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Working Paper 13-2

Generating Historically-Based Stress Scenarios Using Parsimonious Factorization

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Working Paper 13-2—"Generating Historically-Based Stress Scenarios Using Parsimonious Factorization"

Executive Summary

We describe a robust empirical approach to generating plausible historically-based interest rate shocks, which can be applied to *any* market environment. These interest rate shocks can be readily linked to movements in other key risk factors, and used to measure market risk on institutions with large fixed-income portfolios.

Our approach is based upon yield curve parameterization and requires a parsimonious yet flexible factorization model. In the process of selecting a model, we evaluate three variants of the Nelson-Siegel approach to yield curve approximation and find that, in the current low interest rate environment, a 5-factor parameterization developed by Bjork and Christensen (1999) is best suited for accurately translating historical interest rate movements into plausible, current period shocks.

Using the Bjork-Christensen model, we parameterize a time series of historical yield curves and measure interest rate shocks as the historical change in each of the model's factors. We then demonstrate how to add these parameterized shocks to any market environment, while retaining positive rates and plausible credit spreads.

By reducing the dimensionality of the term structure of rates, yield curve parameterization also allows us to effectively model historical, reduced form relationships between rates and other key risk factors through regression analysis. These regression results can be used to estimate plausible joint risk factor movements to accompany each set of stressed rates and spreads. While many additional risk factors can be modeled in this manner, for the sake of brevity we focus on producing plausible co-movements in implied volatility.

Generating Historically-Based Stress Scenarios Using Parsimonious Factorization

Abstract

We describe an empirical approach to generating plausible historically-based interest rate shocks, which can be applied to *any* market environment and readily linked to movements in other key risk factors. Our approach is based upon yield curve parameterization and requires a parsimonious yet flexible factorization model. In the process of selecting a model, we evaluate three variants of the Nelson-Siegel approach to yield curve approximation and find that, in the current low interest rate environment, a 5-factor parameterization developed by Bjork and Christensen (1999) is best suited for accurately translating historical interest rate movements into plausible, current period shocks. Using the Bjork-Christensen model, we parameterize a time series of historical yield curves and measure interest rate shocks as the historical change in each of the model's factors. We then demonstrate how to add these parameterized shocks to any market environment, while retaining positive rates and plausible credit spreads. Given a set of shocked interest rate curves, joint risk factor movements are calculated based upon historical, reduced form dependencies.

Generating Historically-Based Interest Rate Scenarios Using Parsimonious Factorization

1. Introduction

The financial crisis last decade demonstrated that many institutions were not prepared adequately for sudden and large market declines. Even today, much improvement remains for risk management techniques. This paper focuses on one key area of market risk: interest rate shocks. We offer a robust empirical method to generate plausible historically-based interest rate shocks which, when combined with changes in other key risk factors, can be used to measure market risk on financial institutions with large fixed-income portfolios[1].

Currently, interest rate shocks are often generated as historically observed term point specific changes to portfolio relevant interest rate curves. These changes are applied to the current market environment as proportional or absolute shocks. Both approaches have concerning but avoidable flaws.

Proportional shocks are multiplicative and positively covary with the current level of rates. This creates a problem if models are calibrated on a historical period like between 5/31/1999 and 11/30/1999. The 1-month Libor-Swap rate increased from 4.94 to 6.48 percent, giving a proportional increase of 31 percent and an absolute shock of 154 basis points (bps). Applying a 31 percent rate increase to the current 1-month Libor-Swap rate (measured on 9/28/2012) yields a proportional shock of only 7 bps. As this example illustrates, when applied in a low rate environment, proportional shocks often generate interest rate scenarios which imply limited exposure to market movements and are not appropriate for risk measurement.

Absolute shocks are attractive because they are invariant to the current level of rates. When drawn from a period of market tumult, they can produce a wide range of interest rate scenarios regardless of the present-day rate environment. While absolute term point shocks are transparent and easy to calculate, they can lead to interest rate scenarios characterized by multiple “kink” points and implausible credit spreads[2]. Furthermore, these unsmoothed, often disjointed, curves are difficult to link or associate with other key risk factors like implied volatility and housing price appreciation[3].

Fortunately, there is an extensive body of research that offers a tractable solution to simplify yield curve modeling through parsimonious factorization. Exploiting this research, we parameterize a time series of historical yield curves using non-linear, Laguerre based functions of time to maturity. We then measure interest rate shocks as the historical change in each model parameter. These historical parametric shocks are then added to the present-day, parameterized yield curve, and re-converted back into a series of term point specific yields[4]. This approach is similar to the one used by Christensen et al. (2013) who parameterize a time series of historical Treasury curves using a shadow-rate arbitrage-free Nelson Siegel model, which contains a zero lower bound (ZLB) condition. In this respect, their methodology is very similar to Diebold et al. (2008), Loretan (1997), and Rodrigues (1997). Both Diebold et al. (2008) and Christensen et al. (2013) focus on generating shocks to a single interest rate curve and, as a result, there is no focus on inter-curve constraints or linking these interest rate shocks to co-movements in other key market risk factors. Loretan (1997) and Rodrigues (1997) employ principal component analysis to generate shocks to several risk factors, but there is no method to ensure these shocks are plausible when applied in concert[5].

We improve upon the aforementioned problems in several ways. While the yield curves generated using the Christensen et al. (2013) and Diebold et al. (2008) techniques are smooth, they may exhibit negative rates or implausible inter-curve relations[6]. We re-parameterize our initial shock scenarios subject to the constraint of positive rates and plausible credit spreads, in effect identifying the yield curves closest to the initial shock scenarios subject to the aforementioned restrictions.

In sum, yield curve parameterization can generate interest rate scenarios that are realistic and wide ranging. The resulting shock scenarios preserve positive rates and plausible credit spreads without oversimplifying salient characteristics of the curves whose dynamics we wish to capture. Reducing the dimensionality of the term structure of rates also allows us to effectively model historical, reduced form dependencies between rates and other key risk factors, which can be used to generate joint risk factor movements. While additional risk factors can be modeled in this manner, we will focus on producing plausible co-movements in implied volatility. We believe that this paper offers three major innovations: (1) a means to impose intra- and inter-yield curve constraints to ensure plausible Treasury, Agency, and Libor-Swap interest rate movements; (2) a method to link interest rate changes to co-movements in other key market risk factors; and (3) a novel parameterization of the implied volatility surface.

This paper is organized as follows. Section 2 describes several approaches to yield curve parameterization. We examine an easily interpretable 3-factor model whose linear factors correspond to level, slope, and curvature before exploring 4- and 5-factor models, which are less intuitive but offer a greater degree of flexibility. Section 3 discusses the technical details of generating historically based rate shocks using yield curve parameterization. Section 4 describes how to link these interest-rate shocks with associated movements in implied volatility. Section 5 summarizes and concludes.

2. Yield Curve Parameterization

Following Diebold et al. (2008), we explore three variants of the Nelson-Siegel approach to yield curve approximation, each characterized by a differing level of flexibility. Beginning with Durand in 1942, both market participants and academics have worked to identify the principal components or common influences underlying US Treasury yields. Nelson and Siegel (1987) were first to suggest modeling the yield curve using Laguerre functions, a mathematical class of approximating functions consisting of a polynomial multiplied by an exponential decay factor. Using this class of functions, Nelson and Siegel proposed a parsimonious 3-factor forward rate model:

$$f(\mathrm{t}) = \beta_1 + \beta_2 e^{-\lambda t} + \beta_3 \lambda e^{-\lambda t} \tag{1}$$

where λ is an exponential decay factor governing the maturity-dependence of β_2 and β_3. The parameters represent a long term component, β_1, a short term component, β_2, and a medium term component, β_3. This forward rate model can be re-expressed in terms of yield to maturity[7]:

$$y(\mathrm{t}) = \frac{1}{\mathrm{t}}\int_0^{\mathrm{t}} f(u)du = \beta_1 + \beta_2\left(\frac{1-e^{-\lambda t}}{\lambda t}\right) + \beta_3\left(\frac{1-e^{-\lambda t}}{\lambda t} - e^{-\lambda t}\right) \tag{2}$$

< - - - INSERT FIGURE 1 ABOUT HERE - - - >

In 2006, Diebold and Li re-interpreted equation (2), noting that the model's three linear factors are closely related to level, slope, and curvature. As illustrated in Figure (1), the loading attached to β_1 is constant across maturities. This first linear factor equally impacts all term points and can be interpreted as a measure of yield curve level. In contrast, the loading attached to β_2 monotonically decreases with time to maturity. This second linear factor largely impacts short term yields and can be interpreted as a measure of slope. Diebold and Li quantify this association by using equation (2) to solve for an empirical measure of slope in terms of β_1, β_2, and β_3: $y(10\mathrm{yr}) - \mathrm{y}(3\mathrm{m}) = -\mathbf{0.78}\beta_2 + 0.06\beta_3$[8]. As shown, β_2 explains the majority of variation in yield curve slope. The loading attached to the final factor, β_3, increases across the short end of the yield curve, reaches a maximum at t=30m, and then declines with time to maturity. In contrast to the first two linear factors, β_3 largely impacts medium term yields and can be interpreted as a measure of curvature. Diebold and Li again quantify this association, using equation (2) to solve for an empirical measure of curvature: $2 * \mathrm{y}(2\mathrm{yr}) - \mathrm{y}(10\mathrm{yr}) - \mathrm{y}(3\mathrm{m}) = 0.0005\beta_2 + \mathbf{0.37}\beta_3$[9]. This time, β_3 explains the majority of variation in yield curve curvature.

Svensson (1994) added a fourth term to Nelson-Siegel's forward rate model that allows for a second hump shape or curvature term. This 4-factor variant accounts for diverging near and long term monetary policy expectations in Sweden, and permits multiple kinks in the forward rate curve. As documented in Diebold et al. (2008), Svensson's model can be re-expressed as:

$$y(\mathrm{t}) = \beta_1 + \beta_2 \left(\frac{1-e^{-\lambda_1 t}}{\lambda_1 t}\right) + \beta_3 \left(\frac{1-e^{-\lambda_1 t}}{\lambda_1 t} - e^{-\lambda_1 t}\right) + \beta_4 \left(\frac{1-e^{-\lambda_2 t}}{\lambda_2 t} - e^{-\lambda_2 t}\right) \quad (3)$$

The Svensson model is attractive because it retains much of the interpretability of the original Nelson-Siegel model, while adding an additional parameter to improve overall model fit under atypical term structures. β_1 and β_2 can again be interpreted as measures of level and slope, and β_3 and β_4 can be interpreted as measures of near and long term curvature.

While useful in describing historical forward rate dynamics, the Nelson-Siegel parameterization is inconsistent with a number of standard and commonly used interest rate processes including Ho-Lee and Hull-White. To make the Nelson-Siegel approach consistent with these standard interest rate processes, Björk and Christensen (1999) proposed a 5-factor exponential-polynomial generalization:

$$y(\mathrm{t}) = \beta_1 + \beta_2 \left(\frac{t}{2}\right) + \beta_3 \left(\frac{1-e^{-\lambda t}}{\lambda t}\right) + \beta_4 \left(\frac{1-e^{-\lambda t}}{\lambda^2 t} - \frac{e^{-\lambda t}}{\lambda}\right) + \beta_5 \left(\frac{1-e^{-2\lambda t}}{2\lambda t}\right) \quad (4)$$

The parameters attached to the Björk-Christensen model consist of a level factor, β_1, and four shape factors, β_2 to β_5. Although the shape factors are not as easily interpretable as in the Nelson-Siegel and Svensson models, the addition of a fifth parameter greatly increases model flexibility.

In the following subsections, we assess which of the aforementioned variants of the Nelson-Siegel parameterization is most suitable for generating historically based interest rate shocks. To generate historically-based yield curve shocks, we parameterize a time series of historical yield curves and measure interest rate shocks as the change in each of the estimated parameters over historical periods of market stress. These historical parametric shocks are then added to a parameterized current yield curve, and re-converted back into term point specific yields. We document several problems (negative yield curve rates and implausible intra-curve relations) and discuss our method's solutions.

2.1 Accurate Description of Observed Patterns of Yields

We begin by examining how well each model fits historical realizations of a representative interest rate curve. For purposes of exposition, we focus our attention on the Libor-Swap curve[10].

< - - - INSERT FIGURE 2 ABOUT HERE - - - >

Figure 2 displays the observed yield patterns and model fit of Libor-Swap rates for two different dates. Figure 2(a) details how well Diebold-Li (3-factor), Svensson (4-factor), and Björk-Christensen (5-factor) models describe a standard, largely upward sloping yield curve corresponding to the rate environment on 2/20/1998. As illustrated, each model provides a close approximation to the actual interest curve. The adjusted R^2 is above 0.96 for each model with little variation across statistics. Figure 2(b) illustrates model fit for an S-shaped interest rate curve corresponding to the rate environment on 9/29/2006. The 4- and 5-factor models are both relatively successful in capturing the varied, steeply sloped rate trajectory with adjusted R^2 of 0.843 and 0.863, respectively. Alternatively, the 3-factor model mutes the extent of the initial rate decrease and misstates the point of inflection by over 12 months leading to an adjusted R^2 of 0.691. Mathematically, the 3-factor model is not flexible enough for this market date.

< - - - INSERT FIGURE 3 ABOUT HERE - - - >

The superior model fit of the 4- and 5-factor parameterizations is further evidenced in Figure 3, which illustrates adjusted R^2 values from 1995 to 2012[11]. The percentage of trading days with adjusted R^2 values greater than 0.90 is 82.6 percent for the 3-factor model, 87.6 percent for the 4-factor model, and 88.1 percent for the 5-factor model. The explained variation is impressive given the data are cross-sectional (not a time series). If we require a yield curve model to reliably translate *any* historical shock to *any* as of date, the relative flexibility of the 4- and 5-factor models represents a significant asset.

2.2 Flexibility to Handle Intra-Curve Constraints

In order to generate plausible scenarios, it is often necessary to impose an intra-curve non-negativity constraint. Applying historical down shocks to the current low rate environment can lead to sustained periods of negative yield. For instance, we compared the current rate environment as of 9/28/2012 along with the simulated new rate environment if we were to apply absolute term point shocks drawn from the historical period 7/31/1998 to 1/31/1999. The simulation yields negative rates from 0 to 50 months. To incorporate such a down shock while retaining a plausible term structure, negative rates are floored at zero (or slightly above).

< - - - INSERT FIGURE 4 ABOUT HERE - - - >

Figure 4 illustrates the historical down shocks to the Libor-Swap curve generated using a 3-, 4-, and 5-factor model with a simulated term structure in a low current rate environment (from 9/28/2012) having negative rates floored at zero bps. Among the models, only the 5-factor parameterization accurately represents a sustained period of near zero rates. At the 1-month term point, the 3- and 4-factor models generate an up-shock of approximately 30 bps, which contradicts the historically observed decrease of approximately 75 bps. To calculate an applicable fit statistic, we regress the floored shocked rates, depicted as a dashed red line, on the loading factors implied by each yield curve model. The adjusted R^2 statistics for the 3-, 4-, and 5-factor model are 0.778, 0.824, and 0.996 respectively. The fit improves with increased factor parameterization. A higher current rate environment would have fewer down shocks that result in negative par yields, obviating the need for intra-curve constraints. However, until that time, the 5-factor model is best suited for sustained periods of near zero rates.

2.3 *Flexibility to Handle Inter-Curve Constraints*

Along with imposing an intra-curve non-negativity constraint to ensure plausible down shocks, we need an inter-curve constraint to maintain an appropriate credit spread between government and non-government yields. For purposes of exposition, we will focus on the Treasury, Libor-Swap, and Agency curves. Applying a historical shock drawn from a risk-on period where the spread between government and non-government debt narrowed can result in a scenario where simulated Treasury rates are greater than simulated Libor-Swap and Agency rates. To prevent such implausible curve relations we add an inter-curve constraint to our re-parameterization requiring that, at each term point, Treasury rates are equal to or lesser than Libor-Swap and Agency rates.

< - - - INSERT FIGURE 5 ABOUT HERE - - - >

Figure 5 uses three panels to compare the inter-curve constraints across the 3-, 4-, and 5-factor models from 9/30/1998. As illustrated in Figure 5(a), when a historical shock is applied to a low interest rate environment, the 3-factor model is often too inflexible to handle competing constraints—requiring a non-positive spread to Libor-Swap and Agency pushes the Treasury curve down, while requiring rates remain positive pushes the Treasury curve up. Attempting to meet both constraints within the confines of a 3-factor model can yield a nonsensical result, namely a horizontal Treasury yield curve. As the market environment changes and current interest rates rise, this problem significantly abates. Even in low interest rate environments, implausible 3-factor Treasury yields can be remedied by relaxing the inter-curve constraint—e.g. allowing simulated Treasury rates to exceed Libor-Swap and Agency by some pre-determined buffer such as 10 to 15 bps. A review of daily trading data from 01/02/1990 to 12/31/2012 shows that such a buffer is not entirely unreasonable. Historically, we have observed Treasury yields which exceed Libor by up to 7.75 bps, although such occurrences are extremely infrequent. In contrast to the 3-factor model, the 4- and 5-factor models are flexible enough to handle competing constraints, as shown in panels (b) and (c) of Figure 5. They can produce plausible shocked Treasury curves without the aid of a buffer, which is attractive given the customary and expected relationship between government and non-government yields.

2.4 Negative Forward Rates

< - - - INSERT FIGURE 6 ABOUT HERE - - - >

While the added flexibility of the 4- and 5-factor parameterizations substantially improves many aspects of model fit, it can also engender problems, namely negative forward rates. Figure 6 shows this problem and how an intra-curve constraint can provide a solution using market data from 10/29/1999. As illustrated in Figure 6(a), applying certain historical rate changes to the current market environment can yield negative forward rates, particularly at the long end of the interest rate curve[12]. The 4- and 5-factor models more closely adhere to the specifics of historical rate changes, and, if left unconstrained, reproduce similar forward rate trajectories. Alternatively, the 3-factor model abstracts from specifics of historical periods of market stress, and produces fewer incidences of negative forward rates. However, this shortcoming of the 4- and 5-factor models can be remedied by introducing an additional intra-curve constraint which limits the rate of decrease at the long end of the interest rate curve. Figure 6(b) depicts that, when the aforementioned constraint is imposed, the 4- and 5-factor models no longer reproduce the negative forward rate trajectory associated with absolute term point shocks.

3. Generating Historically Based Interest Rate Shocks

Out of the three yield curve models examined, the 5-factor Björk-Christensen parameterization appears to be best suited for historical simulation, at least in the context of the current low rate environment. It offers the closest approximation to historical yield curve realizations and the most flexibility in modeling intra- and inter-yield curve constraints[13]. Even so, there are drawbacks. For example, the 3- and 4-factor models are more economically intuitive and, because of less collinearity across factors, related interest rate shocks can be mapped to a unique and limited set of factor movements. This is potentially useful for loss attribution where a particular type of shock can be decomposed and potentially hedged based upon associated factor movements. The 3- and 4-factor models also require fewer parameters, saving degrees of freedom. Of course, these advantages have to be weighed against (1) a lack of historical accuracy, which is essential in generating historically-based interest rate shocks; and (2), in the current interest rate environment, difficulty in accommodating inter- and intra-curve constraints, which are necessary to ensure plausible interest rate scenarios. Using the 5-factor Björk-Christensen parameterization, our historical simulation technique generates rate shocks corresponding to any desired time horizon H in the following steps:

a. For each trading day (1,…,T), we fit the realized, historical Libor-swap spot curve onto the Björk-Christensen parameters in equation (4) using constrained least-squares.[14] Following Diebold et al. (2008), we specify λ =0.024[15]. To test model fit, we use market quotes for the Libor-swap spot curve on 9/28/2012. The Björk-Christensen factorization describes over 99 percent of the variation in the term structure of rates using five parameters: $\beta_1 = 0.059$, $\beta_2 = -0.001$, $\beta_3 = -0.022$, $\beta_4 = -0.025$, and $\beta_5 = -0.034$.

b. After parameterizing each historical yield curve, we calculate the change in realized Betas over the time horizon H (H $\leq T$)[16]. That is,

$$\Delta\beta_{j,i} = \beta_{j,i} - \beta_{j,i-H} \; \forall i, j \in (1,2,3,4,5) \tag{5}$$

For expository purposes, let H equal 6 calendar months. The change in realized Betas over the 6-calendar month period from 12/31/2008 to 6/30/2009 can be calculated as $\Delta\beta_1 = \beta_{1,6\_2009} - \beta_{1,12\_2008}$ =-0.027,$\Delta\beta_2 = \beta_{2,6\_2009} - \beta_{2,12\_2008}$ =0.002,$\Delta\beta_3 = \beta_{3,6\_2009} - \beta_{3,12\_2008}$ =-0.184,$\Delta\beta_4 = \beta_{4,6\_2009} - \beta_{4,12\_2008}$ =0.059, and$\Delta\beta_5 = \beta_{5,6_2009} - \beta_{5,12_2008}$ =0.207.

c. We generate historical simulation scenarios for Beta, $\tilde{\hat{\beta}}$, by adding these Beta shocks to the current yield curve Betas,

$$\tilde{\hat{\beta}}_{j,base} = \beta_{j,base} + \Delta\beta_{j,i} \; \forall i, j \in (1,2,3,4,5) \tag{6}$$

Define the current market as the rates prevailing on 9/28/2012 and parameterized in step (a). Applying the previously calculated shocks to the current Betas yields:

$$\tilde{\hat{\beta}}_{1,\text{base}} = 0.032, \tilde{\hat{\beta}}_{2,\text{base}} = 0.001, \tilde{\hat{\beta}}_{3,\text{base}} = -0.206, \tilde{\hat{\beta}}_{4,\text{base}} = 0.034, \text{ and } \tilde{\hat{\beta}}_{5,\text{base}} = 0.173$$

After calculating simulated Betas, we can compute associated spot curves using formula (4). Figure 7(a) illustrates both initial (current) and shocked or simulated Libor-swap curves as of 12/31/2008.

< - - - INSERT FIGURE 7 ABOUT HERE - - - >

d. Some shocked spot curves generated in step (c) will exhibit negative yields. To remedy this, we re-parameterize the shocked spot curves subject to non-negative rates using constrained optimization:

$$\underset{\beta}{arg\,min} \sum_t \{\hat{y}_L(\mathrm{t}) - \tilde{y}_L(t)\}^2 \text{ subject to: } \tilde{y}_L(t) \geq 0 \tag{7}$$

where $\hat{y}_L(\mathrm{t})$ denotes the initial shocked Libor-Swap spot curve calculated in step c and $\tilde{y}_L(t)$ denotes the closest possible approximation subject to the constraint of positive rates.

e. We repeat steps (a) thru (d) to simulate the Agency and Treasury spot curves. To ensure realistic credit spreads, shocked Treasury curves are subject to an additional constraint, wherein, at every term point Treasury rates must be less than or equal to Libor-swap and Agency rates:

$$\underset{\beta}{arg\,min} \sum_t \{\hat{y}_T(\mathrm{t}) - \tilde{y}_T(t)\}^2 \text{ subject to: } \tilde{y}_T(t) \geq 0 \text{ and } \tilde{y}_T(t) \leq \min\{\tilde{y}_L(t), \tilde{y}_A(t)\} \tag{8}$$

where $\tilde{y}_T$ denotes shocked Treasury rates, $\tilde{y}_L$ denotes shocked Libor-Swap rates, and $\tilde{y}_A$ denotes shocked Agency rates. Figure 7(b) illustrates the simulated Libor-swap, Agency, and Treasury curves associated with the 12/31/2008 historical 6-calendar month shock. Given the flexibility of the Björk-Christensen parameterization, additional inter-curve constraints could be applied, like specifying a particular relationship between Libor-Swap and Agency curves.

f. We simulate other miscellaneous rates such as the prime rate, Fannie Mae par coupon, mortgage spreads, repurchase rates, and the federal funds rate by modeling each as a function of contemporaneous Agency and Libor-swap Betas. This generates plausible values for several auxiliary rates, which may be required during portfolio revaluation, without simply imposing a fixed spread (e.g. Fannie Mae par coupon = Libor 10 YR + 200 bps). The two steps are described below.

   i. Solve the least-squares minimization problem for the linear parameter set with the best fit between historical realizations of each miscellaneous rate and contemporaneous Agency and Libor-swap Betas, subject to a non-negativity constraint[17].

$$MiscRate_{j,i} = \alpha_0 + \sum_{k=1}^{5} \gamma_{j,k} \beta_{j,k,i}(Agency) + \sum_{k=1}^{5} \xi_{j,k} \beta_{j,k,i}(LiborSwap) + \varepsilon_{j,i} \;\; \forall i, j \qquad (9)$$

   ii. Substitute the simulated Agency and Libor-Swap Betas into equation (9) to solve for the simulated miscellaneous rates corresponding to each historical scenario.

4. Implied Volatility

In addition to the interest-rate term structure and credit spreads, fixed-income investors are affected by other market risk factors, notably housing price appreciation and implied volatility. For the sake of brevity, we focus our attention on linking interest rate shocks to the latter.

Vega, measuring the sensitivity of an option price to the volatility of the underlying asset, is recognized as a significant risk factor in fixed-income portfolios with embedded optionality. Many option contracts are linked to interest rates, making the implied volatility of interest rates a necessary component of any comprehensive shock scenario, as well as a required input in any option valuation software. Implied volatility is a market based measure of future anticipated interest rate volatility and should be affected by the rate environment. For example, a sharp decrease in interest rates, like with monetary easing, is often accompanied by an increase in implied volatility. Similarly, widening credit spreads result in an attendant increase in implied volatility, possibly because investors anticipate future policy interventions. Also, short term interest rates and yield curve slope are important determinants of the premia or compensation required for taking on volatility risk. This volatility risk is embodied in option prices, which, via Black-Scholes, are used to calculate implied volatility. Short term interest rates (e.g. 3-month Libor) influence the payoff of the derivative contract and the yield curve slope acts as an indicator of business cycle expansions/contractions. Fornari (2010) finds that both factors have a significant effect on observed market premia, or implied variance less forecasted historical variance.

We create a time series of implied volatility measures using market quotes from two instruments, at-the-money swaptions and at-the-money interest-rate caps[18]. To ensure sufficient coverage, we include swaption volatilities for 300 different contracts, each defined by a different expiry-tenor combination. Expiries (τ) span 1 to 360 months and indicate an option's expiration date. Tenors (t) span 12 to 360 months and indicate the length of the underlying swap contract. Each day's group of 300 quotes can be viewed as a three-dimensional surface with implied volatility varying over expiry and tenor. We include cap volatilities for 150 different contracts defined by different strike-tenor combinations. For continuity across the time series, we focus on a fixed set of strikes (κ) ranging from 1 to 10 percent. Cap tenors (t) range from 12 to 360 months. Similar to swaptions, groups of cap quotes can be viewed as a three-dimensional surface with implied volatility varying over strike and tenor.

The next several subsections document how we link interest rate shocks to the implied volatility surface. After collecting data and creating a uniform time series across observations, we reduce the dimensionality of the swaption and cap volatility surfaces using a variant of the Nelson-Siegel factorization. This allows us to model the historical, reduced form relationship between implied volatility and interest rates without including hundreds of regressands. These regression results are then used to estimate plausible volatility scenarios to accompany each set of shocked rates and spreads.

4.1 Swaption Volatilities

As alluded to above, reducing the dimensionality of this volatility surface is an essential first step in modeling the historical relationship between implied volatility and the interest rate term structure. Before building a model, we examined the extant literature and found several examples of both parametric and non-parametric approaches to implied volatility surface (IVS) modeling[19]. Some of the seminal works include Heynen et al. (1994), Derman et al. (1996), Dumas et al. (1998), Mixon (2007), Christoffersen and Jacobs (2007), Chalamandaris and Tsekrekos (2011), and Guo (2014). Our specification extends upon Chalamandaris and Tsekrekos (2011), while providing a simple yet flexible parametric model that allows for varying interdependence across option characteristics.

< - - - INSERT FIGURE 8 ABOUT HERE - - - >

In parameterizing the swaption volatility surface it is important to capture: 1) the relationship between expiry and implied volatility; 2) the relationship between tenor and implied volatility, and; 3) interactions between expiry and tenor. These interactions are apparent in daily trading data where the relationship between tenor and implied volatility varies by option expiration date. Figure 8 graphs two cross-sections of the swaption volatility surface from 9/28/2012. As illustrated, the term structure of tenor at the 1 year expiry is substantively different than the term structure of tenor at the 10 year expiry. To fully capture each of these dependencies, we estimate a 9-factor swaption volatility model:

$$\sigma^{Swp}(\tau, t) = B_1^{Swp} + A\{\tau\} + B\{t\} + C\{\tau, t\} + \varepsilon_{\tau,t}, \tag{10}$$

$$A\{\tau\} = B_2^{Swp}\left(\frac{1-e^{-\phi\tau}}{\phi\tau}\right) + B_3^{Swp}\left(\frac{1-e^{-\phi\tau}}{\phi\tau} - e^{-\phi\tau}\right), \tag{11}$$

$$B\{t\} = B_4^{Swp}\left(\frac{1-e^{-\omega t}}{\omega t}\right) + B_5^{Swp}\left(\frac{1-e^{-\omega t}}{\omega t} - e^{-\omega t}\right), \tag{12}$$

$$C\{\tau, t\} = B_6^{Swp}\left(\frac{1-e^{-\phi\tau}-e^{-\omega t}+e^{-(\phi\tau+\omega t)}}{\phi\tau\omega t}\right) + B_7^{Swp}\left(\frac{1-e^{-\phi\tau}-(1+\omega t)\left(e^{-\omega t}-e^{-(\phi\tau+\omega t)}\right)}{\phi\tau\omega t}\right) +$$

$$B_8^{Swp}\left(\frac{1-e^{-\omega t}-(1+\phi\tau)\left(e^{-\phi\tau}-e^{-(\phi\tau+\omega t)}\right)}{\phi\tau\omega t}\right) + B_9^{Swp}\left(\frac{1-e^{-\phi\tau}(1+\phi\tau)-e^{-\omega t}(1+\omega t)+e^{-(\phi\tau+\omega t)}(1+\phi\tau)(1+\omega t)}{\phi\tau\omega t}\right) \tag{13}$$

where $\sigma(\tau, t)$ is implied volatility at expiry τ and tenor t, ϕ and ω are fixed decay parameters[20], and B_1^{Swp} to B_9^{Swp} are estimated using constrained least squares.

$$\underset{\beta}{arg\,min} \sum_{\tau,t}\left\{\sigma^{Swp}(\tau, t) - \widehat{\sigma^{Swp}}(\tau, t)\right\}^2 \text{ subject to: } \widehat{\sigma^{Swp}}(\tau, t) \geq 0 \tag{14}$$

Equation (10) consists of a constant and three functions, $A\{\tau\}$, $B\{t\}$, and $C\{\tau, t\}$. The constant describes average implied volatility across all expiry-tenor combinations. This is used to shift up or shift down the volatility surface in response to changes in market uncertainty and bond yields. Following Chalamandaris and Tsekrekos (2011), $A\{\tau\}$ describes the relationship between implied volatility and expiry. Similarly, $B\{t\}$ describes the relationship between implied volatility and tenor. Both functions employ a Nelson-Siegel factorization. While assessing yield curve models, we discussed the merits (and oftentimes significant advantages) of alternative, higher order factorizations. In parameterizing the volatility surface, these advantages are outweighed by the relative parsimony of the Nelson-Siegel model. As will be described in section 4.3, given a set of rate shocks, we calculate plausible values for the implied volatility surface based upon the historical, reduced form relationship between interest rates and each of our volatility parameters. The more factors included in the volatility parameterization, the more values we need to predict, which increases the possible prediction error. The final function,

$C\{\tau, t\}$, describes non-linear interactions between expiry and tenor (a graphical example of the loading factors attached to the Beta estimates can be provided upon request). These non-linear interactions help us model the varying relationship between implied volatility and tenor across expiry.

< - - - INSERT FIGURE 9 ABOUT HERE - - - >

Figure 9 provides two examples of overall model fit using market quotes from 9/28/2012. The top panel shows a swaption volatility surface while the bottom one (see the next section) focuses on the cap volatility surface. Additionally, tables can be constructed to detail various percentiles of prediction errors, $\sigma^{Swp}(\tau, t) - \widehat{\sigma^{Swp}}(\tau, t)$, for the swaption volatility model as the expiry and tenor vary. A table of the 1st percentile indicates whether the parameterization potentially overstates implied volatility while the 99th percentile speaks to the understatement. Both tables (available upon request) show that short dated contracts have the largest errors; however, median prediction errors are less than two percent for all expiry-tenor combinations (a lack of systematic error or directional bias). As Figure 9(a) concurs, the model fits well the varied historical permutations of the swaption implied volatility surface.

4.2 Cap Volatilities

We parameterize the cap volatility surface using a variant of equations (10) to (13) with term structure components for both strike, $D\{\kappa\}$, and tenor, $E\{t\}$. $F\{\kappa, t\}$ captures non-linear interactions between these two factors and allows us to model the *smile* or *smirk*[21] of the implied volatility curve across in-the-money, at-the-money, and out-of-the-money strikes:

$$\sigma^{Cap}(\kappa, t) = B_1^{Cap} + D\{\kappa\} + E\{t\} + F\{\kappa, t\} + \varepsilon_{\kappa,t}, \tag{15}$$

$$D\{\kappa\} = B_2^{Cap}\left(\frac{1-e^{-\psi\kappa}}{\psi\kappa}\right) + B_3^{Cap}\left(\frac{1-e^{-\psi\kappa}}{\psi\kappa} - e^{-\psi\kappa}\right), \tag{16}$$

$$E\{t\} = B_4^{Cap}\left(\frac{1-e^{-\varphi t}}{\varphi t}\right) + B_5^{Cap}\left(\frac{1-e^{-\varphi t}}{\varphi t} - e^{-\varphi t}\right), \tag{17}$$

$$F\{\kappa, t\} = B_6^{Cap}\left(\frac{1-e^{-\psi\kappa}-e^{-\varphi t}+e^{-(\psi\kappa+\varphi t)}}{\psi\kappa\varphi t}\right) + B_7^{Cap}\left(\frac{1-e^{-\psi\kappa}-(1+\varphi t)\left(e^{-\varphi t}-e^{-(\psi\kappa+\varphi t)}\right)}{\psi\kappa\varphi t}\right) +$$

$$B_8^{Cap}\left(\frac{1-e^{-\varphi t}-(1+\psi\kappa)\left(e^{-\psi\kappa}-e^{-(\psi\kappa+\varphi t)}\right)}{\psi\kappa\varphi t}\right) + B_9^{Cap}\left(\frac{1-e^{-\psi\kappa}(1+\psi\kappa)-e^{-\varphi t}(1+\varphi t)+e^{-(\psi\kappa+\varphi t)}(1+\psi\kappa)(1+\varphi t)}{\psi\kappa\varphi t}\right) \quad (18)$$

where $\sigma^{Cap}(\kappa, t)$ is implied volatility at strike κ and tenor t, ψ and φ are fixed decay parameters[22], and B_1^{Cap} to B_9^{Cap} are estimated using constrained least squares. An example of overall model fit using market quotes from 9/28/2012 is provided in Figure 9(b). Like mentioned before, tables can be constructed for different percentiles of prediction errors for the cap volatility model. Again, the short dated contracts (here, the 12 month caps) are associated with the largest errors but the median prediction error is less than 1.5 percent across all strike-tenor combinations.

4.3 Term Structure of Interest Rates and Implied Volatility

Parameterizing each trading day's swaption and cap volatility surfaces yields a time series of swaption and cap volatility Betas. To model the reduced form relationship between the term structure of interest rates and implied volatility, we regress each volatility Beta on contemporaneous Agency and Libor-swap Björk-Christensen Betas (e.g. using data from 09/1998 to 12/2012, historical estimates of B_1^{Swp} are regressed on (same day) historical estimates of B_1^{Agy} to B_5^{Agy} and B_1^{Libor} to B_5^{Libor}). It is important to note, we are modeling a reduced form relationship. Interest rates have both a direct and indirect effect (through other market factors) on implied volatility. We ignore this hierarchy of influence and group both effects together using a simple linear regression[23]:

$$B_{j,i}^{V} = \alpha_0 + \sum_{k=1}^{5}\gamma_{j,k}\beta_{j,k,i}(Agency) + \sum_{k=1}^{5}\xi_{j,k}\beta_{j,k,i}(LiborSwap) + \varepsilon_{j,i} \;\; \forall i,j \quad (19)$$

where j indexes cap and swaption volatility Betas and i indexes trading days. The adjusted R^2 statistic for the swaption volatility Betas range from 0.38 to 0.93 with a mean of 0.72. The adjusted R^2 statistic for the cap volatility Betas range from 0.57 to 0.88 with a mean of 0.74.

With these regression results in hand, it is straightforward to estimate the volatility Betas corresponding to each shock scenario by substituting the shocked values of the Agency and Libor-swap Betas into equation (19). Once the volatility Betas have been calculated, projecting the volatility surface corresponding to each shock scenario is straightforward using equations (10) to (13) and (15) to (18).

< - - - INSERT FIGURE 10 ABOUT HERE - - - >

Figure 10 compares the same volatility surface (the swaption and cap) as in Figure 9 but shows the historical 6-calendar month simulations. Both of its panels have volatility surfaces corresponding to the simulated Libor-swap, Agency, and Treasury curves presented in section 3 (9/28/2012 base case, 12/31/2008 historical 6-calendar month shock).

4.4 Optimizing Decay Parameters

The swaption and cap volatility parameterizations each contain two fixed decay parameters that are chosen to optimize overall model fit and remain constant across trading days. Both sets are jointly estimated using unconstrained nonlinear optimization to minimize the following objective functions:

$$\sum_i \sum_\tau \sum_t [\sigma_i^{Swp}(\tau,t) - \hat{\sigma}_i^{Swp}(\tau,t)]^2, \tag{20}$$

$$\sum_i \sum_\kappa \sum_t [\sigma_i^{Cap}(\kappa,t) - \hat{\sigma}_i^{Cap}(\kappa,t)]^2, \tag{21}$$

where i indexes trading days and σ is actual volatility. The estimated volatility, $\hat{\sigma}$, is calculated using volatility Betas estimated in equation (19), which are transformed into implied volatility estimates via equations (10) to (13) and (15) to (18). Put differently, $\hat{\sigma}$ is based upon two sets of transformations. We estimate a set of volatility parameters based upon contemporaneous Agency and Libor-Swap Betas. The volatility parameters are entered into our volatility factorization to recover implied volatility estimates.

5. Conclusion

We describe a robust empirical method to generate plausible, historically-based interest rate shocks, which can be applied to *any* interest rate environment. Our approach requires a yield curve parameterization model that can adequately describe historical realizations of several interest rate curves and is flexible enough to handle both intra- and inter-curve constraints. Given these broad requirements, we evaluate three variants of the Nelson-Siegel approach to yield curve approximation.

Out of the three models examined, the 5-factor Björk-Christensen parameterization appears to be best suited for historical simulation. Although the 3- and 4-factor models are more economically intuitive, they (1) lack the 5-factor model's historical accuracy, and in the context of the current low rate environment (2) fail to sufficiently adhere to intra- and inter-curve constraints, which are necessary to ensure plausible interest rate scenarios. Given these significant shortcomings, we believe the Björk-Christensen is most appropriate for generating interest rate scenarios.

Using the Björk-Christensen model, we demonstrate how to apply historical shocks to any current market environment, while retaining positive rates and plausible credit spreads. By regressing a parameterized representation of the implied volatility surface onto the Björk-Christensen yield curve parameter space, we establish a framework to generate the volatility surface implied by any given yield scenario. Together, these joint risk factor movements can be used to measure market risk on institutions with large fixed income portfolios.

As a suggestion for future research, it would be instructive to model the joint dependence of Treasury, Agency, and Libor-Swap yield curve parameters using copulas. By accurately capturing this dependence along with parameter specific marginal distributions, one can generate thousands of plausible yet stressful hypothetical scenarios through simulation. These hypothetical scenarios will help identify yield curve shocks which could potentially occur, but have yet to be observed, resulting in a more robust measure of potential market risk.

The majority of stress scenario methodologies are based upon simulation, and the few that are historically-based result in scenarios characterized by conflicting and sometimes implausible risk factor movements. This paper's methodology improves upon these alternatives by offering a simple way to generate a coherent and internally consistent set of shocks, while reflecting the specifics of historical periods of actual market stress. The use of realistic shock scenarios is important for risk management and, we believe, attractive to practitioners. For example, industry participants may be more willing to set capital against observed changes in market conditions as opposed to potentially implausible simulated or theoretically derived shocks. Another advantage to our approach is its suite of constraints, which ensure a zero lower bound, positive forward rates, and realistic credit spreads. While other papers have imposed some of these constraints individually, none have implemented them simultaneously. These restrictions, in combination with our proposed mechanism of linkage (i.e. between interest rates and implied volatility), guarantee that our stress scenarios consist of plausible joint risk factor movements. Together, these changes should offer a more appealing alternative to industry stake holders while simultaneously promoting better risk management.

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Notes

[1]An accurate measure of market risk can help to inform institutions about the amount of capital needed to withstand a series of adverse market events. A plausible set of shocks is required to ensure market value and cash flow projections are indicative of meaningful market sensitivities.

[2]Measuring market risk using shocked interest rate curves characterized by negative forward rates (engendered by multiple "kink" points) and implausible credit spreads strains credulity among market participants. It is unlikely that sophisticated risk managers would regard market value or cash flow sensitivities to such shocks as actionable information.

[3]Generating plausible co-movements in other key risk factors allows us to build a comprehensive, yet coherent set of stress scenarios. Without a tractable means of linkage, a stress scenario may be characterized by a basket of inconsistent or contradictory shocks (e.g. a 20 percent decrease in housing prices coupled with 10 percent inflation).

[4]While outside the scope of this paper, historical shocks can be rescaled based upon the expected volatility of rates.

[5] We are unaware of any papers on generating historically-based stress scenarios which contain a comparison of the quality of their risk measurement relative to other approaches. That being said, it is possible to compare each methodology's outputs based upon the coherence and internal consistency of the included risk factor movements. When possible, we present explained variation with R^2 values.

[6] This can occur when we seek to replicate a historical risk-on market environment characterized by narrowing credit spreads. In a risk-on market environment, bullish investor sentiment engenders an increase in demand for higher risk investments like commodities, equities, and non-investment grade debt. As investors chase higher returns, demand for relatively low-risk investments like U.S. Treasuries and investment grade debt falls. We last observed such market behavior during the first quarter of 2009. Investors, sensing an end to the recent financial crisis, exited out of Treasuries en masse. Over this three month period, the 10-year Treasury rate increased from 2.5 to 4 percent.

[7]A forward rate model can be converted into a spot rate model as $y(\mathrm{t}) = \frac{1}{\mathrm{t}}\int_0^{\mathrm{t}} f(u)du$.

[8]$y(10\mathrm{yr})\text{-}\,\mathrm{y}(3\mathrm{m}) = \beta_1 + \beta_2\left(\frac{1-\mathrm{e}^{-0.0609*120}}{0.0609*120}\right) + \beta_3\left(\frac{1-\mathrm{e}^{-0.0609*120}}{0.0609*120} - \mathrm{e}^{-0.0609*120}\right) - \beta_1 - \beta_2\left(\frac{1-\mathrm{e}^{-0.0609*3}}{0.0609*3}\right) - \beta_3\left(\frac{1-\mathrm{e}^{-0.0609*3}}{0.0609*3} - \mathrm{e}^{-0.0609*3}\right) = 0.1367\,\beta_2 + 0.1361\,\beta_3 - 0.9140\,\beta_2 - 0.0810\,\beta_3 = -0.7773\,\beta_2 + 0.0551\,\beta_3$

[9]$$2*\mathrm{y}(2\mathrm{yr}) - y(10\mathrm{yr})\text{-}\,\mathrm{y}(3\mathrm{m}) = 2*\left[\beta_1 + \beta_2\left(\frac{1-\mathrm{e}^{-0.0609*120}}{0.0609*120}\right) + \beta_3\left(\frac{1-\mathrm{e}^{-0.0609*120}}{0.0609*120} - \mathrm{e}^{-0.0609*120}\right)\right]$$
$$-2\,\beta_1 - 0.1367\,\beta_2 - 0.1361\,\beta_3 - 0.9140\,\beta_2 - 0.0810\,\beta_3$$
$$= 2\,\beta_1 + 1.0511\,\beta_2 + 0.5874\,\beta_3 - 2\,\beta_1 - 1.0507\,\beta_2 - 0.2171\,\beta_3$$
$$= 0.0005\,\beta_2 + 0.3703\,\beta_3$$

[10] The Libor-swap curve is widely utilized as the basis for discounting cash flows on fixed-income derivatives, and mortgage assets are typically valued using an option-adjusted spread to this curve. The curve is often derived from three market instruments—Libor or short term deposit rates, Eurodollar futures, and swap rates—on the short end, middle, and long end of the curve, respectively.

[11] Only every tenth trading day is graphed when comparing adjusted R^2 values across the three models.

[12]Negative forward rates can arise because of significant differences in the term structure of rates between a historical period of market stress and the current market environment.

[13] Another advantage of the 5-factor Björk-Christensen model is its relative insensitivity to the choice of λ (Hurn et al., 2005). This allows us to fix the model's decay parameter across both trading days and yield curves without significantly eroding overall model fit. In contrast, the 3- and 4-factor models require a targeted and varied choice of λ to accurately fit changing term structures. This makes it difficult to choose a single value for the fixed decay parameter without sacrificing specific periods of historical accuracy.

[14] In the curve fitting process, we impose the constraint of non-negative projected rates.

[15] We explored several different choices of λ including parameter values estimated to optimize initial model fit using unconstrained nonlinear optimization. As documented in Hurn et al. (2005), the 5-factor model proved relatively insensitive to the specified decay parameter. Regardless of our choice, we only observed nominal changes to overall model fit. Because of this insensitivity and in the interest of aligning with extant literature, we chose to use λ =0.024 as specified in Diebold et al. (2008).

[16] Diebold et al. (2008) create a similar set of Beta shocks and partition them into unique clusters using a one-dimensional projection pursuit algorithm. These clusters are then applied to the current market environment to generate stress scenarios. Loretan (1997) and Rodriques (1997) use principal components analysis to identify stressful term structure movements and generate interest rate scenarios.

[17]The miscellaneous rates and Betas are cointegrated and yield a stationary residual series.

[18] All values are downloaded from Bloomberg.

[19] Unfortunately, it is not possible to provide a comprehensive review of industry practices because of the proprietary nature of the models.

[20] As described in section 4.4, ϕ and ω are estimated to optimize overall model fit using unconstrained non-linear optimization. Based upon a time series of swaption volatility quotes from 9/1998 to 12/2012, we estimate ϕ=0.0725 and ω=0.0590.

[21] A volatility smile describes a skew pattern where in- and out-of-the-money options have higher implied volatilities than at-the-money options. A volatility smirk describes a skew pattern where implied volatility decreases with strike.

[22] Again, ϕ and ϕ are estimated to optimize overall model fit. Based upon a time series of cap volatility quotes from 5/2005 to 12/2012, we estimate ϕ =0.3858 and ϕ =0.0480.

[23] The volatility, Agency, and Libor-swap Betas are cointegrated and yield a stationary residual series.

Figure 1. Factor loadings for the Nelson-Siegel spot rate model

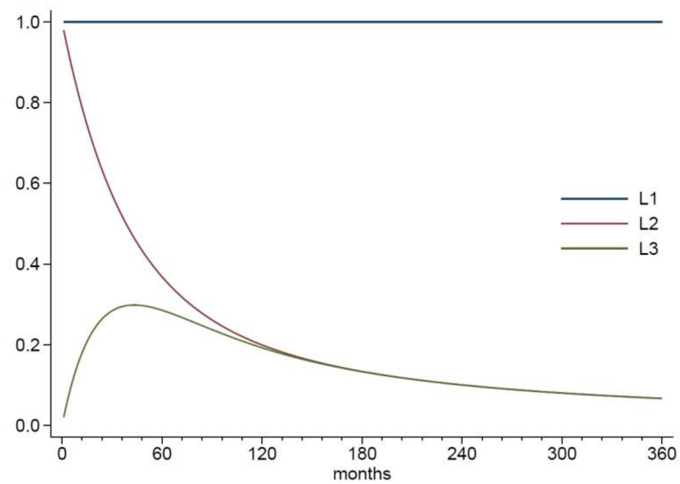


Figure 2. Observed yield patterns and model fit of Libor-Swap rates

(a) from 2/20/1998

(b) from 9/29/2006

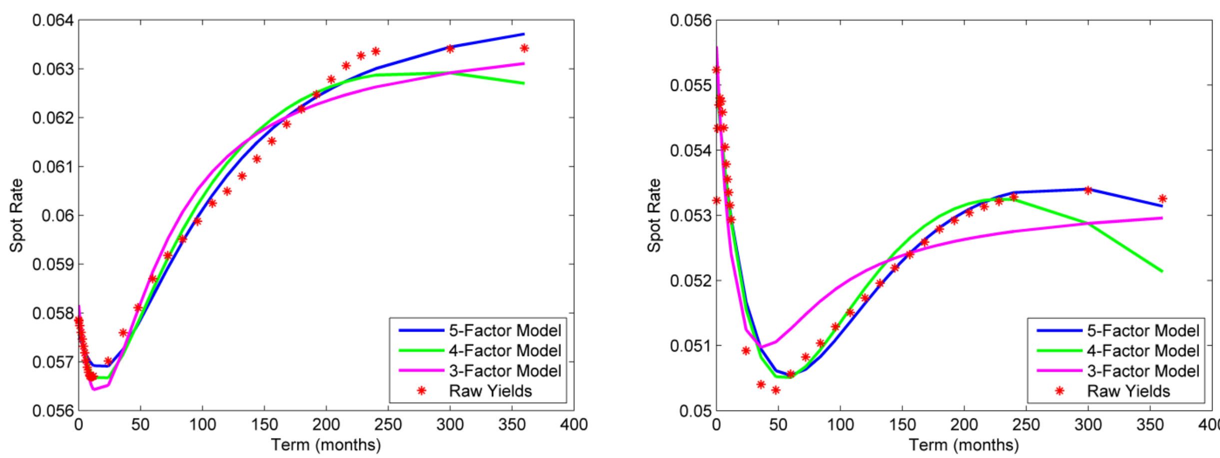


Figure 3. Comparison of model fit for observed Libor-Swap yield patterns from 5/15/1995 to 9/28/2012

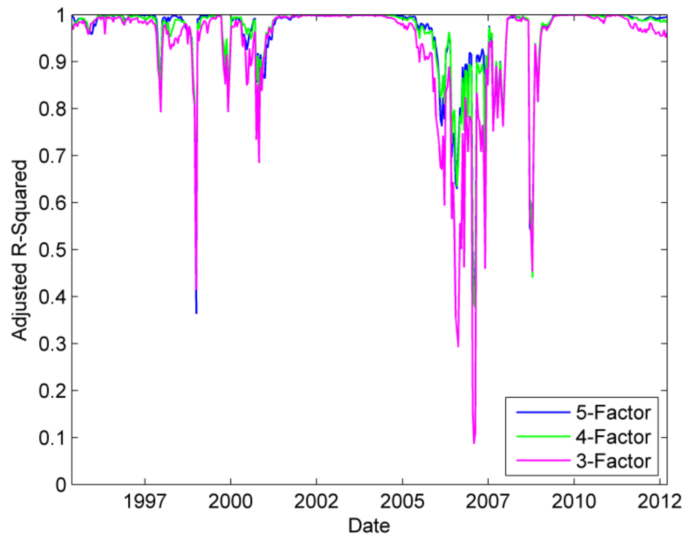


Figure 4. Absolute term point specific down shocks with floored negative rates

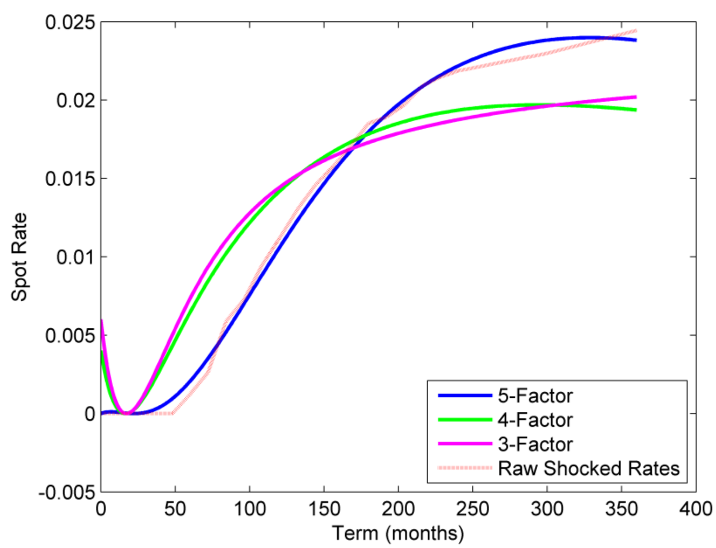
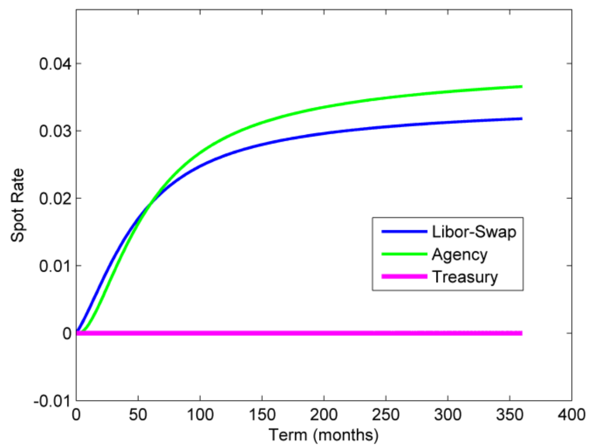
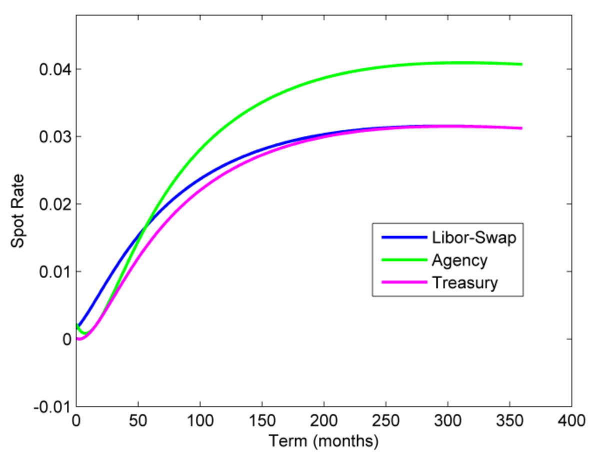


Figure 5. The impact of an inter-curve constraints on different factor parameterizations

(a) 3-factor model



(b) 4-factor model



(c) 5-factor model

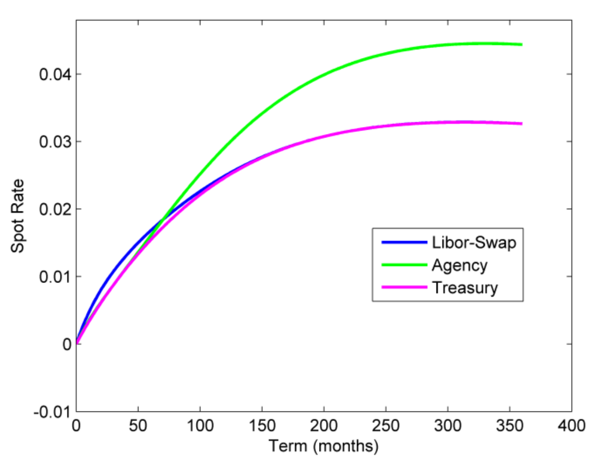
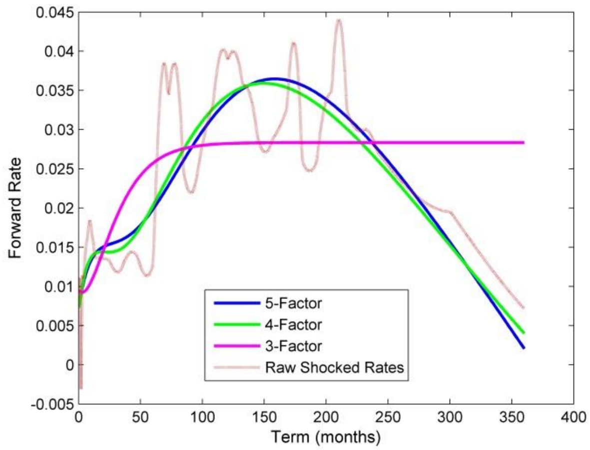


Figure 6. The impact of an intra-curve constraint for the rate of decrease

(a) No intra-curve constraint



(b) Constraining the rate of decrease

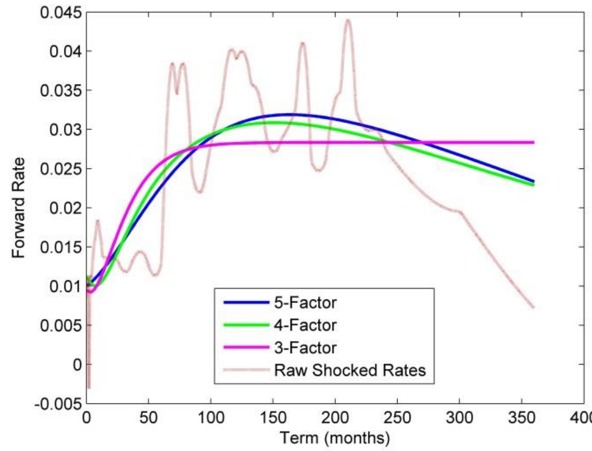
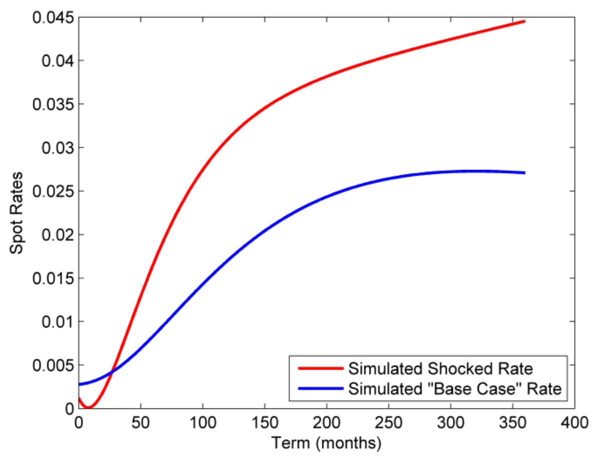


Figure 7. A comparison of the current and shocked Libor-Swap curves

(a) Initial and shocked Libor-Swap curves



(b) Historical shock applied to curves

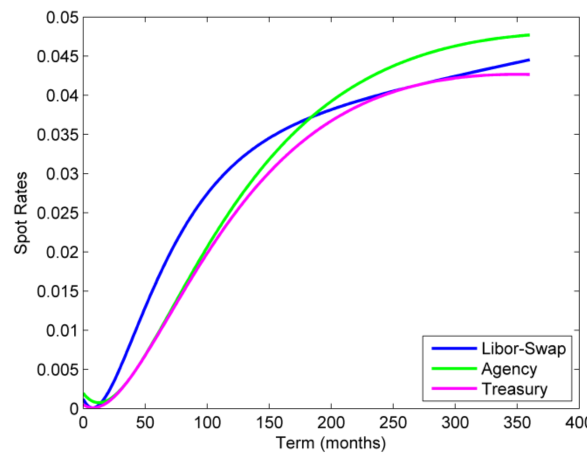


Figure 8. A cross-sectional view of the Swaption volatility surface

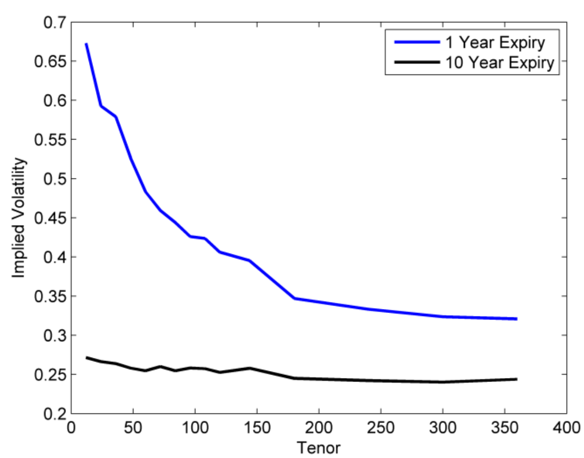
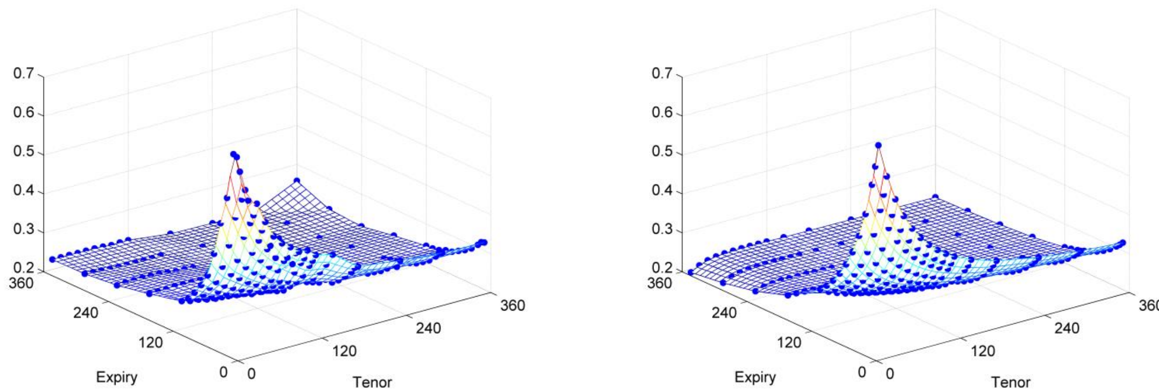


Figure 9. Example of actual vs. projected volatility surfaces

Swaption volatility surface

(a) Actual (b) Projected



Cap volatility surface

(c) Actual (d) Projected

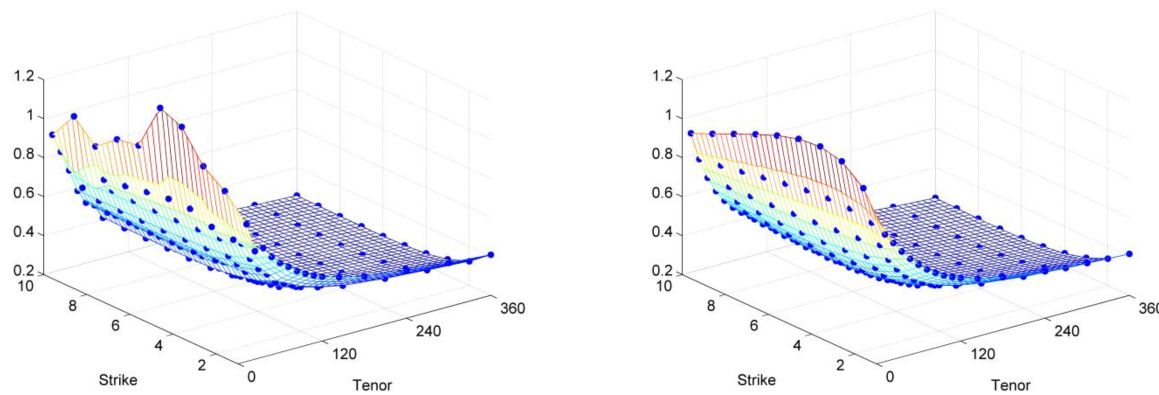


Figure 10. Simulated volatility surfaces (12/31/2008 historical 6-calendar month shock)

(a) Swaption volatility surface (b) Cap volatility surface