presented here also hold for the target VaR and target CVaR strategies presented in Section 3. Further, Section 3.1 presents several advantages of managing downside risk instead of volatility.

First, using the weight given in Equation (3) implies that the weight of the risky asset is decreased in times of high volatility and increased in low volatile times. Since volatility is often associated with risk and investors are typically risk-averse (Scott and Horvath, 1980), targeting a constant level of volatility fits well to these investors' preferences. By choosing an adequate volatility target σ^{target} investors can choose an investment strategy that fits well to their preferences and risk aversion (Bollerslev et al., 2018). Zakamulin (2015) and Moreira and Muir (2017) show that mean-variance investors should, under some assumptions, optimally choose the weight of the risky asset as $w_t = \left(\sigma^{\text{target}}/\sigma_t\right)^2$ (see also Dopfel and Ramkumar (2013)).[73] Building on this result, as suggested by Kirby and Ostdiek (2012), decreasing the sensitivity of w_t to volatility changes, and hence lowering transaction costs, leads to the weight given in Equation (3).[74]

Second, especially during bear markets, which are associated with increases in volatility and correlations, investors seek for risk reduction methods (Ang and Bekaert, 2002).[75] The last financial crises were all accompanied by higher than normal volatilities (Liu et al., 2003, Moreira and Muir, 2017). In particular, times of high volatility typically coincide with times of downward moving markets (Ang et al., 2006b, Campbell and Hentschel, 1992, French et al., 1987). Similarly, Moreira and Muir (2017) find that the probability of a recession is higher in times of high market volatility, i.e. recessions coincide with times of high market volatility. Muir (2017) shows that in financial crises and recessions asset prices decline and stock market

[73] The mean-variance framework is only suitable for elliptical distributions. Since asset returns usually do not follow an elliptical distribution this weighting is not optimal for realistic return distributions (Szegö, 2002, p.1254). We will revisit this issue in Chapter 3 where we present a similar weighting scheme based on risk measures that account for non-normalities in the asset return distribution.

[74] Since volatility can only be estimated with an estimation error, and to lower transaction costs, Kirby and Ostdiek (2012) suggest to scale the weight by a parameter η, called tuning parameter, that determines how aggressively the weight w_t reacts to changes in σ_t. By choosing $\eta = 0.5$ we obtain the weight of the target volatility strategy (see also Zakamulin (2015, p. 91)). Moreira and Muir (2017) compare the weight of the volatility and variance managed strategies and find less extreme weights and lower transaction costs for the volatility managed strategy.

[75] Liu et al. (2003) find that most events with extremely negative returns are accompanied with high increases in volatility. Guidolin and Timmermann (2008) find a bear regime with low returns, negative alphas, high volatility and highly correlated assets and a bull regime with higher returns, positive alphas, lower volatility and less correlated returns (see also Wang et al. (2012, p. 27) and Hocquard et al. (2013)). Similarly, Ang and Bekaert (2002, p. 1139) find "a normal regime with low correlations, low volatilities, and a bear regime with higher correlations, higher volatilities, and lower conditional means." The bear states occurred during financial crises and/or global recessions indicating that periods of market distress are associated with high volatilities and low returns (Muir, 2017).

volatility increases but these effects reverse subsequently. Thus, times of significantly higher volatility coincide with declining asset prices, and hence these times should be avoided by investors. Further, since volatilities and correlations between different equity markets increase simultaneously during bear markets, drawdowns in crises can not simply be managed by diversification (Ang and Bekaert, 2002, Ang and Chen, 2002, Butler and Joaquin, 2002, Karolyi and Stulz, 1996, Longin and Solnik, 2001, Patton, 2004). Chabi-Yo et al. (2018) find that extreme negative returns of stocks are more related than extreme positive returns, i.e. stocks tend to crash simultaneously. In particular, the authors show that the relation of extreme negative returns among stocks increases in crash periods (Chabi-Yo et al., 2018, Figure 2). By incorporating a risk-free asset Ang and Bekaert (2002) find for their model that in the normal regime, the risky asset should be leveraged by being short in the risk-free asset, whereas in the bear regime money should dramatically be shifted to the risk-free asset. Furthermore, they find significant drawbacks of ignoring information about the regime, once the possibility of shifting money to the risk-free asset is introduced (see also Patton (2004)). Bollerslev et al. (2018) find co-movements, spillover effects and simultaneous spikes of volatilities between equities, bonds, commodities and currencies. Thus, risk characteristics between assets and asset classes are quite similar. Jondeau and Rockinger (2003) find that also higher moments like (negative) skewness and kurtosis increase simultaneously between markets during bear regimes. This indicates that the probability of an occurrence of large (negative) returns cannot be reduced by simply combining several risky assets. As a consequence, during high risk periods, the portfolio should be managed by simultaneously decreasing the exposure to a portfolio of risky assets and increasing exposure to the riskless asset, as done by the target volatility strategy. This also solves the problem identified by Ang and Chen (2002) and Longin and Solnik (2001) that investors incorrectly assess the benefits of diversification, and thus typically hold too much equities in bear markets whereas they are underinvested in bull markets. However, by managing volatility an investor is not protected against unpredictable tail events marked by periods with extreme jumps in asset prices.[76]

[76] See Liu et al. (2003) for a study on how jump risk in both equity prices and volatility effects the dynamic asset allocation between a risky and a riskless asset. In order to face jump risk investors should avoid leveraged positions in the risky asset, and hence an equity cap of 100% or a low volatility target should be used (see also Das

Third, an often proclaimed justification of volatility targeting is the relation between volatility and future return. Although classical finance models like the CAPM indicate that higher risk should be compensated by higher expected returns (see Merton (1980) for example), many empirical studies find a negative relation between volatility and returns, i.e. a higher volatility coincides with lower or negative future returns (Glosten et al., 1993).[77] A possible explanation for the negative volatility-return relation is the *volatility feedback* effect, which is sometimes called time-varying risk premium and is opposed to the well-known leverage effect (see Glosten et al. (1993, p. 1786) for an explanation of the leverage effect).[78] Based on this observation an increase in volatility induces an immediate stock decline. In other words, if tomorrow's volatility σ_{t+1} is expected to be higher than today's volatility σ_t, then tomorrow's weight w_{t+1} should be lower than today's weight w_t.[79] However, results in the academic literature on the relation between volatility and future returns are very mixed and a relation between volatility and returns is hard to confirm (Bollerslev et al., 2013, Glosten et al., 1993). Lundblad (2007) shows that for examining the relation of volatility and future returns very long datasets are needed. The author, using a dataset ranging from 1836 to 2003, finds a positive relation between volatility and

and Uppal (2004)). Jarrow and Zhao (2006) show that managing volatility differs from managing downside risk when asset returns exhibit jump risk.

[77] A similar observation has also been found in cross-sectional analyses. See for example Frazzini and Pedersen (2014) who show that buying low beta assets and selling high beta assets produces high returns, although classical finance theory indicates a contrary result. Similarly, Ang et al. (2006b) and Ang et al. (2009) show that assets with high past sensitivity to volatility changes, high idiosyncratic volatility or high total volatility have significantly lower returns than assets with low past sensitivity to volatility changes, low idiosyncratic volatility or low total volatility, respectively. Haugen and Heins (1975) find that the risk-return relation strongly depends on the sample period and whether the sample period is dominated by a bull or bear regime. Using a long data set, the authors find that "over the long run, stock portfolios with lesser variance in monthly returns have experienced greater average returns than their "riskier" counterparts" (Haugen and Heins, 1975, p. 782).

[78] Bollerslev et al. (2006, p. 354) describe the volatility feedback effect as: "If volatility is priced, an anticipated increase in volatility would raise the required rate of return, in turn necessitating an immediate stock-price decline to allow for higher returns. Therefore, the causality underlying the volatility feedback effect runs from volatility to prices, as opposed to the leverage effect that hinges on the reverse causal relationship" (see also Campbell and Hentschel (1992) and Glosten et al. (1993) for an explanation of leverage and volatility feedback effect). For additional studies on the relation between volatility and return see also French et al. (1987), Bali and Peng (2006), Bollerslev et al. (2006), Bollerslev and Zhou (2006), Ghysels et al. (2005), Lundblad (2007), Bollerslev et al. (1992) among others. See Muir (2017) for an examination why risk premiums or expected returns vary over time, rise modestly in recessions and spike in financial crises. See Glosten et al. (1993) on how the leverage effect influences results on the volatility feedback effect.

[79] The volatility feedback effect is reflected by the construction of the target volatility weighting given in Equation (3). More formally, from $\sigma_{t+1} > \sigma_t$ it follows $w_{t+1} = \sigma^{\text{target}}/\sigma_{t+1} < \sigma^{\text{target}}/\sigma_t = w_t$, i.e. an increase in volatility induces a decrease in the weight of the risky asset. Due to this relation, Harvey et al. (2018) find that under the leverage effect, volatility targeting induces momentum, i.e negative returns induce higher future volatilities and lower future weights of the risky asset. The authors find that this observations explains a part of the increase of the Sharpe Ratio of the volatility targeting strategy.

return. Bali and Peng (2006) using high frequency data based volatility measures find a significant and positive relation,[80] whereas Bollerslev et al. (2006) using similar volatility measures find an insignificant or even negative relation. Bollerslev and Zhou (2006, p. 124-125) state that the risk-return relation in empirical investigations strongly depends on the volatility measure used in this investigation, which partly explains the inconsistent results in the academic literature (see also Glosten et al. (1993), Ghysels et al. (2005) and Bollerslev et al. (2013)). Adrian and Rosenberg (2008) show that the risk-return relation depends strongly on the examined time frequency of volatility. In line with the volatility feedback effect, the authors find a negative volatility-return relation for short-term volatility but a positive relation for long-term volatility. To summarize results in the academic literature, the relation between volatility and future returns is hard to identify and results in the literature are too mixed to draw a distinct conclusion (see also Harvey and Siddique (1999) and references therein). Backus and Gregory (1993) theoretically confirm this observation (see also Glosten et al. (1993) who argue that both a positive and negative relation would be consistent with theory). However, Moreira and Muir (2017) show that the relation between volatility and future *risk-adjusted* returns should be of main interest instead of the risk-return relation (see also Dopfel and Ramkumar (2013)). Moreira and Muir (2017) find that the alpha of the volatility managed strategy is mainly driven by the negative relation between volatility and volatility adjusted returns. In particular, they theoretically show that an alpha of zero is obtained if movements of expected returns and volatility coincide. In other words, volatility targeting produces positive alphas, since an increase in volatility is not compensated by an adequate increase in expected return. Hence, high volatility periods exhibit an unattractive risk-return profile and should be avoided by investors. This is also empirically confirmed by the authors: although the authors cannot confirm a negative volatility-return relation, they find that volatility timing increases the Sharpe Ratio. The reason for the increasing Sharpe Ratio is that "changes in volatility are not offset by proportional changes in expected returns" (Moreira and Muir, 2017, p. 1611), i.e. the mean variance trade off is higher in times of low volatility and vice versa. Both observations combined indicate that a high volatility in

[80] Similarly, Ghysels et al. (2005) using daily data to measure monthly volatility by advanced volatility measures find a positive and significant relation.

$t-1$ is related to a low Sharpe Ratio in t. Dachraoui (2018, Eq. (2)) shows that the Sharpe Ratio of the target volatility strategy is given by the Sharpe Ratio of the risky asset and the correlation between the volatility and the risk-adjusted return of the risky asset. In particular, if volatility and risk-adjusted return of the risky asset are negatively correlated, the Sharpe Ratio of the target volatility strategy is higher than the Sharpe Ratio of the risky asset. A sufficient condition for this negative correlation is that volatility and return are negatively correlated or uncorrelated. Moreira and Muir (2019) find that an increase of volatility coincides with higher expected returns, but that the increase in expected return is much more persistent than the increase in volatility. Thus, investors should reduce the weight of the risky asset if short-term volatility increases and then subsequently increase the risky exposure when volatility begins to decline. Barroso and Maio (2016) find that volatility targeting works well since risk and future returns are nearly uncorrelated and risk is highly forecastable due to its persistent nature. Similarly, Harvey et al. (2018) find no clear pattern between volatility of day $t-1$ and the return of day t. However, due to the persistence of volatility, they find that a high volatility in $t-1$ is related to a high volatility in t. Hence, an investor should be higher invested in the risky asset if the risky asset's volatility is low and vice versa and thus should time volatility. These results do not fundamentally contradict the assumption that higher risk is compensated by higher expected returns. In the long run, assets with higher volatility typically earn higher risk premiums, but risk premiums typically fluctuate over time (Lempérière et al., 2017, Muir, 2017). In the long run a higher volatility is related to higher expected return whereas in the short run a higher volatility is related to low or negative returns (Adrian and Rosenberg, 2008). Stocks typically have higher long-term returns than bonds, i.e. investors with a long investment horizon should participate in the stock market. However, most investors fail to capture the long-term potential of stocks, due to too short evaluation periods and the higher volatility of stocks (Benartzi and Thaler, 1995). A nice characteristic of risk targeting is that even highly risk averse investors can participate at the huge long-term return potential of risky assets, where the investor can choose the risk tolerance he is willing to accept. Further, besides making stock market investments available for all investors, volatility targeting can even enhance the risk-adjusted performance for long-term investors by dynamically timing the risky asset's short-term risk (Moreira and Muir, 2019).

Fourth, many studies have shown that volatility timing can add substantial economic value and delivers an enhanced risk-return profile. Fleming et al. (2001), Fleming et al. (2003), Han (2005), Kirby and Ostdiek (2012) and Taylor (2014) examine the utility gain of volatility timing strategies, i.e. strategies that rely on estimates of the covariance matrix solely, in a multivariate setting and find that these strategies are superior to non-volatility managed portfolios even after transaction costs. Moskowitz et al. (2012) and Kim et al. (2016) use volatility timing to manage the risk of the time series momentum strategy. Asness et al. (2013) and Goyal and Jegadeesh (2017) use volatility timing to weight the assets in the momentum portfolio. Although these studies use multivariate data sets they demonstrate how important volatility timing is in the context of portfolio management. Marquering and Verbeek (2004) find substantial increases in Sharpe Ratio and utility if volatility timing is added to return timing when a portfolio of risky assets and a riskless asset in managed (see also Moreira and Muir (2019) who find a similar observation for long-horizon investors). Moreira and Muir (2017) and Bollerslev et al. (2018) using a similar framework as in our paper also demonstrate vast utility gains of volatility targeting. Additionally, in a univariate setting, Barroso and Santa-Clara (2015), Moreira and Muir (2017) and Barroso and Maio (2016) demonstrate the vast potential of volatility timing overlayed on several portfolio strategies, especially in terms of drawdown reduction and improvement of risk-adjusted returns. Harvey et al. (2018) find that volatility targeting works well for risky assets like equities or portfolios that contain risky assets. Busse (1999) examines volatility timing used by mutual funds and finds higher Sharpe Ratios for funds using volatility timing. Volatility targeting strategies often deliver "benchmark-comparable levels of return with lower risk" (Benson et al., 2014, p. 89). Generally, the potential of volatility timing does not only exist for short-horizon investors but also for investors with a long investment horizon as shown by Moreira and Muir (2019). Benartzi and Thaler (1995) show that the evaluation period of long term investors is typically much shorter than their investment horizon. That is, investors with an investment horizon of years act like investors with a horizon of several months. This renders these investor to fully participate at the long term performance potential of stocks, since investors with a short evaluation period are more sensitive to changes in market volatility (Moreira and Muir, 2019). For these long-term investors timing short-term volatility can also be beneficial to capture the long-term potential of stocks and simultaneously manage short-term risk. Ang and Bekaert (2002) show that even for investors with longer horizons it is possible to act myopically, as done by risk targeting, instead of solving complex long term portfolio problems.

Fifth, the target volatility strategy focuses on the risk, measured by volatility of the risky asset solely ignoring information about future returns (Bollerslev et al., 2018). This is appealing since future volatility can be estimated much more precisely than future returns, which minimizes the estimation risk of this approach (Merton, 1980). Kirby and Ostdiek (2012) show that portfolio allocations that rely on estimates of returns and volatilities exhibit very high estimation risk, whereas estimation risk is only small for volatility based allocations. Generally, portfolio allocation under estimation risk, especially for mean returns, is frequently examined in the financial literature (DeMiguel et al., 2009, Garlappi et al., 2006, Kan and Zhou, 2007, Tu and Zhou, 2011). Moreover, a forecast of the whole return distribution is not needed, which is a tenuous task (Aït-Sahalia and Brandt, 2001). Moreira and Muir (2017) find higher utility gains for timing volatility as for timing expected returns. Marquering and Verbeek (2004) examine both return and volatility timing and find that timing return and volatility is superior to strategies that only time return (see also Moreira and Muir (2019)).

Sixth, investors are typically crash averse and dislike periods of huge negative returns (Bollerslev and Todorov, 2011, Chabi-Yo et al., 2018, van Oordt and Zhou, 2016). Volatility timing has proven to be a good and easy method for drawdown protection which makes this approach appealing for investors who typically dislike huge drawdowns (Barroso and Maio, 2016, Barroso and Santa-Clara, 2015, Benson et al., 2014, Moreira and Muir, 2017). Harvey et al. (2018) find that volatility targeting successfully reduces the likelihood of extreme negative returns. Thus, volatility targeting also fits well to the loss-aversion of most investors (Aït-Sahalia and Brandt, 2001, Benartzi and Thaler, 1995). In particular, risk targeting is an easy way to manage portfolio risk dynamically. Cuoco et al. (2008) highlight the importance of managing portfolio risk dynamically, i.e. reevaluating portfolio weights frequently.

Seventh, risk-averse investors typically want to hedge against changes in volatility (see Ang et al. (2006b), Adrian and Rosenberg (2008) and references therein). Bollerslev and Todorov (2011) and Bollerslev et al. (2015) examine the variance risk premium which measures the "compensation for the risk associated with temporal changes in the variation of the price level". Adrian and Rosenberg (2008) find that investors are willing to pay for methods that protect them from changes in volatility. Similarly, in a cross-sectional setting, Baltussen et al. (2018) find that assets with a high volatility of volatility (vol-of-vol) underperform assets with a more constant volatility. Thus, assets with lower volatility changes produce higher returns than assets that exhibit higher volatility changes. Further, higher volatility changes are also related to higher downside risk. The investors' demand to hedge against these changes in volatility has led to the introduction of many new financial instruments like variance swaps (Bollerslev and Todorov, 2011, Footnote 11). Volatility targeting is an easy way to hedge against this volatility risk without using any financial derivatives.

Eighth, liabilities of institutional investors like insurance companies or pension funds are often less volatile than investments in risky assets. Targeting a constant level of volatility can help to match the volatility of the investments with the volatility of the liabilities.[81]

Ninth, a skill of a portfolio manager can typically be separated in his abtility to time the market and the ability to pick the right stocks (see Agarwal and Naik (2004) and references therein). Risk targeting can be used separately as market timing tool, which is independent of the asset selection (Zakamulin, 2015). Traditionally market timing and volatility timing are fundamentally related as documented by Christoffersen and Diebold (2006). The authors state that market timing strategies based on measures of volatility are frequently used by practitioners. Hence, a portfolio manager can focus himself on picking the right assets without accounting for the current market environment, which is separately managed by an overlayed risk targeting strategy. Barroso and Santa-Clara (2015), Barroso and Maio (2016) and Moreira and Muir

[81] Banerjee et al. (2016, p. 1) write that "a risk control strategy may provide a smoother path of asset returns and could more closely align the performance of the institution's assets to the characteristics of its liabilities."

(2017) use volatility targeting for several portfolio strategies.

B Portfolio Risk

B.1 Portfolio Value at Risk

In this section we derive the Value at Risk (VaR) for the portfolio loss given in Equation (10). We denote the conditional cumulative distribution function of the risky asset's loss L_t, based on the information $\mathcal{F}_{t-1}$ available at time $t-1$, by $F_{L_t|\mathcal{F}_{t-1}}$. Moreover, we assume that $F_{L_t|\mathcal{F}_{t-1}}$ is continuous and strictly increasing and denote the corresponding $(1-\alpha)$-quantile by $F^{-1}_{L_t|\mathcal{F}_{t-1}}(1-\alpha)$. For a positive weight w_t the day t VaR of the portfolio loss, denoted by $\mathrm{VaR}^{P,t}_\alpha$, is given by

$$\begin{aligned}
& \mathrm{P}\big(L^P_t \leqslant \mathrm{VaR}^{P,t}_\alpha \mid \mathcal{F}_{t-1}\big) = 1-\alpha \\
\Leftrightarrow\quad & \mathrm{P}\Big(w_t \cdot L_t - (1-w_t) \cdot R^f_t \leqslant \mathrm{VaR}^{P,t}_\alpha \mid \mathcal{F}_{t-1}\Big) = 1-\alpha \\
\Leftrightarrow\quad & \mathrm{P}\Big(L_t \leqslant (\mathrm{VaR}^{P,t}_\alpha + (1-w_t) \cdot R^f_t)/w_t \mid \mathcal{F}_{t-1}\Big) = 1-\alpha \\
\Leftrightarrow\quad & F_{L_t|\mathcal{F}_{t-1}}\Big((\mathrm{VaR}^{P,t}_\alpha + (1-w_t) \cdot R^f_t)/w_t\Big) = 1-\alpha \\
\Leftrightarrow\quad & (\mathrm{VaR}^{P,t}_\alpha + (1-w_t) \cdot R^f_t)/w_t = F^{-1}_{L_t|\mathcal{F}_{t-1}}(1-\alpha) \\
\Leftrightarrow\quad & \mathrm{VaR}^{P,t}_\alpha = w_t \cdot F^{-1}_{L_t|\mathcal{F}_{t-1}}(1-\alpha) - (1-w_t) \cdot R^f_t .
\end{aligned} \tag{77}$$

Since the VaR of the risky asset, denoted by VaR^t_α, is given by the $(1-\alpha)$-quantile of the risky asset's (conditional) loss distribution, i.e. $\mathrm{VaR}^t_\alpha = F^{-1}_{L_t|\mathcal{F}_{t-1}}(1-\alpha)$, the VaR of the portfolio is given by

$$\mathrm{VaR}^{P,t}_\alpha = w_t \cdot \mathrm{VaR}^t_\alpha - (1-w_t) \cdot R^f_t . \tag{78}$$

B.2 Portfolio Conditional Value at Risk

In this section we derive the Conditional Value at Risk (CVaR) for the portfolio loss given in Equation (10). The CVaR of the portfolio loss L^P_t, denoted by $\mathrm{CVaR}^{P,t}_\alpha$, is given by

$$\mathrm{CVaR}^{P,t}_\alpha = \mathbb{E}\big(L^P_t \mid L^P_t \geqslant \mathrm{VaR}^{P,t}_\alpha, \mathcal{F}_{t-1}\big) \tag{79}$$

$$= \mathbb{E}\Big(w_t \cdot L_t - (1-w_t) \cdot R^f_t \mid w_t \cdot L_t - (1-w_t) \cdot R^f_t \geqslant \mathrm{VaR}^{P,t}_\alpha, \mathcal{F}_{t-1}\Big) \tag{80}$$

From Equation (78), i.e. $\mathrm{VaR}_\alpha^{P,t} = w_t \cdot \mathrm{VaR}_\alpha^t - (1-w_t) \cdot R_t^f$, and since the weight w_t and the riskless return R_t^f are $\mathcal{F}_{t-1}$-measurable, it follows

$$\begin{aligned}
\mathrm{CVaR}_\alpha^{P,t} &= \mathbb{E}\Big(w_t \cdot L_t - (1-w_t) \cdot R_t^f \mid L_t \geqslant \mathrm{VaR}_\alpha^t, \mathcal{F}_{t-1}\Big) \\
&= w_t \cdot \mathbb{E}\big(L_t \mid L_t \geqslant \mathrm{VaR}_\alpha^t, \mathcal{F}_{t-1}\big) - (1-w_t) \cdot R_t^f, \\
&= w_t \cdot \mathrm{CVaR}_\alpha^t - (1-w_t) \cdot R_t^f,
\end{aligned} \tag{81}$$

where $\mathrm{CVaR}_\alpha^t := \mathbb{E}\big(L_t \mid L_t \geqslant \mathrm{VaR}_\alpha^t, \mathcal{F}_{t-1}\big)$ denotes the CVaR of the risky asset.

C Backtesting Target Risk Strategies

C.1 Backtesting Target VaR Strategies

By definition, the variable H_t^P is equal to one, if $L_t^P - \mathrm{VaR}_\alpha^{\text{target}} > 0$ and zero else. From Equation (10) it follows that the portfolio loss is given by

$$L_t^P = w_t \cdot L_t - (1-w_t) \cdot R_f^f. \tag{82}$$

Moreover, given the weight w_t, the portfolio VaR equals the predefined VaR level $\mathrm{VaR}_\alpha^{\text{target}}$, and hence from Equation (78) we obtain

$$\mathrm{VaR}_\alpha^{\text{target}} = \mathrm{VaR}_\alpha^{P,t} = w_t \cdot \mathrm{VaR}_\alpha^t - (1-w_t) \cdot R_t^f. \tag{83}$$

Consequently, we have

$$L_t^P - \mathrm{VaR}_\alpha^{\text{target}} = w_t \cdot L_t - w_t \cdot \mathrm{VaR}_\alpha^t = w_t \cdot (L_t - \mathrm{VaR}_\alpha^t). \tag{84}$$

Since the weight w_t is strictly positive, it follows

$$L_t^P - \mathrm{VaR}_\alpha^{\text{target}} > 0 \Leftrightarrow L_t - \mathrm{VaR}_\alpha^t > 0. \tag{85}$$

Therefore, the variable H_t^P is equal to H_t.

C.2 Backtesting Target CVaR Strategies

Given the weight w_t, the target CVaR equals the portfolio CVaR, and hence from Equation (81) it follows

$$\mathrm{CVaR}_\alpha^{\text{target}} = \mathrm{CVaR}_\alpha^{t,P} = w_t \cdot \mathrm{CVaR}_\alpha^t - (1-w_t) \cdot R_t^f. \tag{86}$$

Therefore, the difference between the portfolio loss and the target CVaR is given by

$$L_t^P - \mathrm{CVaR}_\alpha^{\mathrm{target}} = w_t \cdot L_t - w_t \cdot \mathrm{CVaR}_\alpha^t = w_t \cdot (L_t - \mathrm{CVaR}_\alpha^t). \tag{87}$$

Moreover, from Equation (2) and since w_t and R_t^f are $\mathcal{F}_{t-1}$-measurable we obtain

$$\sqrt{\mathrm{var}(R_t^P \mid \mathcal{F}_{t-1})} = \sqrt{\mathrm{var}(w_t \cdot R_t \mid \mathcal{F}_{t-1})} = w_t \cdot \sqrt{\mathrm{var}(R_t \mid \mathcal{F}_{t-1})} = w_t \cdot \sigma_t. \tag{88}$$

Consequently, from Equations (87), (88) and (37) it follows

$$\frac{L_t^P - \mathrm{CVaR}_\alpha^{\mathrm{target}}}{\sqrt{\mathrm{var}(R_t^P \mid \mathcal{F}_{t-1})}} = \frac{w_t \cdot (L_t - \mathrm{CVaR}_\alpha^t)}{w_t \cdot \sigma_t} = \frac{L_t - \mathrm{CVaR}_\alpha^t}{\sigma_t} = L_t^* - \mathrm{CVaR}_\alpha^{t,*}. \tag{89}$$

D Additional Results

D.1 Risk Targeting as Absolute Return Strategy

This section presents additional results for the risk targeting strategies. Figure III demonstrates that downside risk targeting can be used as an alternative to absolute return and other hedge fund strategies as examined in Fung and Hsieh (1997) and Agarwal and Naik (2004). This figure plots a target VaR strategy with a VaR target of $\mathrm{VaR}_\alpha^{\mathrm{target}} = 0.5\%$ and a significance level of $\alpha = 0.4\%$. Hence, this strategy should exhibit a daily return lower than -0.5% only once a year. Days with a return lower than -0.5% are marked with a red cross. As can be seen from the figure, the strategy is successful in mitigating extreme negative returns and produces positive or moderately negative returns with a very high probability. The target VaR strategy's long-term return is as high as the return of the 60/40 portfolio and only slightly lower than the return of the DAX. However, the target risk strategy also takes much lower risk with significantly lower drawdowns. Hence, this strategy is appealing for highly risk-averse or loss-averse investors who nevertheless are interested in capturing the long-term potential of equity markets. We only show the VaR-EWMA-FHS strategy in Figure III since this strategy is easy to estimate and implement, and hence, could be interesting for practitioners. Other risk targeting strategies produce similar or even superior performance charts.

Figure III. Cumulative return of VaR targeting. This figure plots the cumulative return of a target VaR strategy and two benchmark portfolios. The target VaR strategy uses the EWMA volatility model combined with Filtered Historical Simulation (FHS), a VaR target of $\text{VaR}_{\alpha}^{\text{target}} = 0.5\%$ and a significance level of $\alpha = 0.4\%$. Days when the portfolio return is lower than -0.5% are marked with a red cross.

D.2 Downside Risk Targeting for Different Significance Levels

In Table XIII we show additional performance results of risk targeting for significance levels of $1\%, 2.5\%$ and 5%. These significance levels are frequently used in the literature on VaR and CVaR forecasting (see Bali et al. (2008) for example). The target VaR and CVaR levels are recalculated to match the chosen volatility target by using Equations (31) and (46). For reasons of clarity we only report the Sharpe Ratio, the maximum drawdown and the economic value Δ_{MV} of risk timing for a mean-variance investor with a moderate risk aversion of $\gamma = 5$. The economic value Δ_{MV} measures the annualized fee in percent an investor is willing to pay to switch from the 60/40 portfolio to a risk targeting strategy. In line with the results of Tables V and VIII we find that downside risk targeting is superior to volatility targeting in terms of higher Sharpe Ratios, lower drawdowns and a higher economic value. Further, also in line with the previous results, we find that managing risk by conditional models outperforms the strategies based on HSD or Historical Simulation. This result is most pronounced for high significance levels. Generally, downside risk targeting becomes less attractive if a higher significance level α is chosen. This result is in line with Happersberger et al. (2019). The higher the significance level the more downside risk targeting resembles volatility targeting. This result highlights that when portfolio risk is managed, an investor should best manage extreme losses as found by Basak and Shapiro (2001). Hence, an investor should choose CVaR as risk measure and a low significance level. However, even for a significance level of $\alpha = 5\%$ CVaR targeting is typically superior to volatility targeting. Interestingly, the strategies based on the skewed t distribution are very stable in terms of the Sharpe Ratio for different levels of α. However, drawdown and economic value indicate that low levels of α are superior.

Table XIII. Additional performance results for different significance levels

This table reports additional performance results of risk targeting for the DAX as risky asset and significance levels α of $1\%, 2.5\%$ and 5%. SR denotes the annualized Sharpe Ratio, MDD the maximum drawdown in per cent and Δ_{MV} the economic value of risk targeting for a mean-variance investor with risk aversion $\gamma = 5$, i.e. the annualized fee in per cent a mean-variance investor is willing to pay to switch from the 60/40 portfolio to a risk targeting strategy. - marks a negative Sharpe Ratio.

	$\alpha = 1\%$			$\alpha = 2.5\%$			$\alpha = 5\%$		
Model	SR	MDD	Δ_{MV}	SR	MDD	Δ_{MV}	SR	MDD	Δ_{MV}
Vola Hist	0.102	41.237	0.280	0.102	41.237	0.280	0.102	41.237	0.280
Vola EWMA	0.122	40.553	0.669	0.122	40.553	0.669	0.122	40.553	0.669
Vola GARCH	0.116	39.277	0.789	0.116	39.277	0.789	0.116	39.277	0.789
VaR Hist	0.009	35.559	0.191	-	38.415	-0.482	0.022	40.808	-0.697
VaR EWMA FHS	0.126	38.693	1.106	0.119	38.372	1.030	0.124	38.120	0.970
VaR EWMA EVT	0.131	36.365	1.354	0.124	37.873	1.091	0.120	38.902	0.868
VaR EWMA Stsk	0.146	36.304	1.379	0.142	38.030	1.118	0.139	39.464	0.890
VaR GARCH FHS	0.120	38.160	1.085	0.110	38.803	0.928	0.109	38.507	0.821
VaR GARCH EVT	0.116	37.325	1.169	0.108	38.366	0.927	0.106	38.972	0.762
VaR GARCH Stsk	0.148	35.729	1.539	0.148	37.168	1.348	0.146	38.344	1.159
CVaR Hist	0.055	30.298	0.982	0.027	33.217	0.462	0.022	35.590	0.116
CVaR EWMA FHS	0.132	35.754	1.443	0.129	37.056	1.287	0.126	37.421	1.179
CVaR EWMA EVT	0.139	34.685	1.609	0.133	36.002	1.413	0.129	37.018	1.242
CVaR EWMA Stsk	0.152	34.268	1.659	0.147	35.830	1.449	0.143	37.056	1.270
CVaR GARCH FHS	0.126	37.396	1.279	0.120	37.919	1.151	0.115	38.026	1.046
CVaR GARCH EVT	0.132	36.157	1.452	0.121	37.086	1.240	0.115	37.773	1.079
CVaR GARCH Stsk	0.145	34.017	1.707	0.146	35.353	1.570	0.145	36.386	1.445

D.3 Results for US Data and Small Caps

Table XIV shows additional performance results for US data and small caps, proxied by the S&P 500 and the German small cap index SDAX, respectively. The data are also obtained from Datastream, where we use the three month treasury bill rate as risk free rate for the US data as also used by Marquering and Verbeek (2004). Panel A contains results for the S&P 500, which are mainly in line with the results of Tables V and VIII for the DAX. The dynamically managed target risk strategies exhibit higher returns than the 60/40 portfolio with comparable risk. Further, returns of the dynamically managed strategies are slightly lower than the return of the S&P 500 but with only about half of the volatility. The Historical Simulation based strategies perform again significantly worse than the dynamically managed strategies. This is also regarded by the Sharpe Ratios. The downside risk managed strategies, especially the CVaR managed strategies, produce higher Sharpe Ratios than the volatility managed strategies. Only the Historical Simulation based target VaR strategy has a lower Sharpe Ratio than the S&P 500. However, the Sharpe Ratio test of Jobson and Korkie (1981) indicates that only the strategies based on the EWMA model combined with the skewed t distribution produce a statistically higher Sharpe Ratio. Similar results also hold for the maximum drawdown. The highest drawdown reduction, given by ΔMDD, is obtained by the dynamically managed CVaR strategies, whereas statically or volatility managed strategies are less successful in reducing the drawdown. The economic value for a risk aversion of $\gamma = 5$ is even negative for the HSD managed strategy and the target VaR strategy based on Historical Simulation. For $\gamma = 15$ the economic value becomes negative for all target volatility strategies and the Historical Simulation based target VaR strategy. That is, a mean-variance investor with risk aversion $\gamma = 15$ is not willing to pay a positive fee to switch away from the 60/40 portfolio to a volatility managed strategy. In contrast, the economic value of the dynamically downside risk managed strategies is always positive and the highest for the CVaR managed strategies.

Panel B shows the results for the German small cap index SDAX. Interestingly, the dynamically downside risk managed strategies exhibit higher returns with lower risk than the 60/40 portfolio and the SDAX. The volatility managed strategies produce even higher returns than the downside risk managed strategies, but also exhibit higher risk. As before, the highest Sharpe Ratios are obtained by the dynamically managed VaR and CVaR strategies. The test of Jobson and Korkie (1981) shows that three target VaR, four target CVaR strategies but none of the target volatility strategies exhibit statistically significant higher Sharpe Ratios than the HSD managed strategy. Similarly, drawdown reduction is the highest for the CVaR managed strategies, whereas the drawdown reduction of the volatility managed strategies is only small.

Table XIV. Performance results for S&P 500 and SDAX

This table shows additional performance results for the S&P 500 and the SDAX for the period 01.01.2000 to 31.12.2018. Return and Volatility denote the annualized return and volatility in percent. SR denotes the annualized Sharpe Ratio. z_{JK} denotes the test statistic of the Sharpe Ratio test of Jobson and Korkie (1981). MDD and ΔMDD denote the maximum drawdown and the reduction of the maximum drawdown in relation to the drawdown of the risky asset. $\Delta_{MV}^{\gamma=5}$ and $\Delta_{MV}^{\gamma=15}$ denote the economic value of a mean-variance investor with risk-aversion of $\gamma = 5$ and $\gamma = 15$, respectively. Return, Volatility, MDD, ΔMDD, $\Delta_{MV}^{\gamma=5}$ and $\Delta_{MV}^{\gamma=15}$ are given in percent.

	Panel A: Results for S&P 500							
Model	Return	Volatility	SR	z_{JK}	MDD	ΔMDD	$\Delta_{MV}^{\gamma=5}$	$\Delta_{MV}^{\gamma=15}$
Vola Hist	4.758	13.092	0.236	-	36.594	36.753	-0.060	-3.447
Vola EWMA	4.916	12.653	0.257	0.874	34.989	39.527	0.316	-2.530
Vola GARCH	4.356	11.949	0.226	-0.213	33.525	42.058	0.126	-1.863
VaR Hist	3.028	10.147	0.137	-0.834	35.535	38.583	-0.358	-0.367
VaR EWMA FHS	4.617	10.598	0.279	0.976	27.207	52.978	0.990	0.509
VaR EWMA EVT	4.353	10.037	0.268	0.840	26.898	53.512	0.969	1.074
VaR EWMA Stsk	5.015	10.763	0.311	**2.186**	28.338	51.023	1.303	0.642
VaR GARCH FHS	4.402	10.797	0.254	0.337	29.831	48.443	0.697	0.003
VaR GARCH EVT	4.183	10.223	0.247	0.228	27.856	51.855	0.729	0.643
VaR GARCH Stsk	4.339	10.173	0.263	0.520	29.922	48.285	0.900	0.865
CVaR Hist	3.083	9.103	0.159	-0.651	33.242	42.546	0.097	1.102
CVaR EWMA FHS	4.471	9.591	0.293	1.224	23.061	60.144	1.260	1.810
CVaR EWMA EVT	4.369	9.248	0.293	1.134	22.515	61.087	1.292	2.172
CVaR EWMA Stsk	5.123	9.934	0.347	**2.519**	25.619	55.722	1.755	1.965
CVaR GARCH FHS	4.527	10.050	0.285	0.924	26.064	54.953	1.132	1.223
CVaR GARCH EVT	4.404	9.736	0.282	0.863	24.509	57.640	1.138	1.545
CVaR GARCH Stsk	4.550	9.383	0.308	1.222	27.602	52.294	1.417	2.170
S&P 500	4.338	19.220	0.139	-0.750	57.859	-	-4.335	-16.409
60/40	3.395	10.139	0.173	-0.569	33.295	42.455	-	-

	Panel B: Results for SDAX							
Model	Return	Volatility	SR	z_{JK}	MDD	ΔMDD	$\Delta_{MV}^{\gamma=5}$	$\Delta_{MV}^{\gamma=15}$
Vola Hist	8.474	13.524	0.486	-	69.223	-2.530	1.658	-2.309
Vola EWMA	8.388	12.656	0.513	1.021	64.100	5.059	2.039	-0.811
Vola GARCH	8.572	12.109	0.551	1.182	57.488	14.851	2.490	0.309
VaR Hist	4.724	8.937	0.323	-1.619	41.755	38.155	0.185	1.397
VaR EWMA FHS	6.711	9.335	0.519	1.095	50.749	24.834	1.938	2.796
VaR EWMA EVT	6.927	9.162	0.552	**2.176**	49.228	27.086	2.210	3.235
VaR EWMA Stsk	6.527	10.029	0.465	-0.502	52.839	21.737	1.489	1.656
VaR GARCH FHS	8.046	9.908	0.621	**2.380**	44.342	34.323	2.986	3.279
VaR GARCH EVT	7.581	9.559	0.596	**2.039**	46.223	31.537	2.682	3.327
VaR GARCH Stsk	6.439	9.661	0.473	-0.141	47.153	30.159	1.552	2.089
CVaR Hist	4.727	8.172	0.354	-1.296	38.189	43.436	0.451	2.335
CVaR EWMA FHS	6.809	8.667	0.570	**2.263**	46.291	31.436	2.277	3.762
CVaR EWMA EVT	6.673	8.389	0.573	**2.259**	45.036	33.296	2.245	3.975
CVaR EWMA Stsk	5.822	9.139	0.434	-1.106	47.413	29.774	1.162	2.199
CVaR GARCH FHS	7.671	9.204	0.628	**2.543**	43.768	35.173	2.905	3.897
CVaR GARCH EVT	7.487	8.901	0.630	**2.521**	42.341	37.286	2.843	4.121
CVaR GARCH Stsk	5.503	8.861	0.413	-1.116	42.852	36.529	0.958	2.249
SDAX	6.489	15.792	0.293	-1.705	67.515	-	-1.523	-8.496
60/40	5.033	10.190	0.313	-1.534	47.612	29.479	-	-

Interestingly, the drawdown of the HSD managed strategy is even higher than the drawdown of the SDAX. The economic value of risk targeting is again positive for the downside risk managed strategies, the highest for the CVaR managed strategies and negative for the volatility managed strategies when the investor is highly risk-averse. Further, the annualized fees are typically higher when SDAX is used as underlying asset compared to the S&P 500. We also find that the skewed t distribution does not work well for the SDAX although this strategy works well for the DAX and the S&P 500. Hence, the estimation methods can perform quite differently when different assets are used. A possibility to obtain more robust results for different assets would be to combine several forecasting methods (Halbleib and Pohlmeier, 2012, Taylor, 2014). In summary, the additional results for S&P 500 and SDAX confirm the result of the DAX. That is, portfolio risk is best managed by using CVaR as risk measure and a conditional risk model.

Table XV. Performance results for S&P 500 and SDAX: Switching strategies
This table shows additional performance results for the S&P 500 and the SDAX for the period 01.01.2000 to 31.12.2018 for the strategies that switch between volatility and CVaR targeting. The description of the columns is given in Table XIV.

	Panel A: Results for S&P 500							
Model	Return	Volatility	SR	z_{JK}	MDD	ΔMDD	$\Delta_{MV}^{\gamma=5}$	$\Delta_{MV}^{\gamma=15}$
Vola Hist	4.758	13.092	0.236	-	36.594	36.753	-0.060	-3.447
S&P 500	4.338	19.220	0.139	-0.750	57.859	-	-4.335	-16.409
60/40	3.395	10.139	0.173	-0.569	33.295	42.455	-	-
GARCH/CVaR Hist	2.239	12.100	0.051	-0.993	49.617	14.246	-1.975	-4.099
GARCH/CVaR EWMA FHS	4.522	8.753	0.327	0.903	19.646	66.044	1.622	2.964
GARCH/CVaR EWMA EVT	4.445	8.461	0.329	0.822	18.847	67.427	1.649	3.250
GARCH/CVaR EWMA Stsk	6.003	9.493	0.455	**2.440**	23.150	59.989	2.783	3.439
GARCH/CVaR GARCH FHS	4.610	9.264	0.318	1.090	20.855	63.955	1.520	2.387
GARCH/CVaR GARCH EVT	4.462	8.917	0.314	0.954	21.334	63.127	1.504	2.696
GARCH/CVaR GARCH Stsk	5.060	8.833	0.384	1.580	26.067	54.947	2.116	3.391
	Panel B: Results for SDAX							
Model	Return	Volatility	SR	z_{JK}	MDD	ΔMDD	$\Delta_{MV}^{\gamma=5}$	$\Delta_{MV}^{\gamma=15}$
Vola Hist	8.474	13.524	0.486	-	69.223	-2.530	1.658	-2.309
SDAX	6.489	15.792	0.293	-1.705	67.515	-	-1.523	-8.496
60/40	5.033	10.190	0.313	-1.534	47.612	29.479	-	-
GARCH/CVaR Hist	10.271	11.269	0.740	**3.623**	40.745	39.651	4.503	3.294
GARCH/CVaR EWMA FHS	9.897	10.826	0.736	**4.097**	48.113	28.738	4.353	3.655
GARCH/CVaR EWMA EVT	10.012	10.769	0.751	**4.189**	47.135	30.186	4.489	3.855
GARCH/CVaR EWMA Stsk	9.253	11.043	0.665	**3.140**	48.362	28.369	3.644	2.706
GARCH/CVaR GARCH FHS	10.410	11.083	0.765	**4.491**	46.524	31.090	4.723	3.729
GARCH/CVaR GARCH EVT	10.550	11.005	0.783	**4.649**	45.611	32.443	4.891	3.984
GARCH/CVaR GARCH Stsk	10.014	11.079	0.730	**3.903**	44.924	33.461	4.349	3.362

We next assess if our switching approach also works for the S&P 500 and SDAX. Results of the switching strategies are shown in Table XV where we only show results for one indicator δ_t for each asset. The switching strategies are again successful in producing higher returns compared to the individual strategies and thus provide an enhanced risk-return profile. The Sharpe Ratios of the switching strategies in Table XV are higher than the Sharpe Ratios of the individual strategies reported in Table XIV. For the SDAX, the test of Jobson and Korkie (1981) shows that all switching strategies produce statistically higher Sharpe Ratios than the HSD strategy with extremely high values of z_{JK}. The switching strategies not only exhibit higher returns than the individual strategies, they also provide a better drawdown protection measured by ΔMDD. Further, the economic value is also increased by switching between volatility and CVaR targeting. Concluding, our simple switching approach does not only work well for the DAX but also for the S&P 500 and SDAX.

D.4 Risk Targeting in the Long Run

So far, we only examined a period of 18 years which was marked by several crises. To assess if risk targeting is also beneficial in the long run we use data for the US market from 1929 to 2018. Data for the US market and the risk-free rate are obtained from the website of Kenneth French and are also used by Moreira and Muir (2017).[82] Results for the long sample are shown in Table XVI. The strategies that switch between volatility and CVaR targeting exhibit similar levels of return as the US market but take significantly less risk in terms of volatility and drawdown. This translates into a significantly higher Sharpe Ratio and large utility gains for a mean-variance investor. Interestingly, although the skewed t distribution works well for the S&P 500, the same approach does not work well for the long US sample.

Table XVI. Performance results for US market in the long run: Switching strategies
This table shows additional performance results for the US market for the period November 1929 to December 2018 for the strategies that switch between volatility and CVaR targeting. The description of the columns is given in Table XIV.

Model	Return	Volatility	SR	z_{JK}	MDD	ΔMDD	$\Delta_{MV}^{\gamma=5}$	$\Delta_{MV}^{\gamma=15}$
Vola Hist	9.779	13.283	0.481	-	55.886	30.248	1.241	1.671
US market	9.215	16.839	0.347	-2.216	80.121	0.000	-1.424	-6.219
60/40	8.620	13.597	0.387	-1.632	54.299	32.229	-	-
GARCH/CVaR Hist	9.643	11.534	0.543	**1.961**	42.724	46.676	1.996	4.690
GARCH/CVaR EWMA FHS	9.529	10.999	0.559	**3.101**	41.725	47.922	2.136	5.469
GARCH/CVaR EWMA EVT	9.537	10.936	0.563	**3.178**	41.052	48.763	2.172	5.579
GARCH/CVaR EWMA Stsk	9.044	11.264	0.504	1.044	48.097	39.970	1.563	4.568
GARCH/CVaR GARCH FHS	9.399	11.312	0.532	**2.068**	46.561	41.887	1.872	4.829
GARCH/CVaR GARCH EVT	9.416	11.235	0.538	**2.236**	45.571	43.123	1.924	4.973
GARCH/CVaR GARCH Stsk	8.713	11.255	0.476	-0.105	52.580	34.374	1.260	4.266

Similar to Moreira and Muir (2017, Figure 3), Figure IV shows the cumulative return of the US market, the 60/40 portfolio, the HSD strategy and the strategy that switches between volatility and CVaR targeting for a 100$ investment. For a better comparison, we rescale all

[82] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

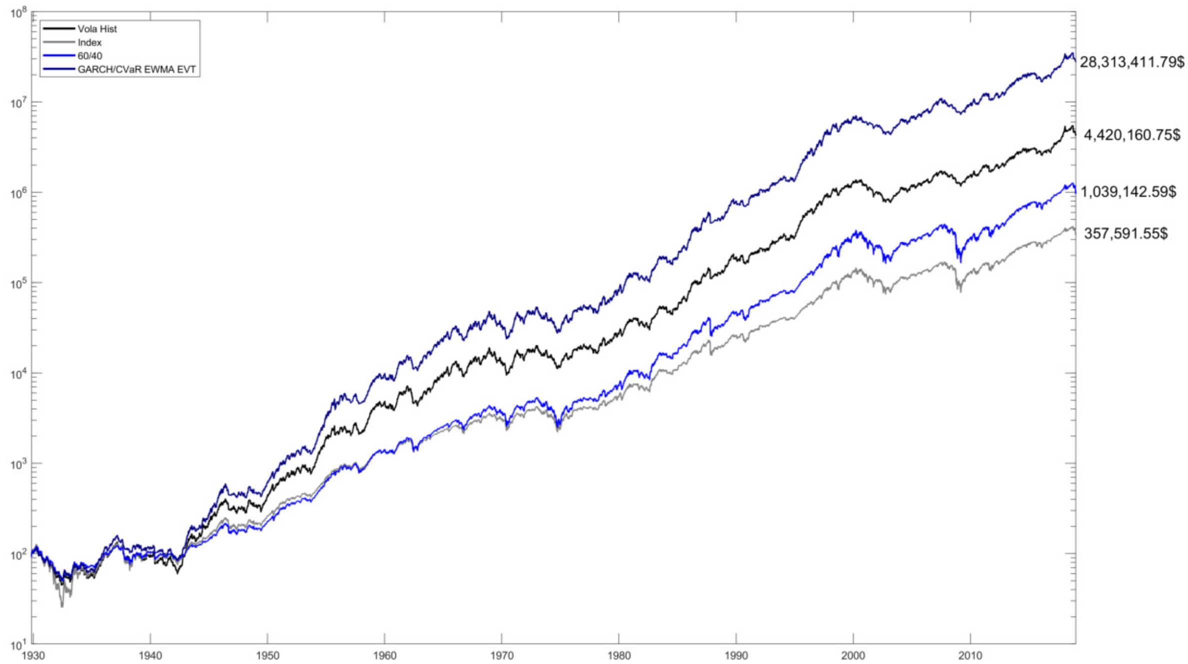

Figure IV. Cumulative return of risk targeting. This figure plots the cumulative return of the US market, the 60/40 portfolio, the HSD target volatility strategy and a strategy that switches between volatility and CVaR targeting for the period 1929 to 2018.

strategies to the volatility of the US market. Figure IV shows the clear outperformance of the risk targeting strategies. Both strategies successfully capture the upside potential of the market while downside risk is limited. However, the strategy that switches between volatility and CVaR targeting clearly outperforms the HSD managed strategy. A 100$ investment in the market portfolio would result in a terminal wealth of 357,591.55$. Invested in the target volatility strategy terminal wealth would increase to 4,420,160.75$. However, the strategy that switches between volatility and CVaR targeting produces a final wealth of even 28,313,411.79$. This is in line with the results of Moreira and Muir (2019) that even long-term investors should time short-term risk.

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