When does portfolio compression reduce systemic risk?

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Abstract

We analyse the consequences of portfolio compression on systemic risk. Portfolio compression is a post-trading netting mechanism that reduces gross positions while keeping net positions unchanged and it is part of the financial legislation in the US (Dodd-Frank Act) and in Europe (European Market Infrastructure Regulation). We derive necessary structural conditions for portfolio compression to be harmful and discuss policy implications. In particular, we show that the potential danger of portfolio compression comes from defaults of firms that conduct portfolio compression. If no defaults occur among those firms that engage in compression, then portfolio compression always reduces systemic risk.

Key words: Systemic risk, portfolio compression, financial networks, cycles, netting.

JEL code: D85, G01, G28, G33.

1 Introduction

Portfolio compression is a mechanism in which multiple offsetting contracts are replaced by fewer contracts to reduce the gross exposure of each institution while keeping its net exposure unchanged. Reducing gross exposure is beneficial for a wide range of reasons including complying with regulatory requirements such as the minimum leverage ratio introduced under the Basel III regulation and margin requirements (Duffie, 2017). The new contracts that replace the old contracts, however, lead to a new network structure of exposures between the market participants. It is not clear, a priori, what the consequences for systemic risk are - this is what we analyse here.

The main contribution of this paper is to derive general theoretical results on the consequences of portfolio compression on systemic risk. To the best of our knowledge these considerations have been absent from the literature so far. The European Securities and Markets Authority (ESMA) has published a consultation paper in 2020, see European Securities and Market Authority (2020), on post trade risk reduction services (PTRR) of which portfolio compression is an important example. They ask: "Would you agree with the description of the benefits (i.e. reduced risks) derived from PTRR services? Are there any missing? Could PTRR services instead increase any of those risks? Are there any other risks you see involved in using PTRR services?" Hence, there remains uncertainty about the risks of post trade risk reduction services such as portfolio compression.

In this paper we derive structural conditions for portfolio compression to be harmful or to reduce systemic risk in a sense that we will formally define in Definition 3.11. Theorem 4.7 contains the main results. It establishes a relationship between defaults and payment abilities of those nodes conducting compression and systemic risk in the system. One conclusion is that

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as long as only nodes that are not at risk of defaulting (in the non-compressed) system conduct portfolio compression, portfolio compression will always reduce systemic risk. We will derive and discuss more fine-tuned results. We will also show both theoretically and in numerical case studies that there are situations under which portfolio compression can indeed be harmful.

We will proceed as follows. In Section 2 we introduce the theoretical model for the financial market and formally define what we refer to as *portfolio compression* (Definition 2.1 and Definition 2.3) building on D'Errico & Roukny (2019). We only consider portfolio compression that (potentially repeatedly) removes one cycle from a network and also consider an optimisation framework for portfolio compression in this context. In Section 3 we explain how we measure systemic risk. We use the framework by Veraart (2020) which generalises the approaches by Eisenberg & Noe (2001) and Rogers & Veraart (2013). We assess how different types of payments obligations associated with derivative positions arise and might lead to loss cascades if a node fails to satisfy their payment obligations. Our analysis includes variation margins become due building on the models by Paddrik et al. (2020) and Ghamami et al. (2020). Section 4 contains all results on the consequences of portfolio compression on systemic risk. Theorem 4.7 is the main result, providing structural conditions that are necessary for portfolio compression to be harmful. We discuss these conditions in detail, derive some additional results that contribute to our understanding of the consequences of portfolio compression and use them to discuss policy implications. In Subsection 4.6 we illustrate the results using some example networks. Section 5 concludes.

1.1 Policy framework and related literature

Understanding the consequences of portfolio compression on systemic risk is of fundamental importance since it is used both in Europe under the European Market Infrastructure Regulation on derivatives, central counterparties and trade repositories (EMIR), and in the US under the Dodd-Frank Wall Street Reform and Consumer Protection Act (Dodd-Frank Act). Under EMIR portfolio compression is one of the risk mitigation mechanisms for non-centrally cleared OTC derivative contracts (European Union, 2012). In the US portfolio compression is used as a risk management tool in the swap market (Commodities Futures Trading Commission, 2012).

Portfolio compression plays an important role in today's financial markets. It is performed by private providers - a well-known one is the company TriOptima. It states that "OTC derivative market participants have eliminated more than \$973 trillion in notional principal through April 2017", TriOptima (2017). Portfolio compression is currently available for "cleared and uncleared interest rate swaps in 28 currencies, cross currency swaps, credit default swaps, FX forwards, and commodity swaps", TriOptima (2017). Its compression service *triReduce* is currently used by over 260 institutions worldwide.

Regulatory reforms such as the Basel III minimum leverage ratio provide strong incentives for market participants to engage in compression activities, see e.g. Duffie (2017). The Basel III leverage ratio is defined as tier 1 capital divided by the total exposure. Since compression reduces the exposure, compression increases the leverage ratio making it easier to satisfy the lower bound, see also Haynes et al. (2019) for further discussions and Remark A.1.

The introduction of margin requirements (also for non-centrally cleared derivatives, see BCBS IOSCO (2015, 2020)) provides incentives for market participants to engage in portfolio compression since lower total exposures are associated with both lower initial margin and also lower variation margin requirements (Duffie, 2017). Despite margin requirements risk of contagion in derivative markets remains as demonstrated empirically in Paddrik et al. (2020) and Bardoscia et al. (2019).

The literature on regulatory reforms and their consequences on systemic risk in the derivatives markets has mainly focussed on the role of central counterparties, see e.g. Duffie & Zhu (2011); Cont & Kokholm (2014). Amini et al. (2016b) analysed different netting mechanisms but not in the context of portfolio compression in centrally cleared markets.

The literature on portfolio compression is still in its infancy. In particular it mainly focuses on the actual algorithms to perform the portfolio compression rather than the potential consequences of portfolio compression. O' Kane (2017) proposes and analyses different multilateral netting algorithms. Among the algorithms studied an algorithm based on the L_1-norm is shown to be particularly beneficial to eliminate a high fraction of bilateral connections and to retain the greatest common divisor of existing positions.
D'Errico & Roukny (2019) introduce different types of portfolio compression mechanisms to show theoretically how the size of over-the-counter markets can be reduced without affecting the net positions of the market participants. In addition they show empirically the large potential for compression to reduce exposure size using a transaction-level dataset for Credit-Default-Swaps (CDS) derivatives. They do not use any risk measures to study the effect of compression on systemic risk.
Schuldenzucker et al. (2018) provide one example that shows that portfolio compression can be harmful for the system but do not provide any general results.
Duffie (2018) used ideas from portfolio compression in his new auction mechanism (*compression auctions*) that convert centrally cleared contracts on the London Interbank Offer Rate to contracts on the Secured Overnight Financing Rate.

2 Portfolio compression

2.1 The network of liabilities

We consider a financial network consisting of N financial institutions with indices in $\mathcal{N} = \{1, 2, \ldots, N\}$ representing the nodes. We denote by X_{ij} the nominal liability of financial institution i to financial institution j and write $X = (X_{ij}) \in [0, \infty)^{N\times N}$ for the corresponding *liabilities matrix*. Furthermore, we assume that $X_{ii} = 0$ for all $i \in \mathcal{N}$, i.e., nodes do not have liabilities to themselves. The set of edges is given by $\mathcal{E} = \{(i, j) \in \mathcal{N}^2 \mid X_{ij} > 0\}$.

The liabilities can arise due to entering into derivative contracts such as Interest Rate Swaps or Credit Default Swaps (CDS), see e.g. Schuldenzucker et al. (2018) [1]. We assume that all these positions are fungible. Our framework would also apply to other types of liabilities such as interbank lending.

We assume that all contracts are established at time $t = 0$ and have the same maturity date $t = T > 0$. A generalisation to the situation with multiple maturities in the spirit of Kusnetsov & Veraart (2019) would also be possible.

We denote by $\bar{L}_i^{(X)} = \sum_{j=1}^N X_{ij}$ the total nominal liabilities of node i and write $\bar{L}^{(X)} \in [0, \infty)^N$ for the vector of total liabilities arising within the network. Similarly, we write $\bar{A}_i^{(X)} = \sum_{j=1}^N X_{ji}$ for the total assets of financial institution i from within the network and write $\bar{A}^{(X)} \in [0, \infty)^N$ for the vector of these total assets. Then we refer to $\bar{A}^{(X)} + \bar{L}^{(X)}$ as gross positions in the network and to $\eta = \bar{A}^{(X)} - \bar{L}^{(X)} \in \mathbb{R}^N$ as net positions in the network.

2.2 Defining portfolio compression

Portfolio compression is a mechanisms that nets trades between two or more counterparties such that the net positions stay the same for all nodes but the gross positions decrease for all market participants. We only consider a method of compression that would be referred to as *conservative portfolio compression* in D'Errico & Roukny (2019). Intuitively, conservative compression is a mechanism that eliminates cycles in networks.

[1]Note that in the context of CDS exposures, we exclude the situation in which nodes write CDSs on each other as in Schuldenzucker et al. (2018). We refer to Schuldenzucker et al. (2020) for more details on clearing in a network with CDSs if this situation is not excluded.

Figure 1 provides an example of a network consisting of four nodes in which three perform multilateral portfolio compression by reducing their exposures along a cycle.

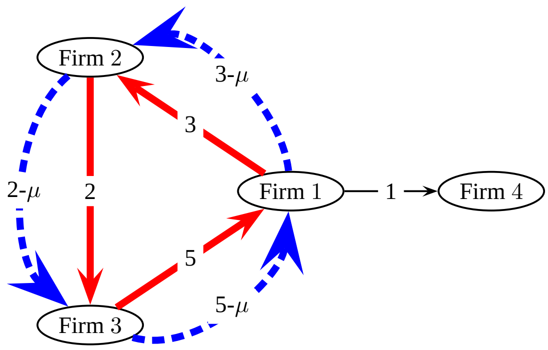


Figure 1: Example of compressing a cycle in a network: The cycle in red with solid lines is replaced by the cycle in blue with dashed lines where $\mu \in (0, 2]$.

Definition 2.1 (Portfolio compression)**.** *Consider a liabilities matrix $X \in [0, \infty)^{N \times N}$ with corresponding nodes $\mathcal{N} = \{1, \dots, N\}$ and edges $\mathcal{E} = \{(i, j) \in \mathcal{N}^2 \mid X_{ij} > 0\}$.*

1. *A* cycle *is a sequence of distinct vertices $\mathcal{C}_{nodes} = \{i_1, \dots, i_n\} \in \mathcal{N}$ with $n \leq N$ with $(i_\nu, i_{\nu+1}) \in \mathcal{E}$ for all $\nu \in \{1, \dots, n-1\}$ and $(i_n, i_1) \in \mathcal{E}$. We denote the corresponding set of edges by $\mathcal{C}_{edges} = \{(i_1, i_2), \dots, (i_{n-1}, i_n), (i_n, i_1)\}$ and write $\mathcal{C} = (\mathcal{C}_{nodes}, \mathcal{C}_{edges})$.*

2. *Let $\mathcal{C}_{nodes}$ be a cycle with $\mathcal{C}_{edges}$ the corresponding set of edges such that*

$$\mu^{\max} = \min_{(i,j) \in \mathcal{C}_{edges}} X_{ij} > 0.$$

   We then refer to $\mathcal{C} = (\mathcal{C}_{nodes}, \mathcal{C}_{edges}, \mu^{\max})$ as a conservative compression network cycle of X with maximal capacity $\mu^{\max}$.

3. *Let $\mathcal{C} = (\mathcal{C}_{nodes}, \mathcal{C}_{edges}, \mu^{\max})$ be a conservative compression network cycle of X with maximal capacity $\mu^{\max}$. For any $0 < \mu \leq \mu^{\max}$, we refer to the matrix $X^{\mathcal{C},\mu}$ with*

$$X_{ij}^{\mathcal{C},\mu} = \begin{cases} X_{ij} - \mu & \text{if } (i,j) \in \mathcal{C}_{edges}, \\ X_{ij} & \text{otherwise}, \end{cases}$$

   as the μ-compressed liabilities matrix *(using cycle $\mathcal{C}$). We refer to $\bar{L}_i^{(X),\mathcal{C},\mu} = \sum_{j \in \mathcal{N}} X_{ij}^{\mathcal{C},\mu}$ as the* total μ-compressed nominal obligations *of node i (using cycle $\mathcal{C}$).*

Conservative compression network cycles may or may not exist for a given liability matrix. Throughout this paper we assume that for any liabilities matrix that we analyse, there exists at least one conservative compression network cycle.

We show in Lemma B.1 in the Appendix that conservative compression does indeed reduce gross positions while keeping net positions fixed. Portfolio compression reduces the size of the balance sheet of a participating node. Its total assets are reduced by μ and its total liabilities are reduced by μ. Its net worth remains the same.

Remark 2.2 (Compression tolerances)**.** In practice firms provide so-called *compression tolerances* to the compression provider (D'Errico & Roukny, 2019). These are restrictions on the changes that can be made to the original positions. Mathematically they can be characterised by requiring that any new position $X_{ij}^{\mathcal{C},\mu}$ that might replace the original position X_{ij} needs to satisfy

$$a_{ij} \leq X_{ij}^{\mathcal{C},\mu} \leq b_{ij} \tag{1}$$

for some $0 \leq a_{ij} \leq b_{ij}$, see (D'Errico & Roukny, 2019, Definition (Compression tolerance)). For example, if a firm does not want to change a particular position it could set $a_{ij} = b_{ij} = X_{ij}$. In such a case we could just set the corresponding $\mu^{\max} = 0$.

Since we only consider conservative compression in this paper we will assume that $b_{ij} = X_{ij}$ for all $i, j \in \mathcal{N}$. This in particular implies that no new edges can be creates as part of a compression exercise (since if $X_{ij} = 0$ then under this assumption $X_{ij} = b_{ij} = 0$). It might be the case that a node does not want to remove an edge completely but only wants to reduce the weight of an edge, i.e., this would correspond to setting $a_{ij} > 0$ as lower bound for the weight of this particular edge. It is clear from the definition of $\mu^{\max}$ that if one sets $\mu = \mu^{\max}$ in a compression exercise then at least one edge (and possibly more) would be removed. To avoid this, one could change the definition of $\mu^{\max}$ by setting $\widetilde{\mu^{\max}} = \min_{(i,j)\in\mathcal{C}_{\text{edges}}}(X_{ij} - a_{ij}) > 0$. Obviously, $\widetilde{\mu^{\max}} \leq \mu^{\max}$. Since all our results will hold for all choices of $\mu \in (0, \mu^{\max}]$ they in particular hold for all $\mu \in (0, \widetilde{\mu^{\max}}]$. Therefore there is no need for us to explicitly add such constraints in our analysis.

These compression tolerances are not considered to be functions of any other parameters of the network and are restrictions on the individual positions. Neither D'Errico & Roukny (2019) nor O' Kane (2017) indicate any global constraints when considering the actual compression mechanism. We will come back to this when we discuss systemic risk in compressed networks.

2.3 Portfolio compression as an optimisation problem

Next we consider conservative portfolio compression as an optimisation problem as described in D'Errico & Roukny (2019). Its objective is to minimise the total gross exposures of all nodes participating in the compression exercise while satisfying some constraints.

Definition 2.3 (Conservative compression optimisation problem). *Let $X \in [0, \infty)^{N\times N}$ be a liability matrix. We refer to the following optimisation problem as the* conservative compression optimisation problem. *It is given by*

$$\min_{\tilde{X}_{ij}, i,j\in\mathcal{N}} \sum_{i=1}^{N}\sum_{j=1}^{N} \tilde{X}_{ij}, \tag{2}$$

subject to

$$\sum_{j=1}^{N}(\tilde{X}_{ji} - \tilde{X}_{ij}) = \sum_{j=1}^{N}(X_{ji} - X_{ij}) \quad \forall i \in \mathcal{N}, \tag{3}$$

$$0 \leq \tilde{X}_{ij} \leq X_{ij} \quad \forall i, j \in \mathcal{N}. \tag{4}$$

This is a linear programming problem and can be solved using standard methods. Since $\tilde{X} = X$ satisfies both constraints ((3) and (4)) a feasible solution to these constraints exists. Furthermore, since the constraint set is bounded, it is clear that a solution exists.

Constraint (3) says that the net exposures of the nodes are not allowed to change by compression. Constraint (4) ensures that the compression is indeed conservative. Since in the new network the value of any edge is between 0 and its original value, this means that this type of compression can only reduce liabilities along existing edges but cannot create new edges. As pointed out in D'Errico & Roukny (2019) the resulting network is therefore a subnetwork of the original one.

One could replace the condition (4) by a condition representing the compression tolerance of the market participants by e.g. requiring that

$$a_{ij} \leq \tilde{X}_{ij} \leq b_{ij} \quad \forall i, j \in \mathcal{N}, \text{for } 0 \leq a_{ij} \leq b_{ij}. \tag{5}$$

Setting e.g., $a_{ij} = 0$ and $b_{ij} = +\infty$ would correspond to non-conservative compression, see D'Errico & Roukny (2019), which would in principle allow for the underlying network to be rewired. Therefore new trading relationships could be established which is not possible under conservative compression. As argued in D'Errico & Roukny (2019), setting $a_{ij} = 0$ and $b_{ij} = X_{ij}$ for all $i, j \in \mathcal{N}$, is "arguably close to the way most compression cycles take place in derivatives markets. We thank Per Sjöberg, founder and former CEO of TriOptima, for fruitful discussion on these particular points", (D'Errico & Roukny, 2019, p. 19). We will therefore focus on this setting.

Remark 2.4 (Relationship of solution to optimisation problem and compressing one cycle)**.** As shown in (D'Errico & Roukny, 2019, Proposition 7) a solution to the optimisation problem in Definition 2.3 is a *directed acyclical graph*, i.e., that is a graph that does not contain any cycles. In particular, one can obtain a solution by repeatedly compressing along one compression network cycle, see (D'Errico & Roukny, 2019, Subsection 12.2 Conservative Algorithm). As discussed there it will matter in which order this is done, since it is possible that some edges are part of several cycles. An algorithm for choosing the ordering is given in (D'Errico & Roukny, 2019, Subsection 12.2 Conservative Algorithm). They always choose $\mu = \mu^{\max}$ in each compression step.

3 Measuring systemic risk

We consider different types of payment obligations that arise from the network of liabilities X and describe how we measure the systemic risk associated with them.

3.1 Payment obligations, margins and liquidity buffer

We assume that all payment obligations that arise from the original liabilities matrix X can be expressed by using a suitable function f^V that maps the original liabilities matrix X into payment obligations $L = f^V(X)$.

Definition 3.1 (Payment function and payment obligation matrix)**.** *Let $X \in [0,\infty)^{N\times N}$ be a liabilities matrix. Consider a function $f^V : [0,\infty)^{N\times N} \to [0,\infty)^{N\times N}$ where $f^V(x) = Vx$ for $V \in [0,\infty)$, $x \in [0,\infty)^{N\times N}$. We refer to f^V as* payment function*.*
We define a matrix $L = (L_{ij}) \in [0,\infty)^{N\times N}$ where each element $L_{ij} = f^V_{ij}(X) = VX_{ij}$ represents the payments that are due from i to j, $i, j \in \mathcal{N}$ at a given point in time. We refer to L as payment obligation matrix*. We refer to $\bar{L} \in [0,\infty)^N$, where $\bar{L}_i = \sum_{j=1}^N L_{ij}$, as the* total payment obligations*.*

Payment obligations can in principle arise throughout the lifetime of the contract. We restrict our analysis to only one point in time at which payments are due. In principle, our analysis could be extended to allow for multiple points in times at which payments are due.

If we set $V = 1$ in the definition of f^V, then $L = X$ and hence the payments due are the original liabilities. In practice, this does not need to be the case. Payment obligations can for example arise from variation margins becoming due (BCBS IOSCO, 2015). One distinguishes between variation and initial margins. Variation margins reflect current exposures and are settled regularly, initial margins reflect potential future exposures and are usually required at the outset of a derivatives transaction. By choosing an appropriate payment function f^V, our payment obligation matrix can represent variation margins that are due on a given day.

Consider for example a situation in which the original network X represents Credit Default Swaps contracts. In particular X_{ij} represents the amount of protection sold from i to j in case of a credit event occurring to the underlying reference entity over a given time period. If there is shock to this reference entity that increases its probability of default for example, variation margins will be due from the seller of the CDS protection to the buyer of the protection since the value of the CDS contracts becomes more valuable to the protection buyer and increases the liabilities of the protection seller. This change is reflected in the variation margin that is then due from the protection seller to the protection buyer, see Paddrik et al. (2020) who discussed such a situation in detail. As in their model, we will also allow for the existence of initial margins.

Definition 3.2 (Initial margins). *Let X be a liabilities matrix and let $J \in [0, \infty)$. The initial margin that node i sets aside to protect its liabilities to node j is given by JX_{ij}, where $i, j \in \mathcal{N}$.*

Setting $J = 0$, would imply that there are no initial margins available, and for $J > 0$ initial margins are available which are proportional to the notional size of the contract. This proportionality assumption is referred to as the *standard schedule* and was introduced in (BCBS IOSCO, 2015). There have been alternative proposals since then, see e.g. Cont (2018) who highlighted that the standard schedule typically overestimates margin requirements. For tractability purposes we will still consider the proportional case.[2]

To complete our modelling framework we assume that at the time when payment obligations become due each node is equipped with external assets, i.e., assets from outside the network, only a part of which, the liquidity buffer, is available to satisfy any payment obligations.

Definition 3.3 (External assets and liquidity buffer). *We denote by $A^{(e)} \in [0, \infty)^N$ the vector of* external assets *and by $b \in [0, A^{(e)}]$ the liquidity buffer.*

We analyse how portfolio compression affects payment obligations and liquidity buffers.

Definition 3.4 (Payment obligations, initial margins, liquidity buffer under compression). *Let X be a liabilities matrix for which there exists a conservative compression network cycle $\mathcal{C} = (\mathcal{C}_{nodes}, \mathcal{C}_{edges})$ of X with maximal capacity $\mu^{\max}$. Let $0 < \mu \leq \mu^{\max}$ and let $X^{\mathcal{C},\mu}$ be the μ-compressed liabilities matrix. Let $L = f^V(X) = VX$ be the payment obligation matrix, where $V \in [0, \infty)$.*

1. *We refer to the matrix $L^{\mathcal{C},\mu}$ with*

$$L_{ij}^{\mathcal{C},\mu} = f_{ij}^V(X^{\mathcal{C},\mu}) = \begin{cases} VX_{ij} - V\mu & \text{if } (i,j) \in \mathcal{C}_{edges}, \\ VX_{ij} & \text{otherwise,} \end{cases} = \begin{cases} L_{ij} - V\mu & \text{if } (i,j) \in \mathcal{C}_{edges}, \\ L_{ij} & \text{otherwise,} \end{cases} \tag{6}$$

   as the μ-compressed payment obligation matrix *(using cycle $\mathcal{C}$). We refer to $\bar{L}_i^{\mathcal{C},\mu} = \sum_{j\in\mathcal{N}} f_{ij}^V(X^{\mathcal{C},\mu}) = V\sum_{j\in\mathcal{N}} X_{ij}^{\mathcal{C},\mu}$ as the* total μ-compressed payment obligations *of node $i \in \mathcal{N}$ (using cycle $\mathcal{C}$).*

2. *The* μ-compressed initial margins *are given by $JX^{\mathcal{C},\mu}$, where*

$$JX_{ij}^{\mathcal{C},\mu} = \begin{cases} JX_{ij} - J\mu & \text{if } (i,j) \in \mathcal{C}_{edges}, \\ JX_{ij} & \text{otherwise.} \end{cases}$$

[2]When modelling initial and variation margins we should keep in mind that for the purpose of this analyses we consider fungible derivative positions, meaning portfolio compression can actually be done since these contracts are completely comparable. Hence, assuming that margin requirements are proportional to exposure size is reasonable. Initial margins are often set as 99% loss quantile for a 10-days period and hence represent a Value-at-risk which is known to be positive homogeneous, i.e., scales with position size. (This would also apply if other risk measures were used such as the expected shortfall.)

3. *The* μ-compressed liquidity buffer $b^{\mathcal{C},\mu,\gamma} \in [0,\infty)^N$*, where* $\gamma \in [0,1]$*, is given by*

$$b_i^{\mathcal{C},\mu,\gamma} = \begin{cases} b_i + \gamma J\mu & \text{if } i \in \mathcal{C}_{nodes}, \\ b_i, & \text{if } i \in \mathcal{N} \setminus \mathcal{C}_{nodes}. \end{cases} \tag{7}$$

Hence, we see that portfolio compression reduces the payment obligations, and therefore variation margins, since the payment function f^V is non-decreasing. Furthermore, portfolio compression also reduces the initial margins. Therefore there are strong incentives for market participants to engage in portfolio compression. This is particularly relevant for initial margins which can never be netted.

Regarding the liquidity buffer we allow for different effects of compression. If $\gamma = 0$, then the liquidity buffer is not affected by portfolio compression, which can be interpreted as the corresponding assets that are no longer tied up in initial margins are considered as illiquid assets. If $\gamma = 1$ then we assume that the liquidity buffer of those nodes taking part in portfolio compression increases by exactly the amount that is no longer required as initial margins since the position was reduced. For $\gamma \in (0,1)$ we have some increase of the liquidity buffer for those nodes taking part in portfolio compression but this is less than the corresponding reduction in initial margins.

We now formally define a payment system in which we will analyse systemic risk.[3]

Definition 3.5 (Payment system)**.** *Let X be a liabilities matrix for which there exists a conservative compression network cycle $\mathcal{C} = (\mathcal{C}_{nodes}, \mathcal{C}_{edges})$ of X with maximal capacity $\mu^{\max}$. Let $0 < \mu \leq \mu^{\max}$ and let $X^{\mathcal{C},\mu}$ be the μ-compressed liabilities matrix. Let $V \in [0,\infty)$ and $L = VX$ the corresponding payment obligation matrix and b the liquidity buffer. We will refer to (L, b) as* payment system *and to $(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma})$ as* μ-compressed *payment system, where $L^{\mathcal{C},\mu}$ and $b^{\mathcal{C},\mu,\gamma}$ are defined in (6) and (7) respectively and $\mu \in [0, \mu^{\max}]$.*

3.2 Clearing the payment obligations

To measure systemic risk we consider a suitable extension of the Eisenberg & Noe (2001) framework for clearing payments in financial networks. In particular, we incorporate ideas developed by Paddrik et al. (2020) and Ghamami et al. (2020) for clearing with collateral (i.e., initial margins) into the the framework developed by Veraart (2020) to measure systemic risk.

In the following we define the notion of equity revaluation which is a slightly modified version of (Veraart, 2020, Definition 2.4) which is related to the approach developed in Barucca et al. (2020).

Definition 3.6 (Re-evaluated equity)**.** *Consider a payment system (L, b) and let $(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma})$ be the corresponding* μ-compressed *payment system where $\mu \in (0, \mu^{\max}]$.*

1. *A* valuation function *is a function $\mathbb{V} : \mathbb{R} \to [0,1]$, given by*

$$\mathbb{V}(y) = \begin{cases} 1, & \text{if } y \geq 1 + k, \\ r(y), & \text{if } y < 1 + k, \end{cases} \tag{8}$$

where $k \geq 0$ and $r : (-\infty, 1+k) \to [0,1]$ is a non-decreasing and right-continuous function.

[3] The payment system characterises all payments due and hence serves as the basis for analysing systemic risk. It is related to the original liabilities X via the payment function f^V. We assumed that $L_{ij} = f^V(X) = VX_{ij}$ for all $i, j \in \mathcal{N}$ and $V \in [0,\infty)$. The analysis on systemic risk does not rely on this proportionality assumption, since it is conducted on the payment system directly. Therefore, one could consider more general functions f^V as long as they are meaningful from an economic perspective. Since we assume that the liabilities X are fungible, the proportionality assumption makes sense and is consistent with approaches used to derive initial margins as outlined before.

2. *Consider a valuation function $\mathbb{V}$. We define a function $\Phi = \Phi(\cdot;\mathbb{V}) : \mathcal{E} \to \mathcal{E}$ where*

$$\Phi_i(E) = \Phi_i(E;\mathbb{V}) = b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i, \tag{9}$$

$\mathcal{M} = \{i \in \mathcal{N} \mid \bar{L}_i > 0\}$, $\mathcal{E} = [b - \bar{L}, b + \bar{A} - \bar{L}]$, $\bar{A}_i = \sum_{j=1}^N L_{ji}$, $\bar{L}_i = \sum_{j=1}^N L_{ij}$ $\forall i \in \mathcal{N}$. *The* re-evaluated equity in the non-compressed network *is a vector $E \in \mathcal{E}$ satisfying*

$$E = \Phi(E). \tag{10}$$

3. *Consider a valuation function $\mathbb{V}$. We define a function $\Phi^{\mathcal{C},\mu,\gamma} = \Phi(\cdot;\mathbb{V}) : \mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{C}}$ where*

$$\Phi_i^{\mathcal{C},\mu,\gamma}(E) = \Phi_i^{\mathcal{C},\mu,\gamma}(E;\mathbb{V}) = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i^{\mathcal{C},\mu}, \tag{11}$$

$\mathcal{M}^{\mathcal{C}} = \{i \in \mathcal{N} \mid \bar{L}_i^{\mathcal{C},\mu} > 0\}$, $\mathcal{E}^{\mathcal{C}} = [b^{\mathcal{C},\mu,\gamma} - \bar{L}^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma} + \bar{A}^{\mathcal{C}} - \bar{L}^{\mathcal{C},\mu}]$, $\bar{A}_i^{\mathcal{C}} = \sum_{j=1}^N L_{ji}^{\mathcal{C},\mu}$, $\bar{L}_i^{\mathcal{C},\mu} = \sum_{j=1}^N L_{ij}^{\mathcal{C},\mu}$ $\forall i \in \mathcal{N}$. *The* re-evaluated equity in the compressed network *is a vector $E \in \mathcal{E}^{\mathcal{C}}$ satisfying*

$$E = \Phi^{\mathcal{C},\mu,\gamma}(E). \tag{12}$$

Since r is non-decreasing and right-continuous, $\mathbb{V}$ is also non-decreasing and right-continuous. Therefore (Veraart, 2020, Theorem 2.5) guarantees the existence of the re-evaluated equities in (10) and (12) as fixed points of Φ and $\Phi^{\mathcal{C},\mu,\gamma}$ respectively.

Similar to Veraart (2020), for a given node $i \in \mathcal{N}$, the function Φ_i models the difference between the liquid assets $b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j+\bar{L}_j}{\bar{L}_j}\right)$ and the total payment obligations $\bar{L}_i$. The liquid assets consist of the liquidity buffer b_i and the payments received from the other financial institutions given by $\sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j+\bar{L}_j}{\bar{L}_j}\right)$. If the function value of $\mathbb{V}$ is strictly less than 1 this implies that not the full amount of payment obligations are paid by j to i which reduces the liquid assets that i has. The interpretation for the terms appearing in $\Phi^{\mathcal{C},\mu,\gamma}$ is the same as for Φ with the only exceptions that the compressed network is considered.

If $L = X$ and $A^{(e)} = b$, then the positive part of the re-evaluated equity would correspond to the *equity* of the node, as in Eisenberg & Noe (2001).

In this re-evaluation approach all payment obligations are treated equally. In particular, no netting takes place prior to clearing. Any type of netting (such as bilateral netting, compression etc.) would effectively introduce a seniority structure, where liabilities that are netted have implicitly a higher seniority than those that are not netted, see Elsinger (2009) for a clearing approach with different seniorities of debt.

As described earlier, we refer to a tuple (L, b) as a payment system, where L is a payment obligation matrix and b is a vector of liquidity buffers. If we use such a system to make any statements about its associated systemic risk based on a valuation function $\mathbb{V}$, we write $(L, b; \mathbb{V})$ and also refer to it as a *payment system*.

We consider special choices of valuation functions next to give some intuition on what they can represent. All results we derive, however, will hold for general valuation functions defined in (8).

Remark 3.7 (Special choices for $\mathbb{V}$ from the literature)**.** 1. The Eisenberg & Noe (2001) model can be recovered[4] by setting $k = 0$ and

$$\mathbb{V}^{\text{EN}}(y) = \min\{1, y^+\}. \tag{13}$$

2. The special case of the model by Rogers & Veraart (2013) with bankruptcy costs parameters $\alpha = \beta \in [0, 1]$ can be recovered by setting $k = 0$ and

$$\mathbb{V}^{\text{RV}}(y) = \begin{cases} 1 & \text{if } y \geq 1, \\ \beta y^+ & \text{if } y < 1. \end{cases} \tag{14}$$

3. In Veraart (2020); Glasserman & Young (2015) it was argued that contagion can be triggered prior to the point where the equity of an institution is zero and it was proposed to consider $k > 0$.

4. The *zero recovery rate valuation function* is defined by

$$\mathbb{V}^{\text{zero}}(y) = \mathbb{I}_{\{y \geq 1+k\}}, \text{ where } k \geq 0. \tag{15}$$

All special choices of valuation functions mentioned so far do not account for initial margins. In the following we define a special case of a valuation function that incorporates initial margins.

Definition 3.8 (Valuation function accounting for initial margins)**.** *Let $J \in [0, \infty)$ and $\beta \in [0, 1]$. We define the* initial margin valuation function *by $\mathbb{V}^{InitialMargin} : \mathbb{R} \to [0, 1]$, where $\forall y \in \mathbb{R}$*

$$\mathbb{V}^{InitialMargin}(y) = \begin{cases} 1, & \text{if } y \geq 1, \\ \min\{1, J + \beta y^+\}, & \text{if } y < 1. \end{cases} \tag{16}$$

One can easily check that $\mathbb{V}^{\text{InitialMargin}}$ is a valuation function. Using (Veraart, 2020, Theorem 2.11), we conclude that higher values of initial margins lead to a better outcome for the system in the following sense: Let $0 \leq J_1 \leq J_2$ be two possible parameters for J in (16), then the greatest re-evaluated equity corresponding to parameter J_2 would be greater or equal than the greatest re-evaluated equity corresponding to J_1. In particular, a system with initial margins has better outcomes than a system without initial margins.

The choice of $\mathbb{V}^{\text{InitialMargin}}$ is motivated by the approaches developed in Paddrik et al. (2020); Ghamami et al. (2020) for clearing in collateralised networks. We are essentially using these ideas but rewrite them to fit a slightly different mathematical framework that is more tractable for the purpose of our analysis. The first difference between Paddrik et al. (2020); Ghamami et al. (2020) and our formulation here is that we express the clearing problem in terms of the re-evaluated equity, i.e., within the framework of Veraart (2020), and not in terms of the clearing payments, since this makes the analysis in the compression context more tractable.[5] The second difference is that we include bankruptcy costs modelled in terms of the parameter $\beta \in [0, 1]$, whereas the other approaches mainly focus on $\beta = 0$ and $\beta = 1$.

[4] In Veraart (2020) it was shown how the corresponding clearing vector considered in Eisenberg & Noe (2001); Rogers & Veraart (2013) can be derived from the re-evaluated equity and how the re-evaluated equity can be derived from the clearing vector. In particular, if E^* is the greatest re-evaluated equity, then $L_i^* = \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right)\bar{L}_i \quad \forall i \in \mathcal{M}$ and $L_i^* = 0$ for all $i \in \mathcal{N} \setminus \mathcal{M}$ denotes the corresponding clearing payments, i.e., these are the total payments that node i makes, which ideally would be its total nominal obligations $\bar{L}_i$ but it could be less than that.

[5] When initial margins are used it is possible that a defaulting node satisfies its payment obligations in full by covering a potential shortfall with the initial margins, see also Ghamami et al. (2020). Therefore, one cannot infer who defaults from the payments made. In the classical Eisenberg & Noe (2001) framework this is indeed possible: There a node defaults if and only if it does not pay its liabilities in full. By analysing the re-evaluated equity we can distinguish between defaulting and non-defaulting nodes and the corresponding payments follow from there.

We provide some intuition for the choice of $\mathbb{V}^{\text{InitialMargin}}$ next. If we set $J = 0$, $\mathbb{V}^{\text{InitialMargin}}$ reduces to $\mathbb{V}^{\text{RV}}$. We therefore consider $J > 0$ for now. Let E^* be a fixed point of Φ. Suppose $y = \frac{E_j^* + \bar{L}_j}{\bar{L}_j} < 1$ for a $j \in \mathcal{N}$. Then, the payment that node j makes to node i is given by

$$L_{ji}\mathbb{V}^{\text{InitialMargin}}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) = \min\left\{L_{ji}, JL_{ji} + \beta L_{ji}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right)^+\right\} =: (\star).$$

Now if $J \leq 1$, then $(\star) \geq JL_{ji}$ implying that node j will always at least pay the amount corresponding to the initial margin to i but possibly even more if $\beta L_{ji}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right)^+ > 0$. Under no circumstances can i receive more than L_{ji} from j. If $J \geq 1$, then the initial margins guarantee full payment of L_{ji}.

3.3 Definition of default, reduction of systemic risk and harmfulness of portfolio compression

In the following we will compute the greatest re-evaluated equity both in the original network and in the compressed network, i.e., we will always consider the greatest fixed point of Φ and $\Phi^{\mathcal{C},\mu,\gamma}$ in (10) and (12) respectively. They correspond to the best possible outcome for the economy. Based on these quantities we can then infer which nodes are in default in the network with compression and in the network without compression. Hence, we take an ex-post point of view. We ask what would happen if no compression takes place and we then evaluate the network at a point in time when payments are due. Then we consider the case where compression has taken place and we then evaluate the network when payments are due and compare the outcome to the situation without compression. We summarise the mathematical setting as follows.

Assumption 3.9 (Market setting).

- Let X be a liabilities matrix for which there exists a conservative compression network cycle $\mathcal{C} = (\mathcal{C}_{\text{nodes}}, \mathcal{C}_{\text{edges}}, \mu^{\max})$ with maximal capacity $\mu^{\max} > 0$.
- Let $\mathbb{V}$ be a valuation function and $k \geq 0$.
- Let $(L, b; \mathbb{V})$ be the corresponding payment system with total payment obligations $\bar{L}$.
- Let $0 < \mu \leq \mu^{\max}$. Let $L^{\mathcal{C},\mu}$ be the μ-compressed payment obligation matrix. Let $\bar{L}^{\mathcal{C},\mu}$ be the total μ-compressed payment obligations.
- Let E^* be the greatest re-evaluated equity in the non-compressed network.
- Let $E^{\mathcal{C},\mu;\gamma;*}$ be the greatest re-evaluated equity in the compressed network with $\gamma \in [0, 1]$.
- Let $E^{\mathcal{C},\mu;0;*}$ be the greatest re-evaluated equity in the compressed network with $\gamma = 0$.

We can now define what it means for an institution to be in default.

Definition 3.10 (Definition of default). *Consider the market setting of Assumption 3.9. Then, the set of* defaulting financial institutions in the non-compressed system *is*

$$\mathcal{D}(L, b; \mathbb{V}) = \{i \in \mathcal{N} \mid E_i^* < k\bar{L}_i\} \tag{17}$$

and the set of defaulting financial institutions in the compressed system *is*

$$\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}) = \{i \in \mathcal{N} \mid E_i^{\mathcal{C},\mu;\gamma;*} < k\bar{L}_i^{\mathcal{C},\mu}\}. \tag{18}$$

The definition above defines *default* (in the non-compressed system) as the point when the quantity $\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) < 1+k$. For $k=0$, this is equivalent to saying that the available liquid assets are strictly smaller than the payment obligations which is equivalent to $E_j^* < 0$. This is the situation we have in mind when considering variation margin payments, i.e., for $\mathbb{V} = \mathbb{V}^{\text{InitialMargin}}$. This is the situation we have in mind when considering variation margin payments, i.e., for $\mathbb{V} = \mathbb{V}^{\text{InitialMargin}}$, as in Paddrik et al. (2020); Ghamami et al. (2020).

If $k > 0$, then the default condition is equivalent to saying that the available liquid assets are strictly less than the required payment obligations plus an additionally required buffer. In Remark A.1 we show that to be able to account for certain capital requirements it is sometimes beneficial to allow for an earlier start point of default, i.e., $k > 0$.

Recall that the payments from j to i are $L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right)$. If j is in default, then we are in the default branch of the valuation function[6], i.e., $\mathbb{V}(y) = r(y)$ and this value can be strictly less than 1 implying that payment obligations from j to i are no longer satisfied completely. In the case of initial margins, i.e., if $\mathbb{V} = \mathbb{V}^{\text{InitialMargin}}$, then it is possible (but this will depend on the magnitude of the initial margins) that $L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) = L_{ji}$ even though j defaults.

We are now in a position to formally define what we mean by saying that a particular compression reduces systemic risk or is harmful. We do this by comparing the defaults in the non-compressed network to the defaults in the compressed network (see Definition 3.10).

Definition 3.11 ((Strong) reduction of systemic risk and harmfulness)**.** *Consider the market setting of Assumption 3.9. We say that the compression network cycle* $\mathcal{C}$ reduces systemic risk *if* $\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}) \subseteq \mathcal{D}(L, b; \mathbb{V})$. *We say that the compression network cycle* $\mathcal{C}$ strongly reduces systemic risk *if* $\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}) \subsetneq \mathcal{D}(L, b; \mathbb{V})$. *We say that the compression network cycle* $\mathcal{C}$ *is* harmful *if* $\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}) \setminus \mathcal{D}(L, b; \mathbb{V}) \neq \emptyset$.

Based on this definition, we say that compression reduces systemic risk if every node that defaults in the compressed network also defaults in the non-compressed network. In the same spirit, we classify compression as harmful if there are nodes in the network that default in the compressed network that would not have defaulted in the non-compressed network.

4 Consequences of portfolio compression on systemic risk

We now analyse the consequences of portfolio compression for systemic risk. One might think that portfolio compression reduces systemic risk since it reduces gross exposures while not changing the net exposures. Indeed we will show that in many realistic scenarios portfolio compression reduces or even strongly reduces systemic risk.

There are circumstances, however, in which compression can be harmful. Portfolio optimisation is an optimisation problem that aims to reduce gross exposures subject to some constraints such as keeping net exposures unchanged (O' Kane, 2017; D'Errico & Roukny, 2019). As long as these constraints do not explicitly account for systemic risk there is no reason why a solution to such an optimisation problem should automatically reduce systemic risk.

4.1 Who can be affected by portfolio compression?

We identify those nodes that can in principle be affected by portfolio compression by defining a compression risk orbit[7].

[6] Veraart (2020) distinguishes between default and distress, but we do not make this distinction here.

[7] A risk orbit for an individual node has been considered in Eisenberg & Noe (2001).

Definition 4.1 (Compression risk orbit)**.** *Consider the market setting of Assumption 3.9. The* compression risk orbit of $\mathcal{C}$ *is*

$$\mathcal{O} = \mathcal{C}_{nodes} \cup \{j \in \mathcal{N} \mid \exists i \in \mathcal{C}_{nodes} \text{ and } \exists \text{ a directed path from } i \text{ to } j \text{ in } \hat{G}\}, \tag{19}$$

where $\hat{G}$ is the graph with nodes $\mathcal{N}$ and edges $\hat{\mathcal{E}} = \{(i,j) \in \mathcal{N}^2 \mid L_{ij} > 0\}$.

The compression risk orbit contains all nodes on the compression network cycle and all nodes that can be reached from nodes on the compression network cycle; compression could in principle affect their outcome (both positively or negatively). All nodes in $\mathcal{N} \setminus \mathcal{O}$ cannot be affected (positively or negatively) by compression, i.e., the greatest re-evaluated equity with or without compression coincides for those nodes.

Proposition 4.2. *Consider the market setting of Assumption 3.9. Then, $E_i^* = E_i^{\mathcal{C},\mu;\gamma;*} \quad \forall i \in \mathcal{N} \setminus \mathcal{O}$, where $\mathcal{O}$ is given in (19).*

The proof of this proposition and proofs of all following results are provided in Appendix B.

4.2 Fundamental versus contagious defaults

We will distinguish between two types of default: fundamental default and contagious default. Fundamental defaults are defaults that occur even if all nodes pay their payment obligations in full. Contagious defaults are all default that are not fundamental defaults. In order to formally define these two types of default we consider the difference between nominal liquid assets and total payment obligations and refer to it as initial equity (even though the assets and liabilities considered here might not reflect the full balance sheet).

Definition 4.3 (Initial equity)**.** *Consider the market setting of Assumption 3.9. For all $i \in \mathcal{N}$ define the* initial equity *in the non-compressed and in the compressed network (for parameters $\gamma = 0$ or $\gamma \in [0,1]$) by*

$$E_i^{(0)} = b_i + \sum_{j \in \mathcal{N}} L_{ji} - \bar{L}_i, \quad E_i^{\mathcal{C}(0);0} = b_i^{\mathcal{C},\mu,0} + \sum_{j \in \mathcal{N}} L_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{\mathcal{C}}, \quad E_i^{\mathcal{C}(0);\gamma} = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j \in \mathcal{N}} L_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{\mathcal{C}}. \tag{20}$$

Hence, the initial equity corresponds to the best possible situation, namely all nodes repaying their payment obligations in full. Recall, that $b^{\mathcal{C},\mu,0} = b$.

Definition 4.4 (Fundamental and contagious defaults)**.** *Consider the market setting of Assumption 3.9. Let $E^{(0)}, E^{\mathcal{C}(0);\gamma}$ be the initial equity defined in (20). We refer to $\mathcal{F} = \{i \in \mathcal{N} \mid E_i^{(0)} < k\bar{L}_i\}$ and $\mathcal{D}(L,b;\mathbb{V}) \setminus \mathcal{F}$ as the fundamental defaults and contagious defaults in the non-compressed network, respectively. Similarly, we refer to $\mathcal{F}^{\mathcal{C}} = \{i \in \mathcal{N} \mid E^{\mathcal{C}(0);\gamma} < k\bar{L}_i^{\mathcal{C},\mu}\}$ and $\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma};\mathbb{V}) \setminus \mathcal{F}^{\mathcal{C}}$ as the fundamental defaults and contagious defaults in the compressed network (where $\gamma \in [0,1]$), respectively.*

We will show in Lemma B.4 that $\mathcal{F} \subseteq \mathcal{D}(L,b;\mathbb{V})$ and $\mathcal{F}^{\mathcal{C}} \subseteq \mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma};\mathbb{V})$. We show how portfolio compression affects fundamental defaults using properties of the initial equity.

Lemma 4.5. *Consider the market setting of Assumption 3.9. The initial equities $E_i^{(0)}$, $E_i^{\mathcal{C}(0);0}$, $E_i^{\mathcal{C}(0);\gamma}$, $i \in \mathcal{N}$, are as in (20). Then, $E_i^{(0)} = E_i^{\mathcal{C}(0);0} \le E_i^{\mathcal{C}(0);\gamma} \quad \forall i \in \mathcal{N}, \forall \gamma \in [0,1]$.*

Proposition 4.6 (Fundamental defaults and compression)**.** *Consider the market setting of Assumption 3.9. Then, $\mathcal{F}^{\mathcal{C}} \subseteq \mathcal{F}$. If $\mathcal{F} \setminus \mathcal{F}^{\mathcal{C}} \neq \emptyset$, then $kV + \gamma J > 0$ and $\mathcal{F} \setminus \mathcal{F}^{\mathcal{C}} \subseteq \mathcal{C}_{nodes}$.*

Hence compression can only improve fundamental defaults, in the sense that every node that is in fundamental default in the compressed network also is in fundamental default in the non-compressed network. For a strict improvement under compression certain parameter constraints are required. For example, if portfolio compression increases the liquidity buffer of a node (which would happen for $\gamma > 0$) then portfolio compression could potentially avoid a fundamental default. Similarly, portfolio compression could move a node further away from the default boundary if $k > 0$ in which case portfolio compression could also potentially avoid a fundamental default.

For compression to be harmful we need at least one firm that defaults in the compressed network that does not default in the non-compressed network. Since according to Proposition 4.6 portfolio compression cannot cause any fundamental defaults, such an additional default would have to be a contagious default.

4.3 Structural conditions for the consequences of portfolio compression

The following theorem contains the main theoretical results of this paper. It identifies three key structural conditions that are necessary for portfolio compression to be harmful.

Theorem 4.7 (Necessary conditions for compression to be harmful)**.** *Consider the market setting of Assumption 3.9. Suppose that compressing cycle $\mathcal{C}$ is harmful. Then,*

1.

$$\mathcal{D}(L, b; \mathbb{V}) \cap \mathcal{C}_{nodes} \neq \emptyset; \tag{21}$$

2. there exists an $i \in \mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}) \cap \mathcal{C}_{nodes}$ such that

$$\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*} + \bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right) < \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right); \tag{22}$$

3. the valuation function satisfies

$$\mathbb{V} \neq \mathbb{V}^{zero}. \tag{23}$$

In the following we discuss the three structural conditions (21), (22) and (23) in more detail.

4.3.1 Defaults on the compression network cycle in the non-compressed network

Condition (21) tells us, that for portfolio compression to be potentially harmful one needs at least one default on the compression network cycle in the non-compressed network. Compressing such a cycle can be harmful. The following proposition is used to prove part 1. of Theorem 4.7 and identifies the relationship between the re-evaluated equity in the compressed and non-compressed network.

Proposition 4.8. *Consider the market setting of Assumption 3.9 and let $\mathcal{D}(L, b; \mathbb{V}) \cap \mathcal{C}_{nodes} = \emptyset$. Then, $E_i^* = E_i^{\mathcal{C},\mu;0;*} \leq E_i^{\mathcal{C},\mu;\gamma;*} \quad \forall i \in \mathcal{N}$ and $\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}) \subseteq \mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,0}; \mathbb{V}) \subseteq \mathcal{D}(L, b; \mathbb{V})$.*

Hence, compression can only increase the re-evaluated equity if there are no defaults on the compression network cycle in the non-compressed system. Under this assumption, if additionally $\gamma = 0$, implying that compression does not increase the liquidity buffer, then the re-evaluated equities with and without compression coincide. In both cases systemic risk is reduced.

We do an ex-post analysis here. In practice, firms would conduct portfolio compression prior to payment obligations becoming due. Hence, at the time compression is done, (21) is a condition on the future state of the network. In this spirit, we conclude that if the probability that condition (21) is satisfied in the future is low, meaning that it is unlikely for firms who took part in compression to default in the future, then it is likely that this compression reduces systemic risk.

4.3.2 Repayment proportions

The condition (22) in Theorem 21, is a statement about repayment proportions of nodes on the compression network cycle that default in the compressed network. The total payments that node i makes to the other nodes in the non-compressed network are

$$\sum_{j\in\mathcal{M}} L_{ij}\mathbb{V}\left(\frac{E_i^*+\bar{L}_i}{\bar{L}_i}\right)=\mathbb{V}\left(\frac{E_i^*+\bar{L}_i}{\bar{L}_i}\right)\bar{L}_i,$$

and hence it repays the proportion $\mathbb{V}\left(\frac{E_i^*+\bar{L}_i}{\bar{L}_i}\right)\bar{L}_i/\bar{L}_i=\mathbb{V}\left(\frac{E_i^*+\bar{L}_i}{\bar{L}_i}\right)$ of its total payment obligations if no compression is used. Similarly, the repayment proportion of node i in the compressed network is $\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*}+\bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right)$. Condition (22) therefore says that there exists a node $i\in\mathcal{C}_{\text{nodes}}$ that repays a smaller proportion of its total payment obligations in the compressed network compared to the non-compressed network. Any node $i\in\mathcal{N}$ satisfying (22) is in $\mathcal{D}(L^{\mathcal{C},\mu},b^{\mathcal{C},\mu,\gamma};\mathbb{V})$, since from (22) $\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*}+\bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right)<\mathbb{V}\left(\frac{E_i^*+\bar{L}_i}{\bar{L}_i}\right)\leq 1$ and hence $\frac{E_i^{\mathcal{C},\mu;\gamma;*}+\bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}<1+k$ by (8).

Hence condition (22) says, that for portfolio compression to be potentially harmful one needs at least one node on the compression network cycle that repays a strictly smaller proportion of its total payment obligations in the compressed network than in the non-compressed network.

Consider such an $i\in\mathcal{D}(L^{\mathcal{C},\mu},b^{\mathcal{C},\mu,\gamma};\mathbb{V})\cap\mathcal{C}_{\text{nodes}}$ satisfying (22). Since i repays a smaller proportion of its debt after compression, it can transmit larger losses to other nodes in the network. The payment that node i makes to any node $j\in\mathcal{N}$ without compression is $L_{ij}\mathbb{V}\left(\frac{E_i^*+\bar{L}_i}{\bar{L}_i}\right)$ and with compression it is $L_{ij}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*}+\bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right)$. Then, for all $j\in\mathcal{N}$ with $L_{ij}^{\mathcal{C},\mu}>0$ it holds that $L_{ij}>0$ and hence

$$L_{ij}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*}+\bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right)\leq L_{ij}\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*}+\bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right)<L_{ij}\mathbb{V}\left(\frac{E_i^*+\bar{L}_i}{\bar{L}_i}\right).$$

Hence, node j receives less from i in the compressed network than in the non-compressed network.

Therefore, as long as all nodes on the compression network cycle repay a greater or equal proportion of their debt in the compressed network compared to the non-compressed network (as stated in 24) then compression reduces systemic risk. From this statement we can derive several implications which are of interest for interpreting the results.

Proposition 4.9. *Consider the market setting of Assumption 3.9. Suppose that at least one of the following three conditions is satisfied:*

1.

$$\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*}+\bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right)\geq\mathbb{V}\left(\frac{E_i^*+\bar{L}_i}{\bar{L}_i}\right)\quad\forall i\in\mathcal{C}_{nodes};\tag{24}$$

2.

$$\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*}+\bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right)=1\quad\forall i\in\mathcal{C}_{nodes};\tag{25}$$

3.

$$\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}) \cap \mathcal{C}_{nodes} = \emptyset. \tag{26}$$

Then, $E_i^* \leq E_i^{\mathcal{C},\mu;\gamma;*}$ *for all* $i \in \mathcal{N}$ *and* $\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}) \subseteq \mathcal{D}(L, b; \mathbb{V})$.

Condition (26) says that for compression to be potentially harmful one needs to have a default on the compression network cycle not just in the non-compressed network (condition (21)) but also in the compressed network. Hence, there cannot be a situation in which financial institutions engage in conservative compression such that none of them defaults in the compressed network but a financial institution outside the compression network cycle is worse off (in the sense that it defaults only in the compressed network). Hence, if compression is harmful for a node outside the compression network cycle, then there must exist a defaulting node on the compression network cycle.

Now consider the situation where at least one node on the compression network cycle defaults in the compressed network. Then condition (25) implies that as long as all nodes on the compression network cycle repay their debt in full - this could happen due to sufficient initial margins - then compression cannot be harmful.

Next we provide an intuitive explanation how portfolio compression can change the distribution of losses in the network. In several approaches in the literature such as Eisenberg & Noe (2001); Rogers & Veraart (2013); Veraart (2020) the valuation function $\mathbb{V}$ is a capped piecewise linear function. Also our newly introduced function $\mathbb{V}^{\text{InitialMargin}}$ falls in this class. A key idea of the Eisenberg & Noe (2001) clearing approach (which also applies to more general capped piecewise linear functions) is that all defaulting nodes repay their debt according to the proportions according to which their nominal payment obligations are distributed. These proportions are specified in terms of a relative liabilities matrix.

Proposition 4.10. *Consider the market setting of Assumption 3.9. Consider the relative payment obligation matrices* $\Pi, \Pi^{\mathcal{C},\mu} \in \mathbb{R}^{N\times N}$, *where*

$$\Pi_{ij} = \begin{cases} \frac{L_{ij}}{\bar{L}_i}, & \text{if } \bar{L}_i > 0, \\ 0, & \text{if } \bar{L}_i = 0, \end{cases} \qquad \Pi_{ij}^{\mathcal{C},\mu} = \begin{cases} \frac{L_{ij}^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}, & \text{if } \bar{L}_i^{\mathcal{C},\mu} > 0, \\ 0, & \text{if } \bar{L}_i^{\mathcal{C},\mu} = 0. \end{cases}$$

For $i \in \mathcal{C}_{nodes}$ *we denote by* $suc(i)$ *(successor) the node in* $\mathcal{C}_{nodes}$ *that satisfies* $(i, suc(i)) \in \mathcal{C}_{edges}$. *Then, for all* $i \in \mathcal{C}_{nodes}$

$$\begin{aligned} \Pi_{isuc(i)}^{\mathcal{C},\mu} &\leq \Pi_{isuc(i)}, \\ \Pi_{ij}^{\mathcal{C},\mu} &\geq \Pi_{ij} \quad \forall j \in \mathcal{N} \setminus \{suc(i)\}; \end{aligned}$$

and for all $i \in \mathcal{N} \setminus \mathcal{C}_{nodes}$ *and for all* $j \in \mathcal{N}$ $\Pi_{ij}^{\mathcal{C},\mu} = \Pi_{ij}$.

We see that for nodes that are not on the compression network cycle, the proportions according to which they distribute their payments to the other nodes in the system do not change. For the nodes on the compression network cycle these proportions do change: Smaller (or equal) proportions are paid to the immediate successor of a node on the compression network cycle. To all other nodes larger (or equal) proportions are used to allocate the payments.

Note that if proportions increase, this can also imply that a larger proportion of losses hits neighbouring nodes and this is where the danger is coming from. As long as there are no defaults on the compression network cycle, the fact that the proportions change for nodes on the compression network cycle is irrelevant because they still satisfy the required payment obligations. As soon as that is no longer the case, and the proportions determine how losses are spread, the change in these proportions starts to matter.

4.3.3 Recovery rates

According to part 3. of Theorem 4.7 one needs non-zero recovery rates for portfolio compression to be potentially harmful.

Proposition 4.11. *Consider the market setting of Assumption 3.9 and assume that* $\mathbb{V} = \mathbb{V}^{zero}$*. Then,* $E_i^* \leq E_i^{\mathcal{C},\mu;\gamma;*}$ *for all* $i \in \mathcal{N}$ *and* $\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}^{zero}) \subseteq \mathcal{D}(L, b; \mathbb{V}^{zero})$*.*

Hence, under zero recovery rates portfolio compression leads to a greater re-evaluated equity. In practice, the recovery rates will depend on the time-horizon considered. Assuming a zero recovery rate is reasonable when considering short-term consequences of default, see e.g. Amini et al. (2016a) and the references therein for a discussion. For mid- to long-term consequences of default it is important to consider models that allow for positive recovery rates.

If recovery rates are positive we can have a *worse default* of a node on the compression cycle meaning that it defaults both in the non-compressed and in the compressed network but it repays a strictly smaller proportion of its debt in the compressed network (satisfying (22)). If recovery rates are zero we cannot have such a worse default.

4.4 Compressing multiple cycles

All results so far (Theorem 4.7, Propositions 4.8, 4.9, Proposition 4.11, are statements about compressing a single cycle. In practice, multiple cycles would/could be compressed. The results, nevertheless, carry over to the multiple cycle case in the following sense. Suppose there are multiple cycles $\mathcal{C}_1, \dots, \mathcal{C}_m$ such that conservative compression could be carried out along those cycles sequentially starting from $\mathcal{C}_1$ and finishing at $\mathcal{C}_m$. This in particular implies that $\mathcal{C}_i$, $i \in \{1, \dots, m\}$ is still a possible compression network cycle after the cycles $\mathcal{C}_j$, $j = 1, \dots, i-1$ have been compressed. If $\mathbb{V} = \mathbb{V}^{\text{zero}}$ then we know from Proposition 4.11 that compressing one cycle after the other cannot be harmful. Suppose now that $\mathbb{V} \neq \mathbb{V}^{\text{zero}}$. Then according to Theorem 4.7 (part 1. and part 2.) we need to check properties of the nodes on the compression cycle. Suppose that at least one of the conditions in part 1. and part 2. is not satisfied for compression cycle $\mathcal{C}_1$, then compressing this cycle cannot be harmful. Next one would need to check the conditions for the nodes on the compression network cycle $\mathcal{C}_2$ after $\mathcal{C}_1$ has been compressed. Again, if at least one of the conditions in part 1. and part 2. is not satisfied, then compressing $\mathcal{C}_2$ cannot be harmful, etc.. This can be formalised as follows.

Proposition 4.12 (Compressing multiple cycles)**.** *Let* X *be a liabilities matrix. Suppose there exist* $m \in \mathbb{N}$ *compression network cycles* $\mathcal{C}^{(1)}, \dots, \mathcal{C}^{(m)}$ *such that conservative compression can be carried out along those cycles sequentially starting from* $\mathcal{C}_1$ *and finishing at* $\mathcal{C}_m$*. This in particular implies that* $\mathcal{C}^{(i)}$*,* $i \in \{1, \dots, m\}$ *is still a possible compression network cycle after the cycles* $\mathcal{C}^{(j)}$*,* $j = 1, \dots, i-1$ *have been compressed. We assume that each compression network cycle* $i \in \{1, \dots, m\}$ *is compressed by a quantity* $\mu_i \in (0, \mu_i^{\max}]$ *where* $\mu_i^{\max}$ *is the maximal compression capacity on cycle* i *after the cycles* $\mathcal{C}^1, \dots, \mathcal{C}^{i-1}$ *have been compressed. Let* $\mathcal{C}^{all}_{nodes} = \mathcal{C}^{(1)}_{nodes} \cup \ldots \cup \mathcal{C}^{(m)}_{nodes}$*. Let* $(L, b; \mathbb{V})$ *denote the corresponding payment system (without compression) and denote by* E^* *the greatest re-evaluated equity in the non-compressed system. We denote by* $E^{\mathcal{C}^1,\dots,\mathcal{C}^i,\star}$ *the greatest re-evaluated equity that corresponds to the payment system in which the cycles* $\mathcal{C}^1, \dots, \mathcal{C}^i$*,* $i \in \{1, \dots, m\}$ *have been compressed. The total payment obligation of node* i *in this system is denoted by* $\bar{L}_i^{\mathcal{C}^1,\dots,\mathcal{C}^i}$*. Suppose at least one of the following three conditions is satisfied:*

1.

$$\mathcal{D}(L, b; \mathbb{V}) \cap \mathcal{C}^{all}_{nodes} = \emptyset; \tag{27}$$

2.

$$\mathbb{V}\left(\frac{E_i^{\mathcal{C}^{(1)},\star}+\bar{L}_i^{\mathcal{C}^{(1)}}}{\bar{L}_i^{\mathcal{C}^{(1)}}}\right)\geq\mathbb{V}\left(\frac{E_i^{*}+\bar{L}_i}{\bar{L}_i}\right)\quad\forall i\in\mathcal{C}^{(1)}_{nodes};$$
$$\forall n\in\{2,\ldots,m\}\text{ it holds that}$$
$$\mathbb{V}\left(\frac{E_i^{\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(n)}\star}+\bar{L}_i^{\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(n)}}}{\bar{L}_i^{\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(n)}}}\right)\geq\mathbb{V}\left(\frac{E_i^{\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(n-1)}\star}+\bar{L}_i^{\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(n-1)}}}{\bar{L}_i^{\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(n-1)}}}\right)\quad\forall i\in\mathcal{C}^{(n)}_{nodes}. \tag{28}$$

3. $\mathbb{V}=\mathbb{V}^{zero}$.

Then, compressing sequentially $\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(m)}$ reduces systemic risk.

By combining the results from Proposition 4.12 above with the results derived in (D'Errico & Roukny, 2019, Section 12) for the characterisation of $\tilde{X}$ we immediately obtain the following corollary (of which statement 1 can be found in (D'Errico & Roukny, 2019, Section 12)).

Corollary 4.13 (Compression as optimisation problem)**.** *Let X be a liabilities matrix and let $\tilde{X}$ be a solution to the conservative compression optimisation problem defined in Definition 2.3. Let $(L,b;\mathbb{V})$ and $(\tilde{L},\tilde{b};\mathbb{V})$ be the payment systems corresponding to X and $\tilde{X}$, respectively.*

1. *There exists a finite sequence of conservative compression network cycles $\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(m)}$, $m\in\mathbb{N}$, such that conservative compression can be carried out along those cycles sequentially starting from $\mathcal{C}_1$ and finishing at $\mathcal{C}_m$, such that $\tilde{X}$ is obtained by sequentially compressing $\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(m)}$ starting from the liabilities matrix X.*

2. *Consider the m conservative compression network cycles from part 1. of this Corollary in Proposition 4.12. If at least one of the three conditions in Proposition 4.12 is satisfied, then the systemic risk in the payment system $(\tilde{L},\tilde{b};\mathbb{V})$ is reduced compared to the systemic risk in the payment system $(L,b;\mathbb{V})$.*

3. *Part 2. of this Corollary remains valid if condition (4) in the Definition 2.3 of $\tilde{X}$ is replaced by $a_{ij}\leq\tilde{X}_{ij}\leq X_{ij}\quad\forall i,j\in\mathcal{N}$, where $a_{ij}\in[0,X_{ij}]\ \forall i,j\in\mathcal{N}$.*

4.5 Policy implications

We have provided necessary conditions for portfolio compression to be harmful. Since we have shown that portfolio compression cannot cause fundamental defaults, these are necessary conditions for portfolio compression to cause contagious defaults. This implies that policy measures that reduce the likelihood and severity of financial contagion automatically mitigate potentially negative effects of portfolio compression.

Key mitigation mechanism for financial contagion are for example sufficient liquidity buffers and sufficient collateral in form of initial margins. Higher levels of liquidity buffers and initial margins would make it less likely that condition (24) would be satisfied in practice, decreasing the probability of portfolio compression having negative consequences.

Could one address possible negative consequences from portfolio compression more directly? We have shown that for portfolio compression to be potentially harmful, we need to have at least one node defaulting in the non-compressed network that takes part in compression (see conditions (21) and (27)). A possible conclusion from this result would be to exclude firms with high default risk from compression activities. While there is currently no regulatory framework to do this, it might not even be desirable. This would severely restrict the possible reduction in gross exposure that can be achieved, which would lead to other disadvantages such as operational risks etc.. We will show that allowing high risk firms to participate in portfolio compression can sometimes even strongly reduce systemic risk.

So a more nuanced approached might be more promising. As discussed in Remark 2.2, institutions participating in compression provide compression tolerances to manage their risk associated with portfolio compression. Currently these tolerances are specified on the individual contract level as in (1), and hence do not account for network spillover effects. To mitigate systemic risk, it would be beneficial to take a network perspective when deciding on compression tolerances and setting constraints in portfolio compression exercises. This is something that usually cannot be done by the individual institution requesting portfolio compression.

We will show in our case study that portfolio compression can be harmful for nodes not taking part in portfolio compression. These nodes would never provide any information or compression tolerances to the compression provider, which shows that there is a need for a financial regulator to oversee such an exercise or to provide a suitable framework for it. This could involve, for example stress testing exercises, checking the validity of conditions like (24) and (28). Alternatively, one could add conditions of this nature to the portfolio compression optimisation problem.

4.6 Illustration of the theoretical results

We illustrate our theoretical results by considering a network that allows for different conservative compressions. Figure 2 highlights the nine different cycles. The liabilities matrix X is defined as

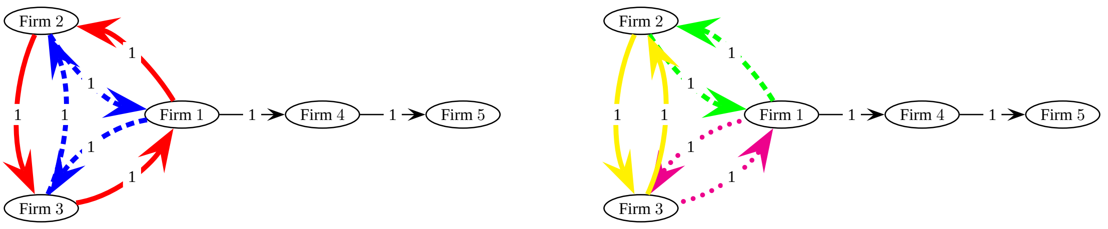
$$X = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The only node that will never default is node 5 since it does not have any liabilities. Whether any of the nodes $1, \dots, 4$ default will depend on the liquidity buffer b, the actual payment obligations arising from X and the choice of compression network cycles. In the following we will assume that the corresponding payment obligations are given by $L = X$ (i.e., $V = 1$ in the definition of f^V). For this liabilities matrix the total net positions are $\bar{A}^{(X)} - \bar{L}^{(X)} = (-1, 0, 0, 0, 1)^\top$. Hence, any compressed network will have the same net positions. In particular we see, that there are nine different ways of how conservative compression could be applied to this particular liabilities matrix. Formally these compression network cycles given by:

- Red cycle (red solid line in Figure 2(a)): $\mathcal{C}_{\text{nodes}} = \{1, 2, 3\}$, $\mathcal{C}_{\text{edges}} = \{(1, 2), (2, 3), (3, 1)\}$;
- Blue cycle (blue dashed line in Figure 2(a)): $\mathcal{C}_{\text{nodes}} = \{1, 2, 3\}, \mathcal{C}_{\text{edges}} = \{(1, 3), (3, 2), (2, 1)\}$;
- Green cycle (green dashed line in Figure 2(b)): $\mathcal{C}_{\text{nodes}} = \{1, 2\}$, $\mathcal{C}_{\text{edges}} = \{(1, 2), (2, 1)\}$;
- Yellow cycle (yellow solid line in Figure 2(b)): $\mathcal{C}_{\text{nodes}} = \{2, 3\}$, $\mathcal{C}_{\text{edges}} = \{(2, 3), (3, 2)\}$;
- Pink cycle (pink dotted line in Figure 2(b)): $\mathcal{C}_{\text{nodes}} = \{1, 3\}$, $\mathcal{C}_{\text{edges}} = \{(1, 3), (3, 1)\}$.

We can compress one or more of these cycles. We only consider compression with the maximal capacity which is $\mu = \mu^{\max} = 1$ for all cycles. We can see that node 1 has fewer internetwork assets than liabilities. Hence, in the absence of any liquidity buffer for node 1 it will default.

We will now show that compression can have very different consequences depending on the liquidity buffer and depending on the recovery rates. Hence, an *optimal compression* in the sense of reducing the number of defaults will not just depend on the network structure but also



(a) Cycles involving 3 financial institutions.

(b) Cycles involving 2 financial institutions.

Figure 2: Illustration of possible compression network cycles.

ν	Defaulting financial institutions when the following cycles are removed									
	none	red	blue	r & b	green	yellow	pink	g & y	g & p	y& p
1										
2	1,2,3	1, 3	1, 2	1,4	1,3,4	1, 2, 3	1,2,4	1, 3,4	1,4	1, 2,4
3	1	1,3,4	1,2,4	1,4	1,3,4	1	1,2,4	1,3,4	1,4	1,2,4

Table 1: Results for $\mathbb{V} = \mathbb{V}^{\text{EN}}$. Colour code: Light gray = strong reduction of systemic risk, dark gray = harmful, white = no difference between compression and no compression in terms of defaults (reduction of systemic risk). $\nu \in \{1, 2, 3\}$ represents three different liquidity buffers $b = A^{(\nu)}$: $A^{(1)} = (1, 0, 0, 0, 0)^\top$, $A^{(2)} = (0, 0.25, 0.25, 0.5, 0)^\top$, $A^{(3)} = (0.25, 0.25, 0.25, 0.25, 0)^\top$.

ν	Defaulting financial institutions when the following cycles are removed									
	none	red	blue	r & b	green	yellow	pink	g & y	g & p	y & p
1										
2	1,2,3,4	1, 2, 3,4	1, 2, 3,4	1,4	1, 2,3,4	1, 2, 3,4	1,2, 3,4	1, 3,4	1,4	1, 2,4
3	1,2,3,4	1, 2, 3,4	1, 2, 3,4	1,4	1, 2,3,4	1, 2, 3,4	1,2, 3,4	1, 3,4	1,4	1, 2,4

Table 2: Results for $\mathbb{V} = \mathbb{V}^{\text{zero}}$ with $k = 0$. Colour code as in Table 1.

on quantities outside the network, e.g. the liquidity buffer. We will highlight the effects of the different structural conditions identified in the previous section.

We will assume that the liquidity buffer corresponds to the external assets in the compressed and non-compressed network, i.e., $b = b^{\mathcal{C},\mu,\gamma} = A^{(e)}$ (where $\gamma = 0$).

Tables 1 and 2 show which financial institutions default for different liquidity buffers b (corresponding to the rows in the tables) and different choices of compression network cycles (corresponding to the different columns in the table). Furthermore, Table 1 reports the results for the Eisenberg & Noe (2001) contagion mechanism, i.e., $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$, whereas Table 2 reports the results for the Rogers & Veraart (2013) contagion mechanism with $\alpha = \beta = 0$, i.e., zero recovery rate in case of default ($\mathbb{V} = \mathbb{V}^{\mathrm{zero}}$ with $k = 0$). Hence, these two tables allow us to compare the effect of the third structural condition - the role of the recovery rate.

The first two structural conditions are concerned with nodes on the compression cycle. We include all nodes that are on a compression network cycle and default both with and without compression in a box, e.g., $\boxed{1, 2}, 3$ indicates that nodes 1 and 2 are on the compression cycle and default with and without compression. Node 3 is either not on the compression network cycle or does not default without compression.

All cells in white that correspond to compressed networks indicate that exactly the same financial institutions default for the compressed network as for the uncompressed network. Cells in light gray indicate that the corresponding compression mechanism strongly reduces systemic risk. Cells in dark gray indicate situations under which compression is harmful.

We consider three different vectors of liquidity buffers and assume that the total liquidity buffer aggregated over all nodes remains the same, i.e., $\sum_{i\in\mathcal{N}} A_i^{(\nu)} = 1$ for all $\nu \in \{1, 2, 3\}$.

Only in the first row corresponding to $b = A^{(1)}$ the liquidity buffer is distributed such that no default occurs (for any choice of compression or no compression). In all other cases node 1 will always default. Since node 5 does not have any liabilities it will never default. For nodes $2, 3, 4$ it depends on the distribution of the liquidity buffer, the recovery rates and the choice of compression whether they default or not. For nodes $1, 2, 3$ there exist cycles that can be used to compress their portfolios whereas for nodes $4, 5$ no such cycles exist.

We observe the following consequences of compression in line with our theoretical results:

Reduction of systemic risk for $\mathbb{V} = \mathbb{V}^{\mathbf{zero}}$: Table 2 contains the results for zero recovery rates. In line with Proposition 4.11 we see a reduction in systemic risk throughout and many examples of a strong reduction in systemic risk indicated by the light gray cells.

Reduction of systemic risk without defaults on compression network cycle in non-compressed financial system: Consistent with Proposition 4.8, for $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$, $b = A^{(3)}$ compressing the yellow cycle (consisting of nodes 2 and 3 which both do not default) makes no difference to the set of defaults and hence reduces systemic risk.

Compression can be harmful for nodes outside the compression network cycle: Let $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$ and $A^{(e)} = A^{(2)}$. Then, nodes $1, 2, 3$ default without compression. When both the red and the blue cycles are compressed, nodes 1 and 4 default. This observation and the example is very similar to the example considered in Schuldenzucker et al. (2018).

Compression can be harmful for nodes on the compression network cycle: When $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$, $A^{(e)} = A^{(3)}$ only node 1 defaults without compression but node 3 on the compression network cycle defaults if the red cycle is compressed and node 4 outside the cycle defaults too.

Different choices of compression network cycles can lead to different outcomes: Let $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$ and $A^{(e)} = A^{(2)}$. Then some compression cycles strongly reduce systemic risk (e.g., compressing only the blue or only the red cycle) whereas other compression cycles are harmful (e.g., compressing both the red and the blue cycle or compressing the green cycle) or make no difference in terms of defaults (e.g., compressing the yellow cycle).

Different distribution of liquidity buffer can lead to different outcomes: Let $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$ and consider compressing the red cycle. For some liquidity buffers (e.g. $\nu = 2$) we observe a strong reduction of systemic risk whereas for others (e.g. $\nu = 3$) this compression is harmful.

There are other cases (e.g. $\nu = 1$) where compression makes no difference in terms of defaults.
Consequences of compression depends on recovery rates: Let $b = A^{(2)}$. By comparing Tables 1 and 2 we find that compressing both the blue and the red cycle is harmful under positive recovery rates, but strongly reduces systemic risk under zero recovery rates. Furthermore compressing only the red or only the blue cycle strongly reduces systemic risk under positive recovery rates (where as it makes no difference under zero recovery rates).
Strong reduction of systemic risk: Let $\mathbb{V} = \mathbb{V}^{\text{EN}}$ and $b = A^{(2)}$. Compressing only the red cycle strongly reduces systemic risk since node 2 no longer defaults.

In the following we illustrate the effects of different valuation functions in more detail.

Example 4.14. We set $b = A^{(2)}$ and compress sequentially first the red cycle referred to as $\mathcal{C}^{(1)}$ and then the blue cycle referred to as $\mathcal{C}^{(2)}$. We assume that $\gamma = 0$, i.e., compression does not increase the liquidity buffer. We consider three different valuation functions $\mathbb{V}^{\text{EN}}$, $\mathbb{V}^{\text{RV}}$ and $\mathbb{V}^{\text{InitialMargin}}$.

First, let $\mathbb{V} = \mathbb{V}^{\text{EN}}$, i.e. we consider the Eisenberg & Noe (2001) model. Table 1 shows that $\mathcal{D}(L, b; \mathbb{V}^{\text{EN}}) = \{1, 2, 3\}$. Compressing the red cycle yields $\mathcal{D}(L^{\mathcal{C}^{(1)}}, b; \mathbb{V}^{\text{EN}}) = \{1, 3\}$ hence a reduction in systemic risk. The repayment proportions for $i \in \mathcal{C}_{\text{nodes}}$ satisfy

$$\mathbb{V}\left(\frac{E_1^* + \bar{L}_1}{\bar{L}_1}\right) = 0.5 = \mathbb{V}\left(\frac{E_1^{\mathcal{C}^{(1)}\star} + \bar{L}_1^{\mathcal{C}^{(1)}}}{\bar{L}_1^{\mathcal{C}^{(1)}}}\right),$$
$$\mathbb{V}\left(\frac{E_2^* + \bar{L}_2}{\bar{L}_2}\right) = 0.75 < 1 = \mathbb{V}\left(\frac{E_2^{\mathcal{C}^{(1)}\star} + \bar{L}_2^{\mathcal{C}^{(1)}}}{\bar{L}_2^{\mathcal{C}^{(1)}}}\right),$$
$$\mathbb{V}\left(\frac{E_3^* + \bar{L}_3}{\bar{L}_3}\right) = 0.75 = \mathbb{V}\left(\frac{E_3^{\mathcal{C}^{(1)}\star} + \bar{L}_3^{\mathcal{C}^{(1)}}}{\bar{L}_3^{\mathcal{C}^{(1)}}}\right).$$

Hence, from Proposition 4.9 we know that this compression reduces systemic risk and here it even strongly reduces systemic risk. Even though nodes 1 and 3 default both in the compressed and the non-compressed network they repay the same relative proportion of their debt in both situations (0.5 and 0.75 respectively) and that is why compression cannot be harmful. Without compression node 1 has a shortfall of $\left(1 - \mathbb{V}\left(\frac{E_1^* + \bar{L}_1}{\bar{L}_1}\right)\right)\bar{L}_1 = 1.5$ and losses of 1.5/3=0.5 hit the three creditors of node 1 (nodes 2, 3, 4). With compression node 1 has a shortfall of 1 and losses of 1/2 hit its two creditors (nodes 3 and 4), i.e., even though the repayment proportions change (see Proposition 4.10), the absolute losses transmitted to nodes 3 and 4 remain the same in this example.
Suppose that after compressing the red cycle, we compress the blue cycle. Then, $\mathcal{D}(L^{\mathcal{C}^{(1)},\mathcal{C}^{(2)}}, b; \mathbb{V}^{\text{EN}}) = \{1, 4\}$, hence node 4 is a new default which is not on any of the compression cycles. Furthermore,

$$\mathbb{V}\left(\frac{E_1^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}\star} + \bar{L}_1^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}}}{\bar{L}_1^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}}}\right) = 0 < 0.5 = \mathbb{V}\left(\frac{E_1^{\mathcal{C}^{(1)}\star} + \bar{L}_1^{\mathcal{C}^{(1)}}}{\bar{L}_1^{\mathcal{C}^{(1)}}}\right),$$
$$\mathbb{V}\left(\frac{E_4^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}\star} + \bar{L}_4^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}}}{\bar{L}_4^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}}}\right) = 0.5 < 1 = \mathbb{V}\left(\frac{E_4^{\mathcal{C}^{(1)}\star} + \bar{L}_4^{\mathcal{C}^{(1)}}}{\bar{L}_4^{\mathcal{C}^{(1)}}}\right).$$

Hence, node 1 always defaults. It repays a strictly smaller proportion of its debt when $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ (the red and blue cycle) are compressed than when only $\mathcal{C}^{(1)}$ (the red cycle) is compressed. Since $\mathbb{V} \neq \mathbb{V}^{\text{zero}}$ all three necessary conditions for compression to be harmful are satisfied. Node 1 pays 0 to node 4 if both the red and the blue cycle are compressed since it has no longer any income. Node 4 cannot cope with this and defaults. When only the red cycle was compressed node, 1 was still able to pay 0.5 to node 4 which was just enough for it not to default.

Second, let $\mathbb{V} = \mathbb{V}^{\text{RV}}$ with $\beta = 0.99$ which is the Rogers & Veraart (2013). We find that $\mathcal{D}(L, b; \mathbb{V}^{\text{RV}}) = \{1, 2, 3, 4\}$. Even the small bankruptcy costs modelled by $\beta = 0.99 < 1$ cause the total collapse of the non-compressed financial system. Compressing the red cycle yields $\mathcal{D}(L^{\mathcal{C}^{(1)}}, b; \mathbb{V}^{\text{RV}}) = \{1, 2, 3, 4\}$, technically a reduction in systemic risk.
If both the blue and the red cycle are compressed, then $\mathcal{D}(L^{\mathcal{C}^{(1)},\mathcal{C}^{(2)}}, b; \mathbb{V}^{\text{RV}}) = \{1, 4\}$. Nodes 2 and 3 can no longer default because they do not have any liabilities any more. Hence, compressing these two cycles strongly reduces systemic risk.

Third, we repeat the analysis with $\mathbb{V} = \mathbb{V}^{\text{InitialMargin}}$ where the parameter for the initial margins is $J = 0.1$. We consider $\beta = 1.0$ (no bankruptcy costs) and $\beta = 0.99$ (small bankruptcy costs). For both choices of β the default sets coincide. They are given by

$$\begin{aligned}\mathcal{D}(L, b; \mathbb{V}^{\text{InitialMargin}}) &= \{1\},\\ \mathcal{D}(L^{\mathcal{C}^{(1)}}, b; \mathbb{V}^{\text{InitialMargin}}) &= \{1, 3\},\\ \mathcal{D}(L^{\mathcal{C}^{(1)},\mathcal{C}^{(2)}}, b; \mathbb{V}^{\text{InitialMargin}}) &= \{1, 4\}.\end{aligned}$$

In line with the ordering results in Veraart (2020), we see that adding initial margins to the Eisenberg & Noe (2001) and the Rogers & Veraart (2013) models yields a better outcome for the system. But even when initial margins are available, i.e., $J > 0$, we find that compressing first the red cycle is harmful since node 3 is a new default and then compressing the blue cycle is also harmful since node 4 is a new default. By increasing J, we could avoid all contagious defaults, but node 1 remains in default since it is a fundamental default.

Remark 4.15 (An optimisation perspective on the numerical example)**.** The optimal solution to the conservative compression optimisation problem (Definition 2.3) for the given example corresponds to the network in which both the red and the blue cycles are removed. There exists liquidity buffers, e.g., $b = A^{(2)}$ or $b = A^{(3)}$ for which this compression is harmful, since node 4 is a new default under compression in the optimally compressed network.

If we consider the non-conservative compression optimisation in this example, i.e., the optimisation problem that has the same objective as the conservative compression optimisation problem and also constraint (3) but does not have constraint (4), then the optimal solution is a network that consists of exactly one edge, the edge from node 1 to node 5, with weight 1. Node 1 remains in default (for all choices of b considered in the example), hence it pays less than 1 to node 5. But node 5 cannot default since its payment obligations are zero, so technically this compression is not harmful. As discussed in D'Errico & Roukny (2019) the non-conservative compression optimisation problem is solved by a bipartite graph, i.e., the nodes can be split into two sets where nodes in one set have only outgoing edges and nodes in the other set have only incoming edges. (It is possible to have nodes that do not have any in- or outgoing edges in which case they can be assigned to any of the two groups.) This is exactly what we get here. Hence, losses can spread from node 1 to node 5, but node 5 cannot transmit them further.

5 Conclusion

When does portfolio compression reduce systemic risk? We have identified three structural conditions that imply a reduction in systemic risk: no defaults on a compression network cycle in the non-compressed financial system, all nodes on the compression network cycle repay a larger proportion of their total payment obligations in the compressed system than in the non-compressed system and zero recovery rates.

Even though there are many situations under which portfolio compression reduces systemic risk, we have shown that there are circumstances under which compression can be harmful. Ultimately the danger from portfolio compression comes from firms at risk of default engaging in portfolio compression. If they then default, losses are spread in a network that now has a different structure compared to the original non-compressed network. In particular, since compression has implicitly changed the seniority structure of the debt, all those debts that were compressed prior to payment obligations becoming due have effectively been paid in full, which is obviously not the case for still outstanding debt. For nodes that do not default this change in seniority structure does not matter, since they continue to be able to satisfy all their payment obligations. For nodes that do default (and who have not posted sufficient initial margins to cover the payment shortfall) compression can imply that they spread losses now differently and some counterparties might be hit by larger losses in the compressed network.

We have shown that portfolio compression cannot cause any fundamental defaults. Portfolio compression might even potentially avoid some fundamental defaults. For compression to be harmful we need at least one firm that defaults in the compressed network that would not have defaulted in the non-compressed network. Hence, such a default would have to be a contagious default. These findings imply that any mechanisms that reduce the likelihood of contagion in financial markets also reduce the likelihood of portfolio compression having a negative outcome. Requiring collateral (initial margins) is an obvious mechanism which reduces the probability of contagion. Nevertheless, a residual risk remains for all not fully collateralised trades and in such a situation portfolio compression can change the outcome for the system. Another mechanism would be to require larger liquidity buffers as they fundamentally determine the likelihood of contagion, see Glasserman & Young (2015) and Paddrik et al. (2020).

Our analysis shows that classical compression tolerances that are meant to provide a safety net for compression to not increase risk, cannot fully achieve this as long as they do not account for network effects. The paths that can transmit losses from a compression network cycle to other nodes in the system are not directly observable for the participants making it difficult for them to assess potential risks from portfolio compression themselves and including them in a meaningful way as part of their compression tolerances.

In general we find that if only firms with low default risk engage in compression activities, then this does not give cause for concern. Whether one should restrict portfolio compression services to low-risk firms is a different question. Any restrictions on who can participate would significantly limit the reduction in gross exposure that can be achieved and the associated benefits that come with it, such as operational benefits. In practice, portfolio compression is done for a wide range of reasons, and we have only considered it from a systemic risk point of view. Even then, we have found situations under which allowing high risk firms to compress their portfolio can sometimes strongly reduce systemic risk.

Ultimately, one would need to conduct a cost-benefit analysis of portfolio compression to decide whether one might want to use such a technology on a large scale or not. Using our analysis within such a cost-benefit analysis would be an interesting avenue for future research.

A Compression and capital requirements

Remark A.1 (Compression and capital requirements)**.** We show that portfolio compression can be beneficial for complying with the minimum leverage ratio under Basel III[8] and this effect can be captured within or model. Under Basel III the leverage ratio is defined as (tier 1 capital)/(balance sheet and off-balance sheet exposures) and is required to be larger than 3% in Europe (slightly higher in the US). Since the leverage ratio uses gross exposures, compression reduces the denominator of the leverage ratio and hence increases it.

We assume for now that $L = X$ and $b = A^{(e)}$. Let $(L, b; \mathbb{V})$ be the corresponding payment system and let E^* be the greatest re-evaluated equity that corresponds to using the valuation

[8]When "regulatory capital charges are aligned with the counterparty exposure risk, the capital charge should not change. However if cruder approaches are being used that do not accurately capture offsetting risks, such as the current exposure method (CEM) or leverage ratio approach, compression will tend to reduce the capital charge", O' Kane (2017).

function $\mathbb{V}$. By using E^* as approximation of the tier 1 capital and $E^* + \bar{L}$ as approximation of the exposures, and requiring that the corresponding leverage ratio is larger than 3%, we obtain $\frac{E_i^*}{E_i^*+\bar{L}_i} \geq 0.03 \Leftrightarrow E_i^* \geq \frac{0.03}{0.97}\bar{L}_i \approx 0.031\bar{L}_i$. Any breach of this inequality could cause default. Within our model, recall that $i \in \mathcal{D}(L, b; \mathbb{V}) \Leftrightarrow E_i^* < k\bar{L}_i$ and therefore we can set $k = \frac{3}{97}$ as threshold in (8) to define such a default event demonstrating the benefit of allowing for $k > 0$. Note, that in the compressed network the corresponding default threshold would satisfy $k\bar{L}_i^{\mathcal{C},\mu} \leq k\bar{L}_i$ for all $i \in \mathcal{N}$ and would therefore be lower (and hence better) than in the non-compressed network. Note that in models with $k = 0$, compression does not affect the default threshold and one can therefore not capture the advantage of portfolio compression for capital requirements.

B Proofs

B.1 Additional notation

Let $\mathcal{C} = (\mathcal{C}_{\text{nodes}}, \mathcal{C}_{\text{edges}}, \mu^{\max})$ be a compression network cycle with maximal capacity $\mu^{\max}$. We will use the notation pred(i) for the node in $\mathcal{C}_{\text{nodes}}$ that is the predecessor of i on the cycle $\mathcal{C}_{\text{nodes}}$, i.e., pred$(i)$ is the node that satisfies $(\text{pred}(i), i) \in \mathcal{C}_{\text{edges}}$. Similarly suc$(i)$ is the successor of i on the cycle, i.e., it is the node in $\mathcal{C}_{\text{nodes}}$ that satisfies $(i, \text{suc}(i)) \in \mathcal{C}_{\text{edges}}$.

B.2 Proofs of the results in Section 2

The following result formalises the claim made that conservative compression keeps net exposures fixed while reducing gross exposures and is therefore in line with the corresponding definition in D'Errico & Roukny (2019).

Lemma B.1. *Let X be a liabilities matrix for which there exists a conservative compression network cycle $\mathcal{C}$ with maximal capacity $\mu^{\max} > 0$. Let $0 < \mu \leq \mu^{\max}$ and let $L^{\mathcal{C},\mu}$ be the μ-compressed liabilities matrix using cycle $\mathcal{C}$. Then,*

1. *$X^{\mathcal{C},\mu}$ is a liabilities matrix, i.e., $X_{ij}^{\mathcal{C},\mu} \geq 0 \ \forall i, j \in \mathcal{N}$ and $X_{ii}^{\mathcal{C},\mu} = 0 \ \forall i \in \mathcal{N}$;*

2. *$X_{ij}^{\mathcal{C},\mu} \leq X_{ij}$ for all $i, j \in \mathcal{N}$;*

3. $$\bar{L}_i^{(X),\mathcal{C},\mu} = \begin{cases} \bar{L}_i^{(X)}, & \text{if } i \notin \mathcal{C}_{nodes}, \\ \bar{L}_i^{(X)} - \mu, & \text{if } i \in \mathcal{C}_{nodes}; \end{cases}$$

4. *the net positions in the compressed network $L^{\mathcal{C},\mu}$ coincide with the net positions in the original network L, i.e., $\eta_i^{\mathcal{C},\mu} = \sum_{j\in\mathcal{N}} X_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{(X),\mathcal{C},\mu} = \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i = \eta_i \ \forall i \in \mathcal{N}$;*

5. *the gross positions in the compressed network $L^{\mathcal{C},\mu}$ are less than or equal to the gross positions in the original network L, i.e., $\sum_{j\in\mathcal{N}} X_{ji}^{\mathcal{C},\mu} + \bar{L}_i^{(X),\mathcal{C},\mu} \leq \sum_{j\in\mathcal{N}} L_{ji} + \bar{L}_i \ \forall i \in \mathcal{N}$;*

6. *compression strictly reduces gross positions of all $i \in \mathcal{C}_{nodes}$, i.e., $\sum_{j\in\mathcal{N}} X_{ji}^{\mathcal{C},\mu} + \bar{L}_i^{(X),\mathcal{C},\mu} < \sum_{j\in\mathcal{N}} X_{ji} + \bar{L}_i^{(X)}$.*

Proof of Lemma B.1. 1. By definition if $(i,j) \notin \mathcal{C}_{\text{edges}}$ then $X_{ij}^{\mathcal{C},\mu} = X_{ij} \geq 0$ and if $(i,j) \in \mathcal{C}_{\text{edges}}$ then $X_{ij}^{\mathcal{C},\mu} = X_{ij} - \mu \geq X_{ij} - \min_{(\nu,\mu)\in\mathcal{C}_{\text{edges}}} X_{\nu\mu} \geq 0$. Since $X_{ii} = 0$ for all $i \in \mathcal{N}$ also $X_{ii}^{\mathcal{C},\mu} = 0$ for all $i \in \mathcal{N}$.

2. This is obvious from the definition of $X^{\mathcal{C},\mu}$.

3. It follows immediately from the definition that if $i \notin \mathcal{C}_{\text{nodes}}$ then $\bar{L}_i^{(X),\mathcal{C},\mu} = \bar{L}_i^{(X)}$. Now let $i \in \mathcal{C}_{\text{nodes}}$. Then,

$$\bar{L}_i^{(X),\mathcal{C},\mu} = \sum_{j\in\mathcal{N}} X_{ij}^{\mathcal{C},\mu} = \underbrace{X_{i\text{suc}(i)}^{\mathcal{C},\mu}}_{=X_{i,\text{suc}(i)}-\mu} + \sum_{j\in\mathcal{N}\setminus\{\text{suc}(i)\}} \underbrace{X_{ij}^{\mathcal{C},\mu}}_{=X_{ij}} = \sum_{j\in\mathcal{N}} X_{ij} - \mu = \bar{L}_i^{(X)} - \mu.$$

4. It is obvious that for $i \notin \mathcal{C}_{\text{nodes}}$ that the net position for the non-compressed network and the compressed network coincides. For $i \in \mathcal{C}_{\text{nodes}}$ this also holds since

$$\eta_i^{\mathcal{C},\mu} = \sum_{j\in\mathcal{N}} X_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{(X),\mathcal{C},\mu} = \underbrace{X_{\text{pred}(i)i}^{\mathcal{C},\mu}}_{X_{\text{pred}(i)i}-\mu} + \sum_{j\in\mathcal{N}\setminus\{\text{pred}(i)\}} \underbrace{X_{ji}^{\mathcal{C},\mu}}_{=X_{ji}} - \underbrace{\bar{L}_i^{(X),\mathcal{C},\mu}}_{\bar{L}_i^{(X)}-\mu} = \sum_{j\in\mathcal{N}} X_{ji} - \bar{L}_i^{(X)} = \eta_i.$$

5. From part 2. and 3. we immediately get that for all $i \in \mathcal{N}$

$$\sum_{j\in\mathcal{N}} \underbrace{X_{ji}^{\mathcal{C},\mu}}_{\leq X_{ji}} + \underbrace{\bar{L}_i^{(X),\mathcal{C},\mu}}_{\leq \bar{L}_i^{(X)}} \leq \sum_{j\in\mathcal{N}} X_{ji} + \bar{L}_i^{(X)}.$$

6. Let $i \in \mathcal{C}_{\text{nodes}}$. Since $\mu > 0$ we get

$$\begin{aligned}\sum_{j\in\mathcal{N}} X_{ji}^{\mathcal{C},\mu} + \underbrace{\bar{L}_i^{(X),\mathcal{C},\mu}}_{=\bar{L}_i^{(X)}-\mu} &= \underbrace{X_{\text{pred}(i)i}^{\mathcal{C},\mu}}_{=X_{\text{pred}(i)i}-\mu} + \sum_{j\in\mathcal{N}\setminus\{\text{pred}(i)\}} \underbrace{X_{ji}^{\mathcal{C},\mu}}_{=X_{ji}} + \bar{L}_i^{(X)} - \mu = \sum_{j\in\mathcal{N}} X_{ji} + \bar{L}_i^{(X)} - 2\mu \\ &< \sum_{j\in\mathcal{N}} X_{ji} + \bar{L}_i^{(X)}.\end{aligned}$$

□

B.3 Proofs of the results in Section 4

The following lemma will be used in several proofs below.

Lemma B.2.
Let X be a liabilities matrix for which there exists a conservative compression network cycle of X with maximal capacity $\mu^{\max}$. Let $(L, b; \mathbb{V})$ be the corresponding payment system, let $0 < \mu \leq \mu^{\max}$ and let $L^{\mathcal{C},\mu}$ be the μ-compressed liabilities matrix. Set

$$\mathcal{M} = \{i \in \mathcal{N} \mid \bar{L}_i > 0\}, \qquad \mathcal{M}^{\mathcal{C}} = \{i \in \mathcal{N} \mid \bar{L}_i^{\mathcal{C}} > 0\}.$$

Let $j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}$. Then the following holds. First, $j \in \mathcal{C}_{nodes}$ and $\bar{L}_j = \mu V$. Second,

$$L_{ji} = \begin{cases} \mu V, & \textit{if } i = \textit{suc}(j), \\ 0, & \textit{otherwise.} \end{cases}$$

Proof of Lemma B.2. First, let $j \in \mathcal{M}\setminus\mathcal{M}^{\mathcal{C}}$. Then by definition of the sets $\bar{L}_j > 0$ and $\bar{L}_j^{\mathcal{C},\mu} = 0$. Hence $\bar{L}_j \neq \bar{L}_j^{\mathcal{C},\mu}$ which implies that $j \in \mathcal{C}_{\text{nodes}}$. Since then $\bar{L}_j^{\mathcal{C},\mu} = \bar{L}_j - \mu V = 0$ we immediately get that $\bar{L}_j = \mu V$.

Second, from part 1. of this lemma we know that $j \in \mathcal{C}_{\text{nodes}}$ and $\bar{L}_j = \mu V$. For fixed $j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}$ we have by definition

$$L_{ji}^{\mathcal{C},\mu} = \begin{cases} L_{ji} - \mu V, & \text{if } i = \text{suc}(j), \\ L_{ji}, & \text{if } i \in \mathcal{N} \setminus \{\text{suc}(j)\}. \end{cases}$$

Since $L_{ji}^{\mathcal{C},\mu} \geq 0$ for all $i \in \mathcal{N}$, in particular, $L_{j\text{suc}(j)} - \mu V \geq 0$ and hence $L_{j\text{suc}(j)} \geq \mu V$. Since,

$$0 = \bar{L}_j^{\mathcal{C},\mu} = \sum_{i\in\mathcal{N}} L_{ji} - \mu V = \underbrace{L_{j\text{suc}(j)}}_{\geq \mu V} + \sum_{i\in\mathcal{N}\setminus\{\text{suc}(j)\}} L_{ji} - \mu V \geq \mu V + \sum_{i\in\mathcal{N}\setminus\{\text{suc}(j)\}} L_{ji} - \mu V$$
$$= \sum_{i\in\mathcal{N}\setminus\{\text{suc}(j)\}} L_{ji},$$

we see that since $L_{ji} \geq 0$ for all $i \in \mathcal{N} \setminus \{\text{suc}(j)\}$ it holds that $L_{ji} = 0$ for all $i \in \mathcal{N} \setminus \{\text{suc}(j)\}$. Furthermore, since $\bar{L}_j = \mu V$ we must have that $L_{j\text{suc}(j)} = \mu V$.

□

Proof of Proposition 4.2. Recall from the definition of E^* that

$$E_i^* = \Phi_i(E^*) = b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \tag{29}$$

for all $i \in \mathcal{N}$. We denote by $\Phi^{\mathcal{C},\mu,\gamma}$ the function that corresponds to the compressed network, i.e., $E^{\mathcal{C},\mu;\gamma;*}$ is the greatest fixed point of $\Phi^{\mathcal{C},\mu,\gamma}$, i.e.,

$$E_i^{\mathcal{C},\mu;\gamma;*} = \Phi_i^{\mathcal{C},\mu,\gamma}(E^{\mathcal{C},\mu;\gamma;*}) = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C},\mu;\gamma;*} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i^{\mathcal{C},\mu} \tag{30}$$

for all $i \in \mathcal{N}$. In the following we show that $E_i^* = E_i^{\mathcal{C},\mu;\gamma;*}$ for all $i \in \mathcal{N} \setminus \mathcal{O}$.

Let $i \in \mathcal{N} \setminus \mathcal{O}$. From (29)

$$\begin{aligned} E_i^* &= \Phi_i(E^*) = b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\ &= b_i + \sum_{j\in\mathcal{M}\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) + \underbrace{\sum_{j\in\mathcal{M}\cap\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right)}_{\overset{\star}{=}0} - \bar{L}_i \\ &= b_i + \sum_{j\in\mathcal{M}\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i =: f_i(E^*_{\mathcal{M}\setminus\mathcal{O}}), \end{aligned} \tag{31}$$

where ($\star$) holds because by assumption $i \in \mathcal{N} \setminus \mathcal{O}$ hence there cannot be a $j \in \mathcal{M} \setminus \mathcal{O}$ with $L_{ji} > 0$ otherwise this would imply that $i \in \mathcal{O}$.

Similarly, from (30)

$$\begin{aligned}
E_i^{\mathcal{C},\mu;\gamma;*} &= \Phi_i^{\mathcal{C},\mu,\gamma}(E^{\mathcal{C},\mu;\gamma;*}) = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C},\mu;\gamma;*}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i^{\mathcal{C},\mu} \\
&= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^{\mathcal{C},\mu;\gamma;*}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i \\
&= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{\mathcal{C},\mu;\gamma;*}+\bar{L}_j}{\bar{L}_j}\right) + \underbrace{\sum_{j\in\mathcal{M}^{\mathcal{C}}\cap\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{\mathcal{C},\mu;\gamma;*}+\bar{L}_j}{\bar{L}_j}\right)}_{\overset{\star}{=}0} - \bar{L}_i \qquad (32)\\
&= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{*}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&\overset{(\star\star)}{=} b_i + \sum_{j\in\mathcal{M}\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{*}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i = f_i(E_{\mathcal{M}\setminus\mathcal{O}}^{\mathcal{C},\mu;\gamma;*}),
\end{aligned}$$

where the justification for $(\star)$ is as before and the justification for $(\star\star)$ is that $L_{ij} = 0$ for all $j \in (\mathcal{M}\setminus\mathcal{M}^{\mathcal{C}})\setminus\mathcal{O}$ since $i \in \mathcal{N}\setminus\mathcal{O}$ (see Lemma B.2).

Let $i \in \mathcal{O}$. From (29) and using ideas from (31)

$$\begin{aligned}
E_i^* &= \Phi_i(E^*) = b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&= \underbrace{b_i + \sum_{j\in\mathcal{M}\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i}_{=f_i(E^*_{\mathcal{M}\setminus\mathcal{O}})} + \underbrace{\sum_{j\in\mathcal{M}\cap\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right)}_{=:g_i(E^*_{\mathcal{M}\cap\mathcal{O}})} = f_i(E^*_{\mathcal{M}\setminus\mathcal{O}}) + g_i(E^*_{\mathcal{M}\cap\mathcal{O}}).
\end{aligned}$$

Similarly, from (30) and using ideas from (32)

$$\begin{aligned}
E_i^{\mathcal{C},\mu;\gamma;*} &= \Phi_i^{\mathcal{C},\mu,\gamma}(E^{\mathcal{C},\mu;\gamma;*}) = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C},\mu;\gamma;*}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i^{\mathcal{C},\mu} \\
&= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{\mathcal{C},\mu;\gamma;*}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \\
&\quad + \underbrace{\sum_{j\in\mathcal{M}^{\mathcal{C}}\cap\mathcal{O}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C},\mu;\gamma;*}+\bar{L}_j}{\bar{L}_j}\right) + \mu(\gamma J + V)\mathbb{I}_{\{i\in\mathcal{C}_{\text{nodes}}\}}}_{=:g_i^{\mathcal{C}}(E^{\mathcal{C},\mu;\gamma;*}_{\mathcal{M}^{\mathcal{C}}\cap\mathcal{O}})} \\
&= b_i + \sum_{j\in\mathcal{M}\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{\mathcal{C},\mu;\gamma;*}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + g_i^{\mathcal{C}}(E^{\mathcal{C},\mu;\gamma;*}_{\mathcal{M}^{\mathcal{C}}\cap\mathcal{O}}) = f_i(E^{\mathcal{C},\mu;\gamma;*}_{\mathcal{M}\setminus\mathcal{O}}) + g_i^{\mathcal{C}}(E^{\mathcal{C},\mu;\gamma;*}_{\mathcal{M}^{\mathcal{C}}\cap\mathcal{O}}).
\end{aligned}$$

The function $f_{\mathcal{N}\setminus\mathcal{O}} : \mathcal{E}_{\mathcal{N}\setminus\mathcal{O}} \to \mathcal{E}_{\mathcal{N}\setminus\mathcal{O}}$ is non-decreasing and its greatest fixed point exists by Tarksi's fixed point theorem. In particular, it coincides with $E^*_{\mathcal{N}\setminus\mathcal{O}}$ and $E^{\mathcal{C},\mu;\gamma;*}_{\mathcal{N}\setminus\mathcal{O}}$ since we have seen that the fixed points E^* and $E^{\mathcal{C},\mu;\gamma;*}$ can be decomposed into a component characterised by f and a component characterised by g or $g^{\mathcal{C}}$ with non-overlapping arguments.

□

The following lemma is a reformulated version of (Veraart, 2020, Theorem 2.6) for the situation with compression. We will use the sequences defined in there in several proofs about the main results (i.e., in the proofs of Propositions 4.8, 4.9, 4.11).

Lemma B.3. *Consider the market setting of Assumption 3.9. Define, the initial equity as in Definition 4.3, i.e., for all $i \in \mathcal{N}$*

$$E_i^{(0)} = b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i \qquad E_i^{\mathcal{C}(0);0} = \underbrace{b_i^{\mathcal{C},\mu,0}}_{=b_i} + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{\mathcal{C}}, \qquad E_i^{\mathcal{C}(0);\gamma} = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{\mathcal{C}}.$$

We define recursively three (N-dimensional) sequences

$$E^{(n)} = \Phi(E^{(n-1)}), \qquad E^{\mathcal{C}(n);0} = \Phi^{\mathcal{C},\mu,0}(E^{\mathcal{C}(n-1);0}), \qquad E^{\mathcal{C}(n);\gamma} = \Phi^{\mathcal{C},\mu,\gamma}(E^{\mathcal{C}(n-1);\gamma}), \tag{33}$$

where $n \in \mathbb{N}$. The functions Φ and $\Phi^{\mathcal{C},\mu,\gamma}$ are defined in (9) and (11) respectively. The function $\Phi^{\mathcal{C},\mu,0}$ is a special case of $\Phi^{\mathcal{C},\mu,\gamma}$ obtained by setting $\gamma = 0$ in the definition of $\Phi^{\mathcal{C},\mu,\gamma}$. Then,

1. *The sequences $(E^{(n)})$, $(E^{\mathcal{C}(n);0})$ and $(E^{\mathcal{C}(n);\gamma})$ are non-increasing, i.e., for all $i \in \mathcal{N}$ and for all $n \in \mathbb{N}_0$ it holds that*

$$E_i^{(n)} \geq E_i^{(n+1)}, \qquad E_i^{\mathcal{C}(n);0} \geq E_i^{\mathcal{C}(n+1);0}, \qquad E_i^{\mathcal{C}(n);\gamma} \geq E_i^{\mathcal{C}(n+1);\gamma}.$$

2. *The three sequences defined in (33) converge to the corresponding greatest re-evaluated equities, i.e., for all $i \in \mathcal{N}$*

$$\lim_{n\to\infty} E_i^{(n)} = E_i^*, \qquad \lim_{n\to\infty} E_i^{\mathcal{C}(n);0} = E_i^{\mathcal{C};0;*}, \qquad \lim_{n\to\infty} E_i^{\mathcal{C}(n);\gamma} = E_i^{\mathcal{C};\gamma;*}.$$

Proof of Lemma B.3. First note that Φ, $\Phi^{\mathcal{C},\mu,0}$ and $\Phi^{\mathcal{C},\mu,\gamma}$ are non-decreasing, see (Veraart, 2020, Lemma A.1). The statements follow directly from (Veraart, 2020, Theorem 2.6). □

Lemma B.4. *Consider the market setting of Assumption 3.9. Let $\mathcal{F}$ be the fundamental defaults in the non-compressed network and let $\mathcal{F}^{\mathcal{C}}$ be the fundamental defaults in the compressed network (see Definition 4.4). Then, $\mathcal{F} \subseteq \mathcal{D}(L, b; \mathbb{V})$ and $\mathcal{F}^{\mathcal{C}} \subseteq \mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V})$.*

Proof of Lemma B.4. Recall, that $\mathcal{F} = \{i \in \mathcal{N} \mid E_i^{(0)} < k\bar{L}_i\}$ and $\mathcal{F}^{\mathcal{C}} = \{i \in \mathcal{N} \mid E^{\mathcal{C}(0);\gamma} < k\bar{L}_i^{\mathcal{C},\mu}\}$. We consider the sequences $(E^{(n)})$ and $(E^{\mathcal{C}(n);\gamma})$ defined in (33). Let $i \in \mathcal{F}$. Then, by Lemma B.3 $\forall m \in \mathbb{N}$ $k\bar{L}_i > E_i^{(0)} \geq E_i^{(m)} \geq \lim_{n\to\infty} E_i^{(n)} = E_i^*$ and hence $i \in \mathcal{D}(L, b; \mathbb{V})$. Similarly, let $i \in \mathcal{F}^{\mathcal{C}}$. Then, by Lemma B.3 $\forall m \in \mathbb{N}$ $k\bar{L}_i^{\mathcal{C},\mu} > E_i^{\mathcal{C}(0);\gamma} \geq E_i^{\mathcal{C}(m);\gamma} \geq \lim_{n\to\infty} E_i^{\mathcal{C}(n);\gamma} = E_i^{\mathcal{C},\mu;\gamma;*}$ and hence $i \in \mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V})$.

□

Proof of Lemma 4.5. Recall that $b_i^{\mathcal{C},\mu,0} = b_i$ for all $i \in \mathcal{N}$ and from the definition of $b^{\mathcal{C},\mu,\gamma}$ it follows immediately that $b_i^{\mathcal{C},\mu,0} \leq b_i^{\mathcal{C},\mu,\gamma}$ for all $i \in \mathcal{N}$ and for all $\gamma \in [0,1]$. If $i \notin \mathcal{C}_{\text{nodes}}$, then one immediately sees that $E_i^{(0)} = E_i^{\mathcal{C}(0);0} \leq E_i^{\mathcal{C}(0);\gamma}$. If $i \in \mathcal{C}_{\text{nodes}}$, then

$$\begin{aligned}
E_i^{\mathcal{C}(0);0} &= b_i^{\mathcal{C},\mu,0} + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{\mathcal{C},\mu} = b_i^{\mathcal{C},\mu,0} + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu} - (\bar{L}_i - \mu V) \\
&= b_i^{\mathcal{C},\mu,0} + \sum_{j\in\mathcal{N}\setminus\{\text{pred}(i)\}} \underbrace{L_{ji}^{\mathcal{C},\mu}}_{=L_{ji}} + \underbrace{L_{\text{pred}(i)i}^{\mathcal{C},\mu}}_{=L_{\text{pred}(i)i}-\mu V} - \bar{L}_i + \mu V \\
&= \underbrace{b_i^{\mathcal{C},\mu,0}}_{=b_i} + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i = E_i^{(0)}.
\end{aligned}$$

Now let $\gamma \in [0,1]$, then

$$E_i^{\mathcal{C}(0);\gamma} = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{\mathcal{C},\mu} \geq b_i^{\mathcal{C},\mu,0} + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{\mathcal{C},\mu} = E_i^{\mathcal{C}(0);0}.$$

□

Proof of Proposition 4.6 . To prove the first statement, let $i \in \mathcal{F}^{\mathcal{C}} \setminus \mathcal{C}_{\text{nodes}}$. Then,

$$E_i^{\mathcal{C}(0);\gamma} = \underbrace{b_i^{\mathcal{C},\mu,\gamma}}_{=b_i} + \sum_{j\in\mathcal{N}} \underbrace{L_{ji}^{\mathcal{C},\mu}}_{=L_{ij}} - \underbrace{\bar{L}_i^{\mathcal{C},\mu}}_{=\bar{L}_i} < k\, \underbrace{\bar{L}_i^{\mathcal{C},\mu}}_{=\bar{L}_i} \iff \qquad E_i^{(0)} = b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i < k\bar{L}_i$$

and hence $i \in \mathcal{F}$. Let $i \in \mathcal{F}^{\mathcal{C}} \cap \mathcal{C}_{\text{nodes}}$. Then,

$$\begin{aligned} E_i^{\mathcal{C}(0);\gamma} = & \underbrace{b_i^{\mathcal{C},\mu,\gamma}}_{=b_i+\gamma\mu J} + \underbrace{\sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu}}_{=\sum_{j\in\mathcal{N}} L_{ji}-\mu V} - \underbrace{\bar{L}_i^{\mathcal{C},\mu}}_{=\bar{L}_i-\mu V} < k \underbrace{\bar{L}_i^{\mathcal{C},\mu}}_{=\bar{L}_i-\mu V} \\ \iff & b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i < k\bar{L}_i \underbrace{-k\mu V - \gamma\mu J}_{=-\mu(kV+\gamma J)} \\ \iff & b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i < k\bar{L}_i \underbrace{-\mu(kV+\gamma J)}_{\leq 0} \leq k\bar{L}_i \end{aligned}$$

and hence $i \in \mathcal{F}$.

To prove the second statement, let $i \in \mathcal{F} \setminus \mathcal{F}^{\mathcal{C}}$. From the arguments used in part 1. it is clear that $i \in \mathcal{C}_{\text{nodes}}$. Furthermore, since $i \notin \mathcal{F}^{\mathcal{C}}$

$$\begin{aligned} E_i^{\mathcal{C}(0);\gamma} = & \underbrace{b_i^{\mathcal{C},\mu,\gamma}}_{=b_i+\gamma\mu J} + \underbrace{\sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu}}_{=\sum_{j\in\mathcal{N}} L_{ji}-\mu V} - \underbrace{\bar{L}_i^{\mathcal{C},\mu}}_{=\bar{L}_i-\mu V} \geq k \underbrace{\bar{L}_i^{\mathcal{C},\mu}}_{=\bar{L}_i-\mu V} \\ \iff & b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i \geq k\bar{L}_i \underbrace{-k\mu V - \gamma\mu J}_{=-\mu(kV+\gamma J)} \\ \iff & b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i \geq k\bar{L}_i \underbrace{-\mu(kV+\gamma J)}_{\leq 0} \end{aligned}$$

Since $i \in \mathcal{F}$, it holds that $E_i^{(0)} = b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i < k\bar{L}_i$. Combining these two inequalities gives $k\bar{L}_i > b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i \geq k\bar{L}_i - \mu(kV+\gamma J)$. For this to hold we need $(kV+\gamma J) > 0$. □

Proof of Theorem 4.7.

1. Assume that $\mathcal{D}(L,b;\mathbb{V}) \cap \mathcal{C}_{\text{nodes}} = \emptyset$. Then by Proposition 4.8 this compression reduces systemic risk, which is a contradiction to it being harmful.

2. Suppose compression is harmful. This means that there exists a node $\nu \in \mathcal{N}$ such that $\nu \in \mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma};\mathbb{V})$ and $\nu \in \mathcal{N} \setminus \mathcal{D}(L,b;\mathbb{V})$, i.e., node ν defaults if compression is done but does not default without compression. This in particular implies that $\nu \in \mathcal{M}^{\mathcal{C}} = \{i \in \mathcal{N} \mid \bar{L}_i^{\mathcal{C},\mu} > 0\}$ and that $E_\nu^{\mathcal{C},\mu;\gamma;*} < k\bar{L}_\nu^{\mathcal{C},\mu} \leq k\bar{L}_\nu \leq E_\nu^*$. Therefore $E_\nu^{\mathcal{C},\mu;\gamma;*} < E_\nu^*$. Then by Proposition 4.9 part 1. this implies that there exists an $i \in \mathcal{C}_{\text{nodes}}$ satisfying

$$\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*} + \bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right) < \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right).$$

Hence, there exists an $i \in \mathcal{C}_{\text{nodes}}$ satisfying

$$\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*} + \bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right) < \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right) \leq 1$$

which implies that $\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*} + \bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right) < 1$ which implies that $i \in \mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V})$.

3. Assume that $\mathbb{V} = \mathbb{V}^{\text{zero}}$. By Proposition 26 we get that this compression reduces systemic risk which is a contradiction to the assumption that it is harmful.

□

Proof of Proposition 4.8. The proof of this statement we consider a fixed point iteration as in Veraart (2020). We consider the three sequences $(E^{(n)})$, $(E^{\mathcal{C}(n);0})$ and $(E^{\mathcal{C}(n);\gamma})$ defined in (33). By Lemma 4.5 we know that $E_i^{\mathcal{C}(0);\gamma} \geq E_i^{\mathcal{C}(0);0} = E_i^{(0)}$ for all $i \in \mathcal{N}$ and for all $\gamma \in [0,1]$. We will prove by induction that if $\{i \in \mathcal{C}_{\text{nodes}} \mid E_i^* < k\bar{L}_i\} = \emptyset$ then

$$E_i^{\mathcal{C}(n);\gamma} \geq E_i^{\mathcal{C}(n);0} = E_i^{(n)} \text{ for all } i \in \mathcal{N} \tag{34}$$

holds for all $n \in \mathbb{N}_0$. Once this has been shown it follows that

$$E_i^{\mathcal{C};\gamma;*} = \lim_{n\to\infty} E_i^{\mathcal{C}(n);\gamma} \geq E_i^{\mathcal{C};0;*} = \lim_{n\to\infty} E_i^{\mathcal{C}(n);0} = \lim_{n\to\infty} E_i^{(n)} = E_i^*$$

for all $i \in \mathcal{N}$ which is the statement of the theorem.

By Lemma B.3 $(E^{(n)})$, $(E^{\mathcal{C}(n);0})$ and $(E^{\mathcal{C}(n);\gamma})$ are non-increasing. This implies that in particular, $E_i^{(n)} \geq \lim_{m\to\infty} E_i^{(m)} = E_i^*$ for all $i \in \mathcal{N}$ and for all $n \in \mathbb{N}_0$ and hence for all $n \in \mathbb{N}_0$ it holds that $E_i^{(n)} \geq E_i^* \geq k\bar{L}_i \quad \forall i \in \mathcal{C}_{\text{nodes}}$ and hence

$$\{i \in \mathcal{C}_{\text{nodes}} \mid E_i^{(n)} < k\bar{L}_i\} = \emptyset. \tag{35}$$

We now start our proof of (34) by induction. Let $n = 0$. Since

$$E_i^{(0)} = b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i, \qquad E_i^{\mathcal{C}(0);0} = \underbrace{b_i^{\mathcal{C},\mu,0}}_{=b_i} + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{\mathcal{C}}, \qquad E_i^{\mathcal{C}(0);\gamma} = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{\mathcal{C}},$$

we are in exactly the same situation as in Lemma 4.5 in which it was shown that indeed $E_i^{\mathcal{C}(0);\gamma} \geq E_i^{\mathcal{C}(0);0} = E_i^{(0)}$ for all $i \in \mathcal{N}$.

Suppose (34) holds for a fixed $n \in \mathbb{N}_0$. We show that it also holds for $n+1$. Then, by the definition of the sequences

$$E_i^{(n+1)} = \Phi_i(E^{(n)}) = b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i,$$

$$E_i^{\mathcal{C}(n+1);0} = \Phi_i^{\mathcal{C};0}(E^{\mathcal{C}(n);0}) = \underbrace{b_i^{\mathcal{C},\mu,0}}_{=b_i} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i^{\mathcal{C},\mu},$$

$$E_i^{\mathcal{C}(n+1);\gamma} = \Phi_i^{\mathcal{C};\gamma}(E^{\mathcal{C}(n);\gamma}) = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i^{\mathcal{C},\mu}.$$

First note that by the monotonicity of $\mathbb{V}$, the definition of $b^{\mathcal{C},\mu,\gamma}$, and the induction hypothesis that $E_i^{\mathcal{C}(n);\gamma} \geq E_i^{\mathcal{C}(n);0}$ for all $i \in \mathcal{N}$, we immediately see that

$$\begin{aligned}E_i^{\mathcal{C}(n+1);\gamma} &= \Phi_i^{\mathcal{C};\gamma}(E^{\mathcal{C}(n);\gamma}) = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i^{\mathcal{C},\mu} \\ &\geq b_i^{\mathcal{C},\mu,0} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i^{\mathcal{C},\mu} = \Phi_i^{\mathcal{C};0}(E^{\mathcal{C}(n);0}) \\ &= E_i^{\mathcal{C}(n+1);0}\end{aligned}$$

holds for all $i \in \mathcal{N}$. Hence, it remains to show that $E_i^{\mathcal{C}(n+1);0} = E_i^{(n+1)}$ for all $i \in \mathcal{N}$.

Let $i \in \mathcal{C}_{\text{nodes}}$. Then,

$$\begin{aligned}E_i^{\mathcal{C}(n+1);0} &= b_i^{\mathcal{C},\mu,0} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \underbrace{\bar{L}_i^{\mathcal{C},\mu}}_{=\bar{L}_i - \mu V} \\ &= b_i^{\mathcal{C},\mu,0} + \sum_{j\in\mathcal{M}^{\mathcal{C}},(j,i)\in\mathcal{C}_{\text{edges}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) \\ &\quad + \sum_{j\in\mathcal{M}^{\mathcal{C}},(j,i)\notin\mathcal{C}_{\text{edges}}} \underbrace{L_{ji}^{\mathcal{C},\mu}}_{=L_{ji}}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i + \mu V.\end{aligned} \tag{36}$$

Note that there exists at most one $j \in \mathcal{M}^{\mathcal{C}}$ with $(j,i) \in \mathcal{C}_{\text{edges}}$. As before we write pred(i) for the predecessor of i on the cycle $\mathcal{C}_{\text{edges}}$, i.e., pred$(i)$ is the index of the node that satisfies $(\text{pred}(i), i) \in \mathcal{C}_{\text{edges}}$.

We distinguish between two cases. First, suppose that $\text{pred}(i) \in \mathcal{M}^{\mathcal{C}}$. Then,

$$\sum_{j\in\mathcal{M}^{\mathcal{C}},(j,i)\in\mathcal{C}_{\text{edges}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) = (L_{\text{pred}(i),i} - \mu V)\mathbb{V}\left(\frac{E_{\text{pred}(i)}^{\mathcal{C}(n);0} + \bar{L}_{\text{pred}(i)}^{\mathcal{C},\mu}}{\bar{L}_{\text{pred}(i)}^{\mathcal{C},\mu}}\right).$$

By the induction hypothesis $E_{\text{pred}(i)}^{\mathcal{C}(n);0} = E_{\text{pred}(i)}^{(n)}$ and by (35) it holds that $E_{\text{pred}(i)}^{(n)} \geq k\bar{L}_{\text{pred}(i)}$ since $\text{pred}(i) \in \mathcal{C}_{\text{nodes}}$. By the definition of $\mathbb{V}$ this implies that

$$\mathbb{V}\left(\frac{E_{\text{pred}(i)}^{\mathcal{C}(n);0} + \bar{L}_{\text{pred}(i)}^{\mathcal{C},\mu}}{\bar{L}_{\text{pred}(i)}^{\mathcal{C},\mu}}\right) = 1 = \mathbb{V}\left(\frac{E_{\text{pred}(i)}^{(n)} + \bar{L}_{\text{pred}(i)}}{\bar{L}_{\text{pred}(i)}}\right).$$

Hence,

$$\begin{aligned}\sum_{j\in\mathcal{M}^{\mathcal{C}},(j,i)\in\mathcal{C}_{\text{edges}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) &= (L_{\text{pred}(i)i} - \mu V) \\ &= L_{\text{pred}(i)i}\mathbb{V}\left(\frac{E_{\text{pred}(i)}^{\mathcal{C}(n);0} + \bar{L}_{\text{pred}(i)}^{\mathcal{C},\mu}}{\bar{L}_{\text{pred}(i)}^{\mathcal{C},\mu}}\right) - \mu V.\end{aligned} \tag{37}$$

Furthermore, since $\text{pred}(i) \notin \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}$ we obtain by Lemma B.2 part 2. that $L_{ji} = 0$ for all

$j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}$ and hence

$$\sum_{j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) = 0. \tag{38}$$

By plugging (37) into (36) we immediately obtain that

$$\begin{aligned}
E_i^{\mathcal{C}(n+1);0} &= b_i^{\mathcal{C},\mu,0} + \sum_{j \in \mathcal{M}^{\mathcal{C}}} L_{ji} \mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i + \mu V - \mu V \\
&= b_i^{\mathcal{C},\mu,0} + \sum_{j \in \mathcal{M}^{\mathcal{C}}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i \qquad \text{(by induction hypothesis)} \\
&= b_i^{\mathcal{C},\mu,0} + \sum_{j \in \mathcal{M}^{\mathcal{C}}, E_j^{(n)} \geq k\bar{L}_j} L_{ji} \underbrace{\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{=1=\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right)} + \sum_{j \in \mathcal{M}^{\mathcal{C}}, E_j^{(n)} < k\bar{L}_j} L_{ji} \underbrace{\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{\overset{(\star)}{=}\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right)} \\
&\quad - \bar{L}_i \\
&= b_i^{\mathcal{C},\mu,0} + \sum_{j \in \mathcal{M}^{\mathcal{C}}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&= \underbrace{b_i^{\mathcal{C},\mu,0}}_{=b_i} + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \underbrace{\sum_{j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right)}_{=0 \text{ by } (38)} - \bar{L}_i = E_i^{(n+1)},
\end{aligned}$$

where $(\star)$ holds because if for an $j \in \mathcal{N}$ it holds that $E_j^{(n)} < k\bar{L}_j$ then $j \in \mathcal{N} \setminus \mathcal{C}_{\text{nodes}}$ since by assumption no defaults occur on the compression cycle. Hence, $\bar{L}_j^{\mathcal{C},\mu} = \bar{L}_j$.

Second, suppose that $\text{pred}(i) \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}$. Then, by Lemma B.2 part 2.

$$\sum_{j \in \mathcal{M}^{\mathcal{C}}, (j,i) \in \mathcal{C}_{\text{edges}}} L_{ji}^{\mathcal{C},\mu} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) = 0. \tag{39}$$

Furthermore, again from Lemma B.2 part 2. and using the assumption that no node on the compression network cycle defaults we get

$$\begin{aligned}
\sum_{j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) &= L_{\text{pred}(i)i} \underbrace{\mathbb{V}\left(\frac{E_{\text{pred}(i)}^{(n)} + \bar{L}_{\text{pred}(i)}}{\bar{L}_{\text{pred}(i)}}\right)}_{=1} \\
&= L_{\text{pred}(i)i} = \mu V,
\end{aligned} \tag{40}$$

where we used the fact that $\text{pred}(i) \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}$.

By plugging (39) into (36) we obtain

$$E_i^{\mathcal{C}(n+1);0} = \underbrace{b_i^{\mathcal{C},\mu,0}}_{=b_i} + \underbrace{\sum_{j \in \mathcal{M}^{\mathcal{C}}, (j,i) \in \mathcal{C}_{\text{edges}}} L_{ji}^{\mathcal{C},\mu} \mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{=0}$$

$$+ \sum_{j\in\mathcal{M}^{\mathcal{C}},(j,i)\notin\mathcal{C}_{\text{edges}}} L_{ji}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i + \mu V$$

$$= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i + \mu V \quad \text{(since pred}(i)\in\mathcal{M}\setminus\mathcal{M}^{\mathcal{C}})$$

$$= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i + \mu V \quad \text{(by induction hypothesis)}$$

$$= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}},E_j^{(n)}\geq k\bar{L}_j} L_{ji}\underbrace{\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{=1=\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)}$$

$$+ \sum_{j\in\mathcal{M}^{\mathcal{C}},E_j^{(n)}<k\bar{L}_j} L_{ji}\underbrace{\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{\overset{=}{(\star)}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)} - \bar{L}_i + \mu V$$

$$= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \mu V$$

$$= b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) - \underbrace{\sum_{j\in\mathcal{M}\setminus\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)}_{=\mu V \text{ by (40)}} - \bar{L}_i + \mu V$$

$$= b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i = E_i^{(n+1)},$$

where the same argument was used in $(\star)$ as before, namely that nodes with $E_j^{(n)} < k\bar{L}_j$ cannot be on the compression network cycle.

Let $i \notin \mathcal{C}_{\text{nodes}}$. Then, using the induction hypothesis in the second line we get

$$E_i^{\mathcal{C}(n+1);0} = \underbrace{b_i^{\mathcal{C},\mu,0}}_{=b_i} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} \underbrace{L_{ji}^{\mathcal{C},\mu}}_{=L_{ji}}\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);0}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \underbrace{\bar{L}_i^{\mathcal{C},\mu}}_{=\bar{L}_i}$$

$$= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \bar{L}_i$$

$$= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}},E_j^{(n)}\geq k\bar{L}_j} L_{ji}\underbrace{\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{=1=\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)}$$

$$+ \sum_{j\in\mathcal{M}^{\mathcal{C}},E_j^{(n)}<k\bar{L}_j} L_{ji} \underbrace{\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{\overset{=}{(\star)}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)} - \bar{L}_i$$

$$= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i$$

$$= b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) - \underbrace{\sum_{j\in\mathcal{M}\setminus\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)}_{=0 \text{(by Lemma B.2 part 2.)}} - \bar{L}_i = E_i^{(n+1)}$$

using again the fact in $(\star)$ that nodes with $E_j^{(n)} < k\bar{L}_j$ cannot be on the compression network cycle.

Hence, we have shown that indeed for all $n \in \mathbb{N}_0$ and for all $i \in \mathcal{N}$ $E_i^{\mathcal{C}(n+1);\gamma} \geq E_i^{\mathcal{C}(n+1);0} = E_i^{(n+1)}$ which completes the induction. Hence, for all $i \in \mathcal{N}$

$$E_i^{\mathcal{C};\gamma;*} = \lim_{n\to\infty} E_i^{\mathcal{C}(n);\gamma} \geq E_i^{\mathcal{C};0;*} = \lim_{n\to\infty} E_i^{\mathcal{C}(n);0} = \lim_{n\to\infty} E_i^{(n)} = E_i^*.$$

From this it follows immediately that $\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}) \subseteq \mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,0}; \mathbb{V}) \subseteq \mathcal{D}(L, b; \mathbb{V})$. □

Proof of Proposition 4.9. We will prove part 1. first and will show that part 2. and part 3. are essentially corollaries from part 1.

1. Suppose that condition (24) is satisfied. We will prove now that compression can only increase the re-evaluated equity. This proof uses similar arguments as in the proof of Proposition 4.8. Again we consider the sequences $(E^{(n)})$ and $(E^{\mathcal{C}(n);\gamma})$ defined in (33).

   Using the same argument as in the proof of Proposition 4.8 we know from Lemma B.3 that $\lim_{n\to\infty} E_j^{(n)} = E_j^*$ and $\lim_{n\to\infty} E_j^{\mathcal{C}(n);\gamma} = E_j^{\mathcal{C},\mu;\gamma;*}$ exist for all $j \in \mathcal{N}$. Furthermore, $(E^{(n)})$ and $(E^{\mathcal{C}(n);\gamma})$ are decreasing sequences, i.e., they converge to their limits from above. In particular, $E_j^{(n)} \geq E_j^*$ and $E_j^{\mathcal{C}(n);\gamma} \geq E_j^{\mathcal{C},\mu;\gamma;*}$ for all $j \in \mathcal{N}$ and for all $n \in \mathbb{N}_0$ and since $\mathbb{V}$ is non-decreasing this implies that

$$\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) \geq \mathbb{V}\left(\frac{E_j^{\mathcal{C},\mu;\gamma;*}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) \quad \forall j \in \mathcal{N}.$$

   Combining this with (24) we obtain that for all $n \in \mathbb{N}_0$

$$\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) \geq \mathbb{V}\left(\frac{E_j^{\mathcal{C},\mu;\gamma;*}+\bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) \geq \mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) \quad \forall j \in \mathcal{C}_{\text{nodes}}. \tag{41}$$

   We will prove by induction that for all $n \in \mathbb{N}_0$

$$E_i^{\mathcal{C}(n);\gamma} \geq E_i^* \qquad \forall i \in \mathcal{N}. \tag{42}$$

   Once this has been shown it follows that $E_i^{\mathcal{C},\mu;\gamma;*} = \lim_{n\to\infty} E_i^{\mathcal{C}(n);\gamma} \geq E_i^* \quad \forall i \in \mathcal{N}$, which is the statement of the proposition.

For the start of the induction we consider $n = 0$. By Lemma 4.5 we know that $E_i^{\mathcal{C}(0);\gamma} \geq E_i^{(0)}$ for all $i \in \mathcal{N}$ and for all $\gamma \in [0,1]$. Since $(E^{(n)})$ is converging to E^* from above this implies that $E_i^{\mathcal{C}(0);\gamma} \geq E_i^{(0)} \geq E_i^*$ for all $i \in \mathcal{N}$.

Next assume that the statement (42) holds for an $n \in \mathbb{N}_0$. We will show that it holds for $n+1$. We distinguish between two cases: Let $i \in \mathcal{N} \setminus \mathcal{C}_{\text{nodes}}$. Then,

$$\begin{aligned}
E_i^{\mathcal{C}(n+1);\gamma} &= \Phi^{\mathcal{C}}(E^{\mathcal{C}(n);\gamma})_i = \underbrace{b_i^{\mathcal{C},\mu,\gamma}}_{=b_i} + \sum_{j \in \mathcal{M}^{\mathcal{C}}} \underbrace{L_{ji}^{\mathcal{C},\mu}}_{=L_{ji}} \mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \underbrace{\bar{L}_i^{\mathcal{C},\mu}}_{=\bar{L}_i} \\
&= b_i + \sum_{j \in \mathcal{M}^{\mathcal{C}} \cap \mathcal{C}_{\text{nodes}}} L_{ji} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{\geq \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) \quad \text{by (41)}} + \sum_{j \in \mathcal{M}^{\mathcal{C}} \setminus \mathcal{C}_{\text{nodes}}} L_{ji} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{=\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j}{\bar{L}_j}\right)} - \bar{L}_i \\
&\geq b_i + \sum_{j \in \mathcal{M}^{\mathcal{C}} \cap \mathcal{C}_{\text{nodes}}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) + \sum_{j \in \mathcal{M}^{\mathcal{C}} \setminus \mathcal{C}_{\text{nodes}}} L_{ji} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j}{\bar{L}_j}\right)}_{\geq \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) \quad \text{by ind. hyp. \& } \mathbb{V} \text{monotone}} - \bar{L}_i \\
&\geq b_i + \sum_{j \in \mathcal{M}^{\mathcal{C}}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&= \underbrace{b_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i}_{=E_i^*} - \sum_{j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) \\
&= E_i^* - \underbrace{\sum_{j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right)}_{=0} = E_i^*.
\end{aligned}$$

Note that $\sum_{j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) = 0$ since $i \in \mathcal{N} \setminus \mathcal{C}_{\text{nodes}}$ and by Lemma B.2 $L_{ji} = 0$ for all $j \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}$.

Let $i \in \mathcal{C}_{\text{nodes}}$. Then,

$$\begin{aligned}
E_i^{\mathcal{C}(n+1);\gamma} &= \Phi^{\mathcal{C}}(E^{\mathcal{C}(n);\gamma})_i = \underbrace{b_i^{\mathcal{C},\mu,\gamma}}_{\geq b_i} + \sum_{j \in \mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu} \mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right) - \underbrace{\bar{L}_i^{\mathcal{C},\mu}}_{=\bar{L}_i - \mu V} \\
&\geq b_i + \sum_{j \in \mathcal{M}^{\mathcal{C}} \cap \mathcal{C}_{\text{nodes}}} L_{ji}^{\mathcal{C},\mu} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{\geq \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) \quad \text{by (41)}} \\
&\quad + \sum_{j \in \mathcal{M}^{\mathcal{C}} \setminus \mathcal{C}_{\text{nodes}}} L_{ji}^{\mathcal{C},\mu} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{=\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j}{\bar{L}_j}\right) \quad \text{since } j \notin \mathcal{C}_{\text{nodes}}} - \bar{L}_i + \mu V
\end{aligned}$$

$$
\begin{aligned}
&\geq b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}\cap\mathcal{C}_{\text{nodes}}} L_{ji}^{\mathcal{C},\mu}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) \\
&\quad + \sum_{j\in\mathcal{M}^{\mathcal{C}}\setminus\mathcal{C}_{\text{nodes}}} \underbrace{L_{ji}^{\mathcal{C},\mu}}_{=L_{ji}} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j}{\bar{L}_j}\right)}_{\geq\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right)\quad\text{by ind. hyp. and} \mathbb{V}\text{ nondecreasing}} - \bar{L}_i + \mu V \\
&\geq b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}\cap\mathcal{C}_{\text{nodes}}\setminus\{\text{pred}(i)\}} \underbrace{L_{ji}^{\mathcal{C},\mu}}_{L_{ji}}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) \\
&\quad + (L_{\text{pred}(i)i} - \mu V)\mathbb{V}\left(\frac{E_{\text{pred}(i)}^* + \bar{L}_{\text{pred}(i)}}{\bar{L}_{\text{pred}(i)}}\right)\mathbb{I}_{\{\text{pred}(i)\in\mathcal{M}^{\mathcal{C}}\}} \\
&\quad + \sum_{j\in\mathcal{M}^{\mathcal{C}}\setminus\mathcal{C}_{\text{nodes}}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \mu V \\
&=: (\star\star)
\end{aligned}
$$

Let $\text{pred}(i) \in \mathcal{M}^{\mathcal{C}}$. Then,

$$
\begin{aligned}
(\star\star) &= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \mu V\left(1 - \mathbb{V}\left(\frac{E_{\text{pred}(i)}^* + \bar{L}_{\text{pred}(i)}}{\bar{L}_{\text{pred}(i)}}\right)\right) \\
&= \underbrace{b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i}_{E_i^*} + \underbrace{\mu V\left(1 - \mathbb{V}\left(\frac{E_{\text{pred}(i)}^* + \bar{L}_{\text{pred}(i)}}{\bar{L}_{\text{pred}(i)}}\right)\right)}_{\geq 0} \\
&\quad - \underbrace{\sum_{j\in\mathcal{M}\setminus\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right)}_{=0\text{ since pred}(i)\in\mathcal{M}^{\mathcal{C}}} \geq E_i^*.
\end{aligned}
$$

Let $\text{pred}(i) \in \mathcal{M} \setminus \mathcal{M}^{\mathcal{C}}$. Then,

$$
\begin{aligned}
(\star\star) &= b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \mu V \\
&= \underbrace{b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i}_{E_i^*} - \underbrace{\sum_{j\in\mathcal{M}\setminus\mathcal{M}^{\mathcal{C}}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right)}_{=\mu V\mathbb{V}\left(\frac{E_{\text{pred}(i)}^* + \bar{L}_{\text{pred}(i)}}{\bar{L}_{\text{pred}(i)}}\right)} + \mu V \\
&= E_i^* + \underbrace{\mu V\left(1 - \mathbb{V}\left(\frac{E_{\text{pred}(i)}^* + \bar{L}_{\text{pred}(i)}}{\bar{L}_{\text{pred}(i)}}\right)\right)}_{\geq 0} \geq E_i^*.
\end{aligned}
$$

Hence, this completes the induction and the result follows.

2. Suppose that (25) holds. Then by the definition of $\mathbb{V}$ we immediately get that

$$1 = \mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*} + \bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right) \geq \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right) \quad \forall i \in \mathcal{C}_{\text{nodes}}.$$

Hence, the result follows with part 1. of this Proposition.

3. Suppose that (26) holds, then $\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*}+\bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right) = 1 \quad \forall i \in \mathcal{C}_{\text{nodes}}$ and hence the statement follows directly from part 2. of this Proposition since condition (25) is satisfied.

□

Proof of Proposition 4.10. Let $i \in \mathcal{C}_{\text{nodes}}$. Hence, $\bar{L}_i > 0$ and $\bar{L}_i^{\mathcal{C},\mu} = \bar{L}_i - \mu$. Let $j = \text{suc}(i) \in \mathcal{N}$ and first suppose that $\bar{L}_i^{\mathcal{C},\mu} > 0$. Then,

$$\Pi_{ij}^{\mathcal{C},\mu} = \Pi_{i\text{suc}(i)}^{\mathcal{C},\mu} = \frac{L_{i\text{suc}(i)}^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}} = \frac{\text{Ł}_{i\text{suc}(i)} - \mu}{\bar{L}_i - \mu} \leq \frac{L_{i\text{suc}(i)}}{\bar{L}_i} = \Pi_{i\text{suc}(i)},$$

since

$$\frac{\text{Ł}_{i\text{suc}(i)} - \mu}{\bar{L}_i - \mu} \leq \frac{L_{i\text{suc}(i)}}{\bar{L}_i} \quad \Leftrightarrow \text{Ł}_{i\text{suc}(i)}\bar{L}_i - \mu\bar{L}_i \quad \leq L_{i\text{suc}(i)}\bar{L}_i - L_{i\text{suc}(i)}\mu \Leftrightarrow \quad 0 \leq \mu(\bar{L}_i - L_{i\text{suc}(i)})$$

is always satisfied. Second, suppose that $\bar{L}_i^{\mathcal{C},\mu} = 0$. Then, $\Pi_{i\text{suc}(i)}^{\mathcal{C},\mu} = 0 \leq \frac{L_{i\text{suc}(i)}}{\bar{L}_i} = \Pi_{i\text{suc}(i)}$. Now let $j \in \mathcal{N} \setminus \{\text{suc}(i)\}$. Then, $\Pi_{ij}^{\mathcal{C},\mu} = \frac{L_{ij}^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}} = \frac{L_{ij}}{\bar{L}_i} = \Pi_{ij}$.

Let $i \in \mathcal{N} \setminus \mathcal{C}_{\text{nodes}}$ and $j \in \mathcal{N}$. Then, $\bar{L}_i^{\mathcal{C},\mu} = \bar{L}_i$. If $\bar{L}_i > 0$, then $\bar{L}_i^{\mathcal{C},\mu} = \bar{L}_i > 0$ and $\Pi_{ij}^{\mathcal{C},\mu} = \frac{L_{ij}^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}} = \frac{L_{ij}}{\bar{L}_i} = \Pi_{ij}$; and if $\bar{L}_i = 0$, then $\bar{L}_i^{\mathcal{C},\mu} = \bar{L}_i = 0$ and $\Pi_{ij}^{\mathcal{C},\mu} = 0 = \Pi_{ij}$. □

We will use the following Lemma to prove Theorem 4.11.

Lemma B.5. *Let* $E_i^{\mathcal{C}(n);0}, E_i^{(n)}, \bar{L}_i^{\mathcal{C},\mu}, \bar{L}_i \in \mathbb{R}$, $E_i^{\mathcal{C}(n);0} \geq E_i^{(n)}$, $\bar{L}_i^{\mathcal{C},\mu} \leq \bar{L}_i$ *and* $k \geq 0$. *Then,*

$$\mathbb{V}^{zero}\left(\frac{E_i^{\mathcal{C}(n);0} + \bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right) = \mathbb{I}_{\{E_i^{\mathcal{C}(n);0} \geq k\bar{L}_i^{\mathcal{C},\mu}\}} \geq \mathbb{I}_{\{E_i^{(n)} \geq k\bar{L}_i\}} = \mathbb{V}^{zero}\left(\frac{E_i^{(n)} + \bar{L}_i}{\bar{L}_i}\right). \tag{43}$$

Proof of Lemma B.5. Note that $k\bar{L}_i \geq k\bar{L}_i^{\mathcal{C},\mu}$. Suppose $\mathbb{I}_{\{E_i^{\mathcal{C}(n);0} \geq k\bar{L}_i^{\mathcal{C},\mu}\}} = 1$. Then, $1 \geq \mathbb{I}_{\{E_i^{(n)} \geq k\bar{L}_i\}}$. Suppose $\mathbb{I}_{\{E_i^{\mathcal{C}(n);0} \geq k\bar{L}_i^{\mathcal{C},\mu}\}} = 0$. Then, $E_i^{(n)} \leq E_i^{\mathcal{C}(n);0} < k\bar{L}_i^{\mathcal{C},\mu} \leq k\bar{L}_i$ and therefore $\mathbb{I}_{\{E_i^{\mathcal{C}(n);0} \geq k\bar{L}_i^{\mathcal{C},\mu}\}} = 0 = \mathbb{I}_{\{E_i^{(n)} \geq k\bar{L}_i\}}$. □

Proof of Proposition 4.11. We proceed similarly as in the proof of Proposition 4.8. We consider two sequences $(E^{(n)})$ and $(E^{\mathcal{C}(n);\gamma})$ defined in (33) but now assume that $\mathbb{V} = \mathbb{V}^{\text{zero}}$.

We will prove by induction that

$$E_i^{\mathcal{C}(n);\gamma} \geq E_i^{(n)} \text{ for all } i \in \mathcal{N} \tag{44}$$

holds for all $n \in \mathbb{N}_0$. Once this has been shown it follows that $E_i^{\mathcal{C},\mu;\gamma;*} = \lim_{n\to\infty} E_i^{\mathcal{C}(n);\gamma} \geq \lim_{n\to\infty} E_i^{(n)} = E_i^*$ for all $i \in \mathcal{N}$ which is the statement of the theorem.

Let $n = 0$. Then the result follows directly from Lemma 4.5.

We now assume that (44) holds true for an $n \in \mathbb{N}$. We show that (44) is true for $n+1$. Consider

$$E_i^{(n+1)} = \Phi_i(E^{(n)}) = b_i + \sum_{j\in\mathcal{M}} L_{ji} \mathbb{V}^{\text{zero}} \left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j} \right) - \bar{L}_i,$$

$$E_i^{\mathcal{C}(n+1);\gamma} = \Phi_i^{\mathcal{C};\gamma}(E^{\mathcal{C}(n);\gamma}) = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu} \mathbb{V}^{\text{zero}} \left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}} \right) - \bar{L}_i^{\mathcal{C},\mu}.$$

By the induction hypothesis (44) $E_i^{\mathcal{C}(n);\gamma} \geq E_i^{(n)}$ for all $i \in \mathcal{N}$ and hence

$$\mathbb{V}^{\text{zero}} \left(\frac{E_i^{\mathcal{C}(n);\gamma} + \bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}} \right) = \mathbb{I}_{\{E_i^{\mathcal{C}(n);\gamma} \geq k\bar{L}_i^{\mathcal{C},\mu}\}} \geq \mathbb{I}_{\{E_i^{(n)} \geq k\bar{L}_i\}} = \mathbb{V}^{\text{zero}} \left(\frac{E_i^{(n)} + \bar{L}_i}{\bar{L}_i} \right)$$

by Lemma (B.5). Hence,

$$\begin{aligned} E_i^{\mathcal{C}(n+1);\gamma} &= b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{M}^{\mathcal{C}}} L_{ji}^{\mathcal{C},\mu} \mathbb{V}^{\text{zero}} \left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}} \right) - \bar{L}_i^{\mathcal{C},\mu} \\ &= \underbrace{b_i^{\mathcal{C},\mu,\gamma}}_{\geq b_i} + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu} \mathbb{I}_{\{E_j^{\mathcal{C}(n);\gamma} \geq k\bar{L}_j^{\mathcal{C},\mu}\}} - \bar{L}_i^{\mathcal{C},\mu} \\ &\geq b_i + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu} \mathbb{I}_{\{E_j^{(n)} \geq k\bar{L}_j\}} - \bar{L}_i^{\mathcal{C},\mu} = (*), \end{aligned}$$

where we used (43) to derive the inequality. If $i \notin \mathcal{C}_{\text{nodes}}$, then

$$(*) = b_i + \sum_{j\in\mathcal{N}} L_{ji} \mathbb{I}_{\{E_j^{(n)} \geq k\bar{L}_j\}} - \bar{L}_i \qquad = b_i + \sum_{j\in\mathcal{M}} L_{ji} \mathbb{V}^{\text{zero}} \left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j} \right) - \bar{L}_i = E_i^{(n+1)}.$$

If $i \in \mathcal{C}_{\text{nodes}}$, then

$$\begin{aligned} (*) &= b_i + \sum_{j\in\mathcal{N}\setminus\{\text{pred}(i)\}} L_{ji} \mathbb{I}_{\{E_j^{(n)} \geq k\bar{L}_j\}} + (L_{\text{pred}(i)i} - \mu V) \mathbb{I}_{\{E_{\text{pred}(i)}^{(n)} \geq k\bar{L}_{\text{pred}(i)}\}} - (\bar{L}_i - \mu V) \\ &= b_i + \sum_{j\in\mathcal{N}} L_{ji} \mathbb{I}_{\{E_j^{(n)} \geq k\bar{L}_j\}} - \bar{L}_i + \underbrace{\mu V (1 - \mathbb{I}_{\{E_{\text{pred}(i)}^{(n)} \geq k\bar{L}_{\text{pred}(i)}\}})}_{\geq 0} \\ &\geq b_i + \sum_{j\in\mathcal{N}} L_{ji} \mathbb{I}_{\{E_j^{(n)} \geq k\bar{L}_j\}} - \bar{L}_i = b_i + \sum_{j\in\mathcal{M}} L_{ji} \mathbb{V}^{\text{zero}} \left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j} \right) - \bar{L}_i = E_i^{(n+1)}. \end{aligned}$$

Hence, $E_i^{\mathcal{C}(n+1)} \geq E_i^{(n+1)}$ for all $i \in \mathcal{N}$ which completes the induction.

To see that indeed systemic risk is reduced by compression here, we use the results of the first part, namely $E_i^{\mathcal{C},\mu;\gamma;*} \geq E_i^*$ for all $i \in \mathcal{N}$. Suppose $\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}^{\text{zero}}) \neq \emptyset$ otherwise there is nothing to show. Let $i \in \mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}^{\text{zero}})$. Then, $E_i^{\mathcal{C},\mu;\gamma;*} < k\bar{L}_i^{\mathcal{C},\mu}$ and hence $E_i^* \leq E_i^{\mathcal{C},\mu;\gamma;*} < k\bar{L}_i^{\mathcal{C},\mu} \leq k\bar{L}_i$. This implies that $i \in \mathcal{D}(L, b; \mathbb{V}^{\text{zero}})$. Hence,

$$\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}^{\text{zero}}) = \{i \in \mathcal{N} \mid E_i^{\mathcal{C},\mu;\gamma;*} < k\bar{L}_i^{\mathcal{C},\mu}\} \subseteq \{i \in \mathcal{N} \mid E_i^* < k\bar{L}_i\} = \mathcal{D}(L, b; \mathbb{V}^{\text{zero}}).$$

□

Proof of Proposition 4.12. Suppose condition 1. i.e., formula (27) is satisfied, i.e., $\mathcal{D}(L, b; \mathbb{V}) \cap$

$\mathcal{C}_{\text{nodes}}^{\text{all}} = \emptyset$. Then, in particular $\mathcal{D}(L, b; \mathbb{V}) \cap \mathcal{C}_{\text{nodes}}^{(1)} = \emptyset$. Then, by Proposition 4.8, compressing $\mathcal{C}^{(1)}$ reduces systemic risk. In particular, $\mathcal{D}(L^{\mathcal{C}^{(1)}}, b^{\mathcal{C}^{(1)}}; \mathbb{V}) \subseteq \mathcal{D}(L, b; \mathbb{V})$. Combining this results with (27) implies that $\mathcal{D}(L^{\mathcal{C}^{(1)}}, b^{\mathcal{C}^{(1)}}; \mathbb{V}) \cap \mathcal{C}_{\text{nodes}}^{(2)} = \emptyset$. Then, applying Proposition 4.8 to the system $\mathcal{D}(L^{\mathcal{C}^{(1)}}, b^{\mathcal{C}^{(1)}}; \mathbb{V})$ by compressing cycle $\mathcal{C}^{(2)}$ yields $\mathcal{D}(L^{\mathcal{C}^{(1)},\mathcal{C}^{(2)}}, b^{\mathcal{C}^{(1)},\mathcal{C}^{(2)}}; \mathbb{V}) \subseteq \mathcal{D}(L^{\mathcal{C}^{(1)}}, b^{\mathcal{C}^{(1)}}; \mathbb{V})$. By repeating these arguments, we obtain that

$$\begin{aligned}\mathcal{D}(L^{\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(m)}}, b^{\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(m)}}; \mathbb{V}) &\subseteq \mathcal{D}(L^{\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(m-1)}}, b^{\mathcal{C}^{(1)},\ldots,\mathcal{C}^{(m-1)}}; \mathbb{V}) \\ &\subseteq \ldots \subseteq \mathcal{D}(L^{\mathcal{C}^{(1)}}, b^{\mathcal{C}^{(1)}}; \mathbb{V}) \subseteq \mathcal{D}(L, b; \mathbb{V})\end{aligned}$$

and hence indeed compressing sequentially $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(m)}$ reduces systemic risk.

Suppose the second condition, i.e., (28) holds, then Proposition 4.9 yields the statement. If the third condition, i.e., $\mathbb{V} = \mathbb{V}^{\text{zero}}$ holds, the statement follows from Proposition 4.11.

□

Proof of Corollary 4.13.

1. This statement and its proof is given in (D'Errico & Roukny, 2019, Section 12)).
2. This statement follows directly from Proposition 4.12 by using the sequence of cycles to obtain $\tilde{X}$ that is guaranteed to exist from part 1. of this Corollary.
3. The algorithm developed in (D'Errico & Roukny, 2019, Section 12)) to determine the sequence of cycles $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(m)}$ can still be used if a lower bound $a_{ij} \geq 0$ is introduced. The results derived in Proposition 4.12 hold for all possible compression volumes and not just for the original $\mu_i^{\max}$, $i \in \{1, \ldots, m\}$. Hence, in line with Remark 2.2 the results remain valid for the case of a lower bound that is not necessarily 0.

□

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