A STATE-DEPENDENT DUAL RISK MODEL

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Abstract. In a dual risk model, the premiums are considered as the costs and the claims are regarded as the profits. The surplus can be interpreted as the wealth of a venture capital, whose profits depend on research and development. In most of the existing literature of dual risk models, the profits follow the compound Poisson model and the cost is constant. In this paper, we develop a state-dependent dual risk model, in which the arrival rate of the profits and the costs depend on the current state of the wealth process. Ruin probabilities are obtained in closed-forms. Further properties and results will also be discussed.

1. Introduction

The classic risk model is based on the surplus process $U_t = u + \rho t - \sum_{i=1}^{N_{t-}} C_i$, where the insurer starts with the initial reserve u and receives the premium at a constant rate ρ and C_i are the claims. A central problem is to study the ruin probability, i.e., the probability that the surplus process will ever hit zero. In recent years, a dual risk model has attracted many attentions, in which the surplus process is modeled as

$$U_t = u - \rho t + \sum_{i=1}^{N_{t-}} C_i \tag{1.1}$$

where C_i are i.i.d. positive random variables distributed according to $Q(dc)$ independent of N_t, which is a Poisson process with intensity λ. We assume $\lambda\mathbb{E}[C_1] > \rho$. The surplus can be interpreted as the wealth of a venture capital, whose profits depend on research and development. The profits are uncertain and modeled as a jump process and the costs are more predictable and are modeled as a deterministic process. The company pays expenses continuously over time for the research and development and gets profits at random discrete times in the future. Many properties have been studied for the dual risk model. The ruin probability $\psi(u) = \mathbb{P}(\tau < \infty | U_0 = u)$, where

$$\tau = \inf\{t > 0 : U_t \leq 0\}, \tag{1.2}$$

satisfies the equation, see e.g. Afonso et al. [2]

$$\psi(u) = e^{-\lambda \frac{u}{\rho}} + \int_0^{\frac{u}{\rho}} \lambda e^{-\lambda t} \int_0^\infty \psi(u - \rho t + c) Q(dc) dt. \tag{1.3}$$

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It is well known that $\psi(u) = e^{-\alpha u}$ where α is the unique positive solution to the equation:

$$\lambda\left(\int_0^\infty e^{-\alpha x}Q(dx) - 1\right) = -\rho\alpha. \tag{1.4}$$

Avanzi et al. [5] worked on optimal dividends in the dual risk model where the wealth process follows a Lévy process and the optimal strategy is a barrier strategy. Albrecher et al. [3] studied a dual risk model with tax payments. For general interclaim time dsitributions and exponentially distributed C_i's, an expression for the ruin probability with tax is obtained in terms of the ruin probability without taxation. When the interclaim times are exponential or mixture of exponentials, explicit expressions are obtained. Ng [18] considered a dual model with a threshold dividend strategy, with exponential interclaim times. Afonso et al. [2] worked on dividend problem in the dual risk model, assuming exponential interclaim times. They presented a new approach for the calculation of expected discounted dividends. and studied ruin and dividend probabilities, number of dividends, time to a dividend. and the distribution for the amount of single dividends. Avanzi et al. [4] studied a dividend barrier strategy for the dual risk model whereby dividend decisions are made only periodically, but still allow ruin to occur at any time. Cheung [10] studied the Laplace transform of the time of recovery after default, amongst other concepts for a dual risk model. Cheung and Drekic [11] studied dividend moments in the dual risk model. They derived integro-differential equations for the moments of the total discounted dividends which can be solved explicitly assuming the jump size distribution has a rational Laplace transform. Rodríguez et al. [21] worked on a dual risk model with Erlang interclaim times, studied the ruin probability, the Laplace transform of the time of ruin for generally distributed C_i's. They also studied the expected discounted dividends assuming the profits follow a Phase Type distribution. When the profits are Phase Type distributed, Ng [19] also studied the cross probabilities. Yang and Sendova [23] studied the Laplace transform of the ruin time, expected discounted dividends for the Sparre-Andersen dual model. The dual risk model has also been used in the context of venture capital investments and some optimization problems have been studied, see e.g. Bayraktar and Egami [7]. In Fahim and Zhu [13], they studied the optimal control problem for the dual risk model, which is the minimization of the ruin probability of the underlying company by optimizing over the investment in research and development.

In this paper, we develop a state-dependent dual risk model. The innovations of a company may have self-exciting phenomena, i.e., an innovation or breakthrough will increase the chance of the next innovation and breakthrough. Also, when the wealth process increases, the company will be in a better shape to innovate and hence the arrival rate of the profits, may depend on the state of the wealth rather than simply being Poisson. Also, the expenses that a company pays for research and develop may also increase after the company receive more profits. For the high tech and fast-growing companies, the running cost and the revenues of a company grow in line with the size of the company, see e.g. Table 1, where we considered the annual total revenues, cost of total revenues and the gross profits[1] in the years 2011-2014 [2]. We can see the upward trend of growth for Google. Therefore, for a

[1]Gross profit is the difference between the revenue and the cost of the revenue.
[2]Available on Google Finance

high tech company for Google, the usual constant assumption for running cost, the intensity of profits arrivals in the dual risk model might be too simplistic. On the other hand, for a traditional company like Coca-Cola, the annual total revenues, cost of total revenues and the gross profits do not vary too much year over year, see e.g. Table 2, where we considered the annual total revenues, cost of total revenues and the gross profits in the years 2011-2014 [3]. That might also be the pattern for a high tech company that has already matured and no longer has stellar growth. Therefore, the dual risk model in the existing literature might be a good model when the financials of a company do not change too much over time. A state-dependent dual risk model might be more appropriate when the underlying company has phenomenal growth.

Full Year	2011	2012	2013	2014
Revenue (millions)	\$37,905	\$46,039	\$55,519	\$66,001
Cost of Revenue (millions)	\$13,188	\$17,176	\$21,993	\$25,313
Gross Profit (millions)	\$24,717	\$28,863	\$33,526	\$40,688

TABLE 1. Revenue and Cost by Google during 2011-2014.

Full Year	2011	2012	2013	2014
Revenue (millions)	\$46,542	\$48,017	\$46,854	\$45,998
Cost of Revenue (millions)	\$18,215	\$19,053	\$18,421	\$17,889
Gross Profit (millions)	\$28,327	\$28,964	\$28,433	\$28,109

TABLE 2. Revenue and Cost by Coca-Cola during 2011-2014.

So it will be reasonable to assume that the costs depend on the state of the wealth of the company. Indeed, it is not only possible that the company spends more capital on research and development when the profits increase, it is also quite common in the technology industry to increase the capital spending on research when the company is lagging behind its pairs so that it is fighting for survival and catch-up. When we assume that the cost is constant, the wealth process of the company is illustrated Figure 1 till the ruin time. If we allow the cost to depend linearly on the wealth, the wealth process of the company is illustrated in Figure 2. When the dual risk model uses the classical compound Poisson as the wealth process, the probability that the company eventually ruins decays exponentially in terms of the initial wealth of the company. As we will see later in the paper, e.g. Figure 3 and Figure 4, by allowing the costs and arrival rates of the profits depending on the state of the wealth process, the model becomes much more robust, and the ruin probability can decay superexponentially in terms of the initial wealth, i.e., Figure 3, Table 3, and it can also decay polynomially in terms of the initial wealth, i.e., Figure 4, Table 4.

We are interested to develop a state-dependent dual risk model, which still leads to closed-form solutions to the ruin probabilities. Let us assume that the wealth process U_t satisfies the dynamics

$$dU_t = -\eta(U_t)dt + dJ_t, \qquad U_0 = u, \tag{1.5}$$

[3]Available on Google Finance

where $J_t = \sum_{i=1}^{N_t} C_i$ and N_t is a simple point process with intensity $\lambda(U_{t-})$ at time t. Here, $\eta(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\lambda(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are both continuously differentiable. Throughout the paper, unless specified otherwise, we assume that C_i are i.i.d. exponentially distributed with parameter $\gamma > 0$. While allowing $\eta(\cdot)$ and $\lambda(\cdot)$ to be general, the drawback of our model is that we restrict C_i's to be exponentially distributed for the paper. It will be an interesting future research project to investigate generally distributed C_i's. For the wealth process U_t in (1.5), we will obtain closed-form expressions for the ruin probability and further properties will also be studied.

It is worth noting that the U_t process in (1.5) is an extension of the Hawkes process with exponential kernel and exponentially distributed jump sizes. that is, a simple point process N_t with intensity $\lambda\left(ue^{-\beta t} + \sum_{i:\tau_i<t} C_i e^{-\beta(t-\tau_i)}\right)$, where C_i are i.i.d. exponentially distributed independent of $\mathcal{F}_{\tau_i-}$. If we let $U_t = ue^{-\beta t} + \sum_{i:0<\tau_i<t} C_i e^{-\beta(t-\tau_i)}$. Then U_t satisfies the dynamics (1.5) with $\eta(u) := \beta u$. When $\lambda(\cdot)$ is linear, it is called linear Hawkes process, named after Hawkes [15]. The linear Hawkes process can be studied via immigration-birth representation, see e.g. Hawkes and Oakes [16]. When $\lambda(\cdot)$ is nonlinear, the Hawkes process is said to be nonlinear and the nonlinear Hawkes process was first introduced by Brémaud and Massoulié [9]. The limit theorems for linear and nonlinear Hawkes processes have been studied in e.g. [6, 8, 28, 29, 24, 27, 25, 17]. The applications of Hawkes processes to insurance have been studied in e.g. [12, 22, 26]. As a by-product and corollary of the ruin probabilities results obtained in this paper, the first-passage time for nonlinear Hawkes process with exponential kernel and exponentially distributed jump sizes is therefore also analytically tractable, which is of independent interest and is a new contribution to the theory of Hawkes processes.

The paper is organized as follows. In Section 2, we will derive the ruin probability for the wealth process U_t in closed-forms. Expected dividends, first and second moments of the wealth process, Laplace transform of the ruin time and expected ruin time, will also be studied. We will illustrate our results by many examples for which more explicit formulas are obtained. We will also give numerical examples. The proofs will be provided in Section 3.

2. Main Results

2.1. Ruin Probability.

Theorem 1. *Assume that* $\int_0^\infty \frac{\lambda(v)}{\eta(v)} e^{\gamma v - \int_0^v \frac{\lambda(w)}{\eta(w)} dw} dv$ *exists and is finite. Then, the ruin probability* $\psi(u) = \mathbb{P}(\tau < \infty | U_0 = u)$ *is given by*

$$\psi(u) = \mathbb{P}(\tau < \infty) = \frac{\int_u^\infty \frac{\lambda(v)}{\eta(v)} e^{\gamma v - \int_0^v \frac{\lambda(w)}{\eta(w)} dw} dv}{\int_0^\infty \frac{\lambda(v)}{\eta(v)} e^{\gamma v - \int_0^v \frac{\lambda(w)}{\eta(w)} dw} dv}. \tag{2.1}$$

Figure 1 and Figure 2 are illustrations of the wealth process again time till the time when the company is ruined. In Figure 1, $\eta(z)$ is a constant and we can see that the wealth process always decays with the constant rate. In Figure 2, $\eta(z)$ is linear in z, i.e. $\eta(z) = \alpha + \beta z$, for some $\alpha, \beta > 0$ and the wealth process decays exponentially and might get ruined. A nonparametric approach to the decay function $\eta(z)$ gives us more flexibility.

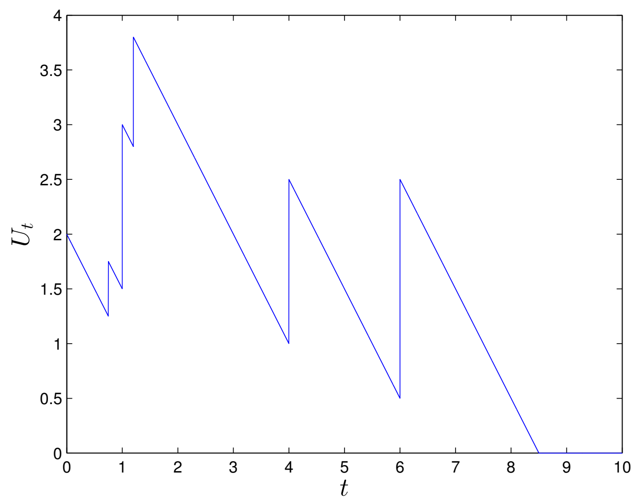

FIGURE 1. An illustration of the wealth process against time till the company is ruined.

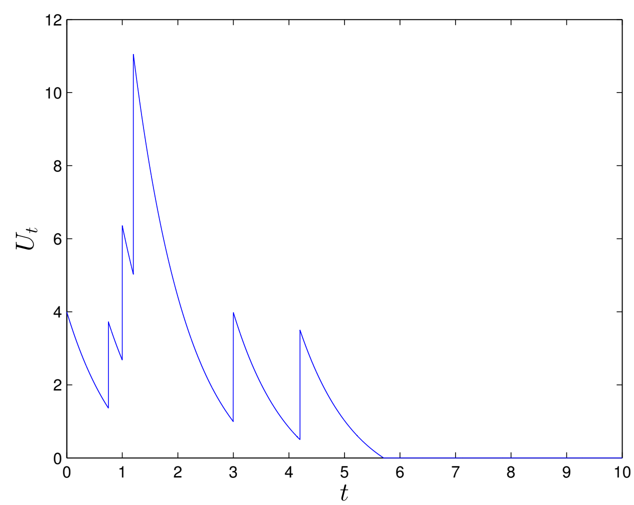

FIGURE 2. An illustration of the wealth process against time till the company is ruined.

Example 2. *Assume that* $U_t = u - \rho t + \sum_{i=1}^{N_t} C_i$, *where for any* $t \leq \tau$, N_t *is a simple point process whose intensity depends linearly on the wealth process, i.e. with intensity* $\alpha + \beta U_{t-}$ *for some* $\alpha, \beta > 0$. *That is* $\eta(v) = \rho$ *and* $\lambda(v) = \alpha + \beta v$ *for* *any* $v \geq 0$. *Hence, the ruin probability is given by*

$$
\begin{aligned}
\psi(u) &= \frac{\int_u^\infty \frac{\alpha+\beta v}{\rho} e^{\gamma v - \int_0^v \frac{\alpha+\beta w}{\rho} dw} dv}{\int_0^\infty \frac{\alpha+\beta v}{\rho} e^{\gamma v - \int_0^v \frac{\alpha+\beta w}{\rho} dw} dv} \\
&= \frac{\int_u^\infty (\alpha+\beta v) e^{\gamma v - \frac{1}{2\rho\beta}(\alpha+\beta v)^2} dv}{\int_0^\infty (\alpha+\beta v) e^{\gamma v - \frac{1}{2\rho\beta}(\alpha+\beta v)^2} dv} \\
&= \frac{-\rho e^{-\frac{\alpha^2}{2\rho\beta} - \alpha x - \frac{\beta}{2\rho} x^2 + \gamma x} + \frac{\sqrt{\pi}\gamma}{4\beta^2} (2\rho\beta)^{3/2} e^{\frac{\gamma(\rho\gamma - 2\alpha)}{2\beta}} erf\left(\frac{\alpha+\beta x - \gamma\rho}{\sqrt{2\beta\rho}}\right) \Big|_u^\infty}{-\rho e^{-\frac{\alpha^2}{2\rho\beta} - \alpha x - \frac{\beta}{2\rho} x^2 + \gamma x} + \frac{\sqrt{\pi}\gamma}{4\beta^2} (2\rho\beta)^{3/2} e^{\frac{\gamma(\rho\gamma - 2\alpha)}{2\beta}} erf\left(\frac{\alpha+\beta x - \gamma\rho}{\sqrt{2\beta\rho}}\right) \Big|_0^\infty} \\
&= \frac{\frac{\sqrt{\pi}\gamma}{4\beta^2} (2\rho\beta)^{3/2} e^{\frac{\gamma(\rho\gamma - 2\alpha)}{2\beta}} \left[1 - erf\left(\frac{\alpha+\beta u - \gamma\rho}{\sqrt{2\beta\rho}}\right)\right] + \rho e^{-\frac{\alpha^2}{2\rho\beta} - \alpha u - \frac{\beta}{2\rho} u^2 + \gamma u}}{\frac{\sqrt{\pi}\gamma}{4\beta^2} (2\rho\beta)^{3/2} e^{\frac{\gamma(\rho\gamma - 2\alpha)}{2\beta}} \left[1 - erf\left(\frac{\alpha - \gamma\rho}{\sqrt{2\beta\rho}}\right)\right] + \rho e^{-\frac{\alpha^2}{2\rho\beta}}},
\end{aligned}
\tag{2.2}
$$

where $erf(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ *is the error function.*

Example 3. *Assume that* $\lambda(v) = \mu\eta(v)$ *for some constant* $\mu > \gamma$. *Then, the ruin probability is explicitly given by*

$$\psi(u) = \frac{\int_u^\infty e^{-(\mu-\gamma)v} dv}{\int_0^\infty e^{-(\mu-\gamma)v} dv} = e^{-(\mu-\gamma)u}. \tag{2.3}$$

In general, if we assume that the profits C_i *are i.i.d. with probability distribution* $Q(dc)$, *then,*

$$\mathcal{A}f(u) = -\eta(u)\frac{\partial f}{\partial u} + \lambda(u) \int_0^\infty [f(u+c) - f(u)] Q(dc) = 0. \tag{2.4}$$

We aim to find f *such that* $\mathcal{A}f = 0$. *In general, this may not yield closed form solutions. In the special case that* $\lambda(u) = \mu\eta(u)$, *for some* $\mu > \frac{1}{\mathbb{E}^Q[c]}$, *then,* $\mathcal{A}f(u) = 0$ *reduces to*

$$-f'(u) + \mu \int_0^\infty [f(u+c) - f(u)] Q(dc) = 0. \tag{2.5}$$

Let us try the Ansatz $f(u) = e^{\theta u}$. *Then, we get*

$$-\theta + \mu(\mathbb{E}^Q[e^{\theta c}] - 1) = 0. \tag{2.6}$$

The function $F(\theta) := -\theta + \mu(\mathbb{E}^Q[e^{\theta c}] - 1)$ *is convex in* θ *and* $F(0) = 0$. *Since* $F'(0) = -1 + \mu\mathbb{E}^Q[c] > 0$, *we conclude that* $F(\theta) = 0$ *has a unique negative solution* θ^*. *Then,* $f(u) = e^{\theta^* u}$ *and* $f(\infty) = 0$. *Therefore,*

$$\psi(u) = \mathbb{P}(\tau < \infty) = e^{\theta^* u}. \tag{2.7}$$

Example 4. *Assume that* $\lambda(v) = (\alpha + \frac{\beta}{\sqrt{v}})\eta(v)$ *for some constant* $\alpha > \gamma$ *and* $\beta > 0$. *Then, the ruin probability is given by*

$$
\begin{aligned}
\psi(u) &= \frac{\int_u^\infty (\alpha + \frac{\beta}{\sqrt{v}})e^{\gamma v - \alpha v - 2\beta\sqrt{v}}dv}{\int_0^\infty (\alpha + \frac{\beta}{\sqrt{v}})e^{\gamma v - \alpha v - 2\beta\sqrt{v}}dv} \\
&= \frac{-\frac{\sqrt{\pi}\beta\gamma}{(\alpha-\gamma)^{3/2}}e^{\frac{\beta^2}{\alpha-\gamma}}\, erf(\frac{\sqrt{x}(\alpha-\gamma)+\beta}{\sqrt{\alpha-\gamma}}) - \frac{\alpha}{\alpha-\gamma}e^{-\alpha x - 2\beta\sqrt{x}+\gamma x}\Big|_u^\infty}{-\frac{\sqrt{\pi}\beta\gamma}{(\alpha-\gamma)^{3/2}}e^{\frac{\beta^2}{\alpha-\gamma}}\, erf(\frac{\sqrt{x}(\alpha-\gamma)+\beta}{\sqrt{\alpha-\gamma}}) - \frac{\alpha}{\alpha-\gamma}e^{-\alpha x - 2\beta\sqrt{x}+\gamma x}\Big|_0^\infty} \\
&= \frac{\frac{\sqrt{\pi}\beta\gamma}{(\alpha-\gamma)^{3/2}}e^{\frac{\beta^2}{\alpha-\gamma}}\left[erf(\frac{\sqrt{u}(\alpha-\gamma)+\beta}{\sqrt{\alpha-\gamma}}) - 1\right] + \frac{\alpha}{\alpha-\gamma}e^{-\alpha u - 2\beta\sqrt{u}+\gamma u}}{\frac{\sqrt{\pi}\beta\gamma}{(\alpha-\gamma)^{3/2}}e^{\frac{\beta^2}{\alpha-\gamma}}\left[erf(\frac{\beta}{\sqrt{\alpha-\gamma}}) - 1\right] + \frac{\alpha}{\alpha-\gamma}},
\end{aligned}
\tag{2.8}
$$

where $erf(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ *is the error function.*

Example 5. *Assume that* $\lambda(v) = (\alpha v^\beta + \gamma)\eta(v)$, *for some constants* $\alpha, \beta > 0$. *Then, the ruin probability is given by*

$$
\begin{aligned}
\psi(u) &= \frac{\int_u^\infty (\alpha v^\beta + \gamma)e^{-\frac{\alpha}{\beta+1}v^{\beta+1}}dv}{\int_0^\infty (\alpha v^\beta + \gamma)e^{-\frac{\alpha}{\beta+1}v^{\beta+1}}dv} \\
&= \frac{-\frac{\gamma}{\beta+1}(\frac{\alpha}{\beta+1})^{-\frac{1}{\beta+1}}\Gamma(\frac{1}{\beta+1}, \frac{\alpha x^{\beta+1}}{\beta+1}) - e^{-\frac{\alpha}{\beta+1}x^{\beta+1}}\Big|_u^\infty}{-\frac{\gamma}{\beta+1}(\frac{\alpha}{\beta+1})^{-\frac{1}{\beta+1}}\Gamma(\frac{1}{\beta+1}, \frac{\alpha x^{\beta+1}}{\beta+1}) - e^{-\frac{\alpha}{\beta+1}x^{\beta+1}}\Big|_0^\infty} \\
&= \frac{e^{-\frac{\alpha}{\beta+1}u^{\beta+1}} + \frac{\gamma}{\beta+1}(\frac{\alpha}{\beta+1})^{-\frac{1}{\beta+1}}\Gamma(\frac{1}{\beta+1}, \frac{\alpha u^{\beta+1}}{\beta+1})}{1 + \frac{\gamma}{\beta+1}(\frac{\alpha}{\beta+1})^{-\frac{1}{\beta+1}}\Gamma(\frac{1}{\beta+1}, 0)},
\end{aligned}
\tag{2.9}
$$

where $\Gamma(s, x) := \int_x^\infty t^{s-1}e^{-t}dt$ *is the incomplete gamma function.*

Example 6. *Assume that* $\lambda(v) = (\alpha - \frac{\beta}{1+v})\eta(v)$, *for some constants* $\alpha > \gamma$ *and* $\alpha > \beta > 1$. *Then, the ruin probability is given by*

$$
\begin{aligned}
\psi(u) &= \frac{\int_u^\infty (\alpha - \frac{\beta}{1+v})e^{-(\alpha-\gamma)v + \beta\log(v+1)}dv}{\int_0^\infty (\alpha - \frac{\beta}{1+v})e^{-(\alpha-\gamma)v + \beta\log(v+1)}dv} \\
&= \frac{\int_u^\infty [\alpha(v+1)^\beta - \beta(v+1)^{\beta-1}]e^{-(\alpha-\gamma)v}dv}{\int_0^\infty [\alpha(v+1)^\beta - \beta(v+1)^{\beta-1}]e^{-(\alpha-\gamma)v}dv} \\
&= \frac{\beta(\alpha-\gamma)\Gamma(\beta, (\alpha-\gamma)(x+1)) - \alpha\Gamma(\beta+1, (\alpha-\gamma)(x+1))\Big|_u^\infty}{\beta(\alpha-\gamma)\Gamma(\beta, (\alpha-\gamma)(x+1)) - \alpha\Gamma(\beta+1, (\alpha-\gamma)(x+1))\Big|_0^\infty} \\
&= \frac{-\beta(\alpha-\gamma)\Gamma(\beta, (\alpha-\gamma)(u+1)) + \alpha\Gamma(\beta+1, (\alpha-\gamma)(u+1))}{-\beta(\alpha-\gamma)\Gamma(\beta, (\alpha-\gamma)) + \alpha\Gamma(\beta+1, (\alpha-\gamma))},
\end{aligned}
\tag{2.10}
$$

where $\Gamma(s, x) := \int_x^\infty t^{s-1}e^{-t}dt$ *is the incomplete gamma function.*

Example 7. *Assume that* $\lambda(v)=(\gamma+\frac{\beta}{1+v})\eta(v)$, *for some constant* $\beta>1$. *Then, the ruin probability is given by*

$$\begin{aligned}\psi(u)&=\frac{\int_u^\infty(\gamma+\frac{\beta}{1+v})e^{-\beta\log(v+1)}dv}{\int_0^\infty(\gamma+\frac{\beta}{1+v})e^{-\beta\log(v+1)}dv}\\&=\frac{\gamma}{\gamma+\beta-1}\frac{1}{(1+u)^{\beta-1}}+\frac{\beta-1}{\gamma+\beta-1}\frac{1}{(1+u)^{\beta}}.\end{aligned}\tag{2.11}$$

Remark 8. *One way to interpretate the formula for the ruin probability is to write it as*

$$\psi(u)=\frac{\mathbb{E}[e^{\gamma V}1_{V\geq u}]}{\mathbb{E}[e^{\gamma V}]},\tag{2.12}$$

where V *is a positive random variable with probability density function* $\frac{\lambda(v)}{\eta(v)}e^{-\int_0^v\frac{\lambda(w)}{\eta(w)}dw}$.

2.2. **Expected Dividends.** One can also study the single dividend payment problem. Let $b>U_0$ be the barrier of the dividend. For the first time that the wealth process U_t goes above the barrier b, say at the first-passage time $\tau_b:=\inf\{t>0:U_t\geq b\}$, a dividend of the amount $D=U_{\tau_b}-b$ is paid out. No dividend is paid out if the company is ruined before ever hitting the barrier b. We are interested to compute that expected value of the dividend to be paid out $\mathbb{E}[D1_{\tau_b<\tau}]$.

Note that under the assumption that the C_i's are i.i.d. exponentially distributed with parameter $\gamma>0$, from the memoryless property of exponential distribution, $U_{\tau_b}-b$ is also exponentially distributed with parameter $\gamma>0$. Therefore,

$$\mathbb{E}[D1_{\tau_b<\tau}]=\frac{1}{\gamma}\mathbb{P}(\tau_b<\tau).\tag{2.13}$$

Hence, the problem reduces to compute the probability that the dividend will be paid out before the company is ruined.

Theorem 9. *Assume that* $\int_0^\infty\frac{\lambda(v)}{\eta(v)}e^{\gamma v-\int_0^v\frac{\lambda(w)}{\eta(w)}dw}dv$ *exists and is finite. Then, the probability* $\phi(u,b):=\mathbb{P}(\tau_b<\tau|U_0=u)$ *is given by*

$$\phi(u,b)=\frac{\int_0^u\frac{\lambda(v)}{\eta(v)}e^{\gamma v-\int_0^v\frac{\lambda(w)}{\eta(w)}dw}dv}{\int_0^\infty\gamma e^{-\gamma c}\int_0^{b+c}\frac{\lambda(v)}{\eta(v)}e^{\gamma v-\int_0^v\frac{\lambda(w)}{\eta(w)}dw}dvdc},\tag{2.14}$$

and the expected dividend is given by

$$\mathbb{E}[D1_{\tau_b<\tau}]=\frac{1}{\gamma}\frac{\int_0^u\frac{\lambda(v)}{\eta(v)}e^{\gamma v-\int_0^v\frac{\lambda(w)}{\eta(w)}dw}dv}{\int_0^\infty\gamma e^{-\gamma c}\int_0^{b+c}\frac{\lambda(v)}{\eta(v)}e^{\gamma v-\int_0^v\frac{\lambda(w)}{\eta(w)}dw}dvdc}.\tag{2.15}$$

One can consider multiple dividend payments as follows:

$$\tau_b^{(1)}:=\inf\{t>0:U_t>b\},\qquad\tau_b^{(i)}:=\inf\{t>\tau_b^{(i)}:U_t>b\},i\geq 2.\tag{2.16}$$

Then $\tau_b^{(i)}$ is the ith payment of the dividend if $\tau_b^{(i)}<\tau$.

Let N be the total number of dividends to be paid out before the ruin occurs and $\sum_{i=1}^N D_i$ be the total value of dividends to be paid out before the ruin occurs.

Recall that $\phi(u,b) = \mathbb{P}(\tau_b < \tau | U_0 = u)$, was computed in Theorem 9 with closed-form formulas. It is easy to see that

$$\mathbb{P}(N=0) = 1-\phi(u,b), \tag{2.17}$$

$$\mathbb{P}(N=n) = \phi(u,b)\phi(b,b)^{n-1}(1-\phi(b,b)), \qquad n \geq 1. \tag{2.18}$$

Therefore,

$$\mathbb{E}\left[\sum_{i=1}^{N} D_i\right] = \frac{1}{\gamma}\mathbb{E}[N] = \frac{1}{\gamma}\frac{\phi(u,b)}{1-\phi(b,b)}. \tag{2.19}$$

One can also compute the Laplace transform of the total amount of dividends to be paid out. For any $\theta > 0$,

(2.20)

$$\begin{aligned}\mathbb{E}\left[e^{-\theta\sum_{i=1}^{N} D_i}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-\theta\sum_{i=1}^{N} D_i}\middle| N\right]\right] \\ &= \mathbb{E}\left[\left(\frac{\gamma}{\gamma+\theta}\right)^{N}\right] \\ &= 1-\phi(u,b)+\phi(u,b)\sum_{n=1}^{\infty}\left(\frac{\gamma}{\gamma+\theta}\right)^{n}\phi(b,b)^{n-1}(1-\phi(b,b)) \\ &= 1-\phi(u,b)+\phi(u,b)(1-\phi(b,b))\frac{\frac{\gamma}{\gamma+\theta}}{1-\frac{\gamma}{\gamma+\theta}\phi(b,b)}.\end{aligned}$$

Example 10. *Assume that $\lambda(v) = \mu\eta(v)$ for some constant $\mu > \gamma$. Then, we can take $f(x) = e^{-(\mu-\gamma)x}$ and*

$$\int_0^\infty f(b+c)\gamma e^{-\gamma c}dc = \frac{\gamma}{\mu}e^{-\mu b}. \tag{2.21}$$

Example 11. *Assume that $\eta(v) = \rho$ and $\lambda(v) = \alpha + \beta v$. Then, we can take*

$$f(x) = -\rho e^{-\frac{\alpha^2}{2\rho\beta}-\alpha x-\frac{\beta}{2\rho}x^2+\gamma x} + \frac{\sqrt{\pi}\gamma}{4\beta^2}(2\rho\beta)^{3/2}e^{\frac{\gamma(\rho\gamma-2\alpha)}{2\beta}} erf\left(\frac{\alpha+\beta x-\gamma\rho}{\sqrt{2\beta\rho}}\right). \tag{2.22}$$

We can compute that

$$\int_0^\infty f(b+c)\gamma e^{-\gamma c}dc = \frac{\sqrt{\pi}\gamma}{\sqrt{2\beta}}\rho^{3/2}e^{\frac{\gamma(\rho\gamma-2\alpha)}{2\beta}} erf\left(\frac{\alpha+\beta b-\gamma\rho}{\sqrt{2\beta\rho}}\right). \tag{2.23}$$

Example 12. *Assume that $\lambda(v) = (\alpha - \frac{\beta}{1+v})\eta(v)$, for some constants $\alpha > \gamma$ and $\alpha > \beta > 1$. Then, we can take*

$$f(x) = \beta(\alpha-\gamma)\Gamma(\beta,(\alpha-\gamma)(x+1)) - \alpha\Gamma(\beta+1,(\alpha-\gamma)(x+1)), \tag{2.24}$$

and we can compute that

$$\begin{aligned}\int_0^\infty & f(b+c)\gamma e^{-\gamma c}dc \\ &= -\beta(\alpha-\gamma)e^{(b+1)\gamma}(\alpha-\gamma)^{\beta}(\beta-1)\alpha^{-\beta}\Gamma(\beta-1,(b+1)\alpha) \\ &\quad -\beta(\alpha-\gamma)e^{(b+1)\gamma}(\alpha-\gamma)^{\beta}(b+1)^{\beta-1}\alpha^{-1}e^{-\alpha(b+1)} \\ &\qquad +\beta(\alpha-\gamma)\Gamma(\beta,(b+1)(\alpha-\gamma)) \\ &\quad +\alpha^{-\beta}e^{(b+1)\gamma}(\alpha-\gamma)^{\beta+1}\Gamma(\beta+1,(b+1)\alpha)-\alpha\Gamma(\beta+1,(b+1)(\alpha-\gamma)).\end{aligned} \tag{2.25}$$

Example 13. *Assume that* $\lambda(v)=(\alpha+\frac{\beta}{\sqrt{v}})\eta(v)$ *for some* $\alpha>\gamma$ *and* $\beta>0$. *Then, we can take*

$$f(x)=\frac{\sqrt{\pi}\beta\gamma}{(\alpha-\gamma)^{3/2}}e^{\frac{\beta^2}{\alpha-\gamma}}\mathit{erf}\left(\frac{\sqrt{x}(\alpha-\gamma)+\beta}{\sqrt{\alpha-\gamma}}\right)-\frac{\alpha}{\alpha-\gamma}e^{-\alpha x-2\beta\sqrt{x}+\gamma x}. \tag{2.26}$$

And we can compute that

$$\begin{aligned}\int_0^\infty & f(b+c)\gamma e^{-\gamma c}dc \\ =&\frac{\sqrt{\pi}\beta\gamma}{(\alpha-\gamma)^{3/2}}e^{\frac{\beta^2}{\alpha-\gamma}}\left[e^{b\gamma}\frac{e^{-\frac{\beta^2\gamma}{(\alpha-\gamma)(\gamma+1)}}}{\sqrt{\gamma+1}}\mathit{erfc}\left(\frac{\sqrt{b}(\gamma+1)+\frac{\beta}{\sqrt{\alpha-\gamma}}}{\sqrt{\gamma+1}}\right)+\mathit{erf}\left(\sqrt{b}+\frac{\beta}{\sqrt{\alpha-\gamma}}\right)\right] \\ &-\frac{\alpha}{\alpha-\gamma}\gamma\left[-\frac{\sqrt{\pi}\beta e^{\frac{\beta^2}{\alpha}+b\gamma}}{\alpha^{3/2}}\mathit{erfc}\left(\frac{\alpha\sqrt{b}+\beta}{\sqrt{\alpha}}\right)+\frac{e^{b\gamma-\alpha b-2\beta\sqrt{b}}}{\alpha}\right].\end{aligned} \tag{2.27}$$

where $\mathit{erfc}(x):=1-\mathit{erf}(x)$ *is the complementary error function.*

2.3. **First and Second Moments of the Wealth Process.** We are also interested to study the first and second moments of the wealth process U_t. Note that since the wealth process is defined only up to the ruin time τ, we should evaluate $\mathbb{E}[U_{t\wedge\tau}]$ and $\mathbb{E}[U_{t\wedge\tau}^2]$, which in general is a challenge to compute since it will require us to know explicitly the distribution of the ruin time. We derive the first and second moments of the wealth process U_t for a special case instead. Let $\eta(u)\equiv\rho+\mu u$ and $\lambda(u)=\alpha+\beta u$, for some $\alpha,\beta\geq 0$, i.e.

$$dU_t=-(\rho+\mu U_t)dt+dJ_t \tag{2.28}$$

where $J_t=\sum_{i=1}^{N_t}C_i$, where N_t is a simple point process with intensity $\lambda(U_{t-})=\alpha+\beta U_{t-}$ and C_i are i.i.d. with distribution $Q(dc)$.

In this case, $\tau\geq T_0$, where T_0 is the time that the ODE

$$du_t=-(\rho+\mu u_t)dt,\qquad u_t=u, \tag{2.29}$$

hits zero. It is easy to solve the above ODE and get

$$u_t=\left(\frac{\rho}{\mu}+u\right)e^{-\mu t}-\frac{\rho}{\mu},\qquad T_0=\frac{1}{\mu}\log\left(1+\frac{\mu u}{\rho}\right). \tag{2.30}$$

Then, for any $t<T_0$, $t\wedge\tau=t$.

Proposition 14. *For any* $t < \frac{1}{\mu}\log\left(1+\frac{\mu u}{\rho}\right)$,

$$\mathbb{E}[U_t] = \left(\frac{\rho-\alpha\mathbb{E}^Q[c]}{\mu-\beta\mathbb{E}^Q[c]}+u\right)e^{-(\mu-\beta\mathbb{E}^Q[c])t} - \frac{\rho-\alpha\mathbb{E}^Q[c]}{\mu-\beta\mathbb{E}^Q[c]}, \tag{2.31}$$

and

$$\begin{aligned}\mathbb{E}[U_t^2] &= ue^{-2\beta(\mathbb{E}^Q[c]-\mu)t} + \alpha\mathbb{E}^Q[c^2]\frac{1-e^{-2\beta(\mathbb{E}^Q[c]-\mu)t}}{2\beta(\mathbb{E}^Q[c]-\mu)} \\ &\quad - (2(\alpha\mathbb{E}^Q[c]-\rho)+\beta\mathbb{E}[c^2])\frac{\rho-\alpha\mathbb{E}^Q[c]}{\mu-\beta\mathbb{E}^Q[c]}\frac{1-e^{-2\beta(\mathbb{E}^Q[c]-\mu)t}}{2\beta(\mathbb{E}^Q[c]-\mu)} \\ &\quad + (2(\alpha\mathbb{E}^Q[c]-\rho)+\beta\mathbb{E}[c^2])\left(\frac{\rho-\alpha\mathbb{E}^Q[c]}{\mu-\beta\mathbb{E}^Q[c]}+u\right)\frac{e^{-\beta(\mathbb{E}^Q[c]-\mu)t}-e^{-2\beta(\mathbb{E}^Q[c]-\mu)t}}{\beta(\mathbb{E}^Q[c]-\mu)}.\end{aligned}$$

2.4. **Laplace Transform of Ruin Time.** In the ruin theory of the dual risk models, it is of great interest to study the Laplace transform of the ruin time,

$$\psi(u,\delta) = \mathbb{E}\left[e^{-\delta\tau}1_{\tau<\infty}\right], \tag{2.32}$$

where $\delta > 0$. Note that $\psi(u,\delta)$ can also be interpreted as a perpetual digit option, with payoff 1 dollar at the time of ruin, with discount coefficient $\delta > 0$, which can be taken as the risk-free rate.

Theorem 15. *Assume that the equation*

$$\begin{aligned}&\lambda(u)\eta(u)f''(u) + [\lambda(u)\eta'(u)+\lambda^2(u)-\gamma\eta(u)\lambda(u)-\lambda'(u)\eta(u)+\delta\lambda(u)]f'(u) \\ &\qquad - (\gamma\lambda(u)+\lambda'(u))\delta f(u) = 0\end{aligned} \tag{2.33}$$

has a uniformly bounded positive solution $f(u)$ *that satisfies* $f(\infty)=0$. *Then, we have* $\psi(u,\delta) = f(u)/f(0)$.

In general, a second-order linear ODE with non-constant coefficients do not yield closed-form solutions. Nevertheless, there are a wide range of special cases that do yield analytical solutions.

Example 16. $\lambda(u)\equiv\lambda$, $\eta(u)=\rho+\mu u$. *Then, we have*

$$[\lambda\rho+\lambda\mu u]f''(u) + [(\lambda\mu+\lambda^2-\gamma\lambda\rho+\delta\lambda)-\gamma\lambda\mu u\lambda]f'(u) - \gamma\lambda\delta f(u) = 0. \tag{2.34}$$

This is a special 2nd order ODE that has a solution, see e.g. Polyanin and Zaitsev [20]

$$f(u) = e^{\gamma u}J\left(\frac{\mu+\lambda}{\mu}, \frac{\mu+\lambda+\delta}{\mu}; -\gamma u - \frac{\rho\gamma}{\mu}\right), \tag{2.35}$$

where $J(a,b;x)$ *is a solution to the degenerate hypergeometric equation*

$$xy''(x) + (b-x)y'(x) - ay(x) = 0, \tag{2.36}$$

which has the solution in the case $b > a > 0$:

$$J(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{b-a-1}dt, \tag{2.37}$$

where $\Gamma(\cdot)$ *is the gamma function.*

Example 17. *Assume that* $\lambda(u) = \mu e^{-\gamma u}$. *Then,*

$$\eta(u)f''(u) + [\eta'(u) + \mu e^{-\gamma u} + \delta]f'(u) = 0, \tag{2.38}$$

which yields that

$$f(u) = \int_0^u \frac{1}{\eta(v)} e^{-\int_0^v \frac{\mu e^{-\gamma w}+\delta}{\eta(w)}dw} dv. \tag{2.39}$$

Example 18. *Assume that* $\lambda(u) = \gamma\eta(u)$. *Then,*

$$\eta^2(u)f''(u) + \delta\eta(u)f'(u) - (\gamma\eta(u) + \eta'(u))\delta f(u) = 0. \tag{2.40}$$

Further assume that $\eta(u) = \frac{1}{\alpha+\beta u}$, *where* $\alpha, \beta > 0$. *Then,*

$$f''(u) + (\delta\beta u + \delta\alpha)f'(u) + (-\delta\gamma\beta u + \delta\beta - \delta\gamma\alpha)f(u) = 0, \tag{2.41}$$

which yields the solution

$$f(u) = e^{\gamma u} J\left(\frac{\gamma^2 + \delta\beta}{2\delta\beta}, \frac{1}{2}; \frac{-\delta\beta}{2}\left(u + \frac{2\gamma + \delta\alpha}{\delta\beta}\right)^2\right), \tag{2.42}$$

where J *was defined in* (2.37).

Example 19. *Assume that* $\eta(u) \equiv \eta$ *is a constant and* $\lambda(u) = \mu e^{\lambda u}$ *for some* $\mu, \lambda > 0$. *Then, we get*

$$f''(u) + \left[\frac{\mu}{\eta}e^{\lambda u} - \gamma - \lambda + \frac{\delta}{\eta}\right] f'(u) - \left[\frac{\gamma}{\eta} + \frac{\lambda}{\eta}\right] \delta f(u) = 0, \tag{2.43}$$

and it is equivalent to

$$f''(u) + (ae^{\lambda u} + b)f'(u) + cf(u) = 0, \tag{2.44}$$

where

$$a := \frac{\mu}{\eta}, \quad b := -\gamma - \lambda + \frac{\delta}{\eta}, \quad c := -\left[\frac{\gamma}{\eta} + \frac{\lambda}{\eta}\right]. \tag{2.45}$$

By letting $\xi = e^u$, (2.44) *reduces to*

$$\xi^2 f_{\xi\xi} + (a\xi^\lambda + b + 1)\xi f_\xi + cf_\xi = 0, \tag{2.46}$$

and by letting $z = \xi^\lambda$, $w = fz^{-k}$, *where* k *satisfies the quadratic equation*

$$\lambda^2 k^2 + \lambda bk + c = 0, \tag{2.47}$$

we have that (2.46) *reduces to*

$$\lambda^2 zw_{zz} + [\lambda az + 2k\lambda^2 + \lambda(\lambda + b)]w_z + k\lambda a = 0, \tag{2.48}$$

which has solution, see e.g. [20]

$$w(z) = J\left(k, 2k + 1 + \frac{b}{\lambda}; -\frac{a}{\lambda}z\right), \tag{2.49}$$

where J *was defined in* (2.37).

2.5. **Expected Ruin Time.** We have already computed the ruin probability $\mathbb{P}(\tau < \infty)$ in Theorem 1 under certain assumptions. Note that when $\mathbb{P}(\tau < \infty) < 1$, $\mathbb{E}[\tau] = \infty$. In the case that the ruin occurs with probability one, i.e., $\mathbb{P}(\tau < \infty) = 1$, we can also compute that expected time that the ruin occurs.

Theorem 20. *Assume that* $\mathbb{P}(\tau < \infty) = 1$ *and let us define*

$$f(u) := \int_0^u \frac{\lambda(v)}{\eta(v)} \int_0^v [-\lambda'(w) - \gamma\lambda(w)] \frac{1}{\lambda(w)^2} e^{\gamma(v-w) - \int_w^v \frac{\lambda(r)}{\eta(r)} dr} dw dv + g(0) \int_0^u \frac{\eta(0)}{\eta(v)} \frac{\lambda(v)}{\lambda(0)} e^{\gamma v} e^{-\int_0^v \frac{\lambda(w)}{\eta(w)} dw} dv, \tag{2.50}$$

where

$$g(0) := \frac{1 + \lambda(0) \int_0^\infty \int_0^c \frac{\lambda(v)}{\eta(v)} \int_0^v [-\lambda'(w) - \gamma\lambda(w)] \frac{1}{\lambda(w)^2} e^{\gamma(v-w) - \int_w^v \frac{\lambda(r)}{\eta(r)} dr} \gamma e^{-\gamma(c-u)} dw dv dc}{\eta(0) - \lambda(0) \int_0^\infty \int_0^c \frac{\eta(0)}{\eta(v)} \frac{\lambda(v)}{\lambda(0)} e^{\gamma v} e^{-\int_0^v \frac{\lambda(w)}{\eta(w)} dw} \gamma e^{-\gamma(c-u)} dv dc}. \tag{2.51}$$

Assume that $\sup_{0<u<\infty} f(u) < \infty$. *Then,* $\mathbb{E}[\tau] = f(u)$.

2.6. **Numerical Examples.** In this section, we illustrate the ruin probability $\psi(u)$ obtained in Theorem 1 by some numerical examples. The summary statistics of the ruin probability $\psi(u)$ for the case $\lambda(u) = (\alpha u^\beta + \gamma)\eta(u)$ with fixed $\alpha = \gamma = 1.0$ and $\beta = 0.0, 0.5, 1.0, 1.5$ and 2.0 are given in Figure 3 and Table 3. The summary statistics of the ruin probability $\psi(u)$ for the case $\lambda(u) = (\gamma + \frac{\beta}{1+u})\eta(u)$ with fixed $\gamma = 1.0$ and $\beta = 1.5, 2.0, 2.5, 3.0$ and 3.5 are given in Figure 4 and Table 4. As we can see from Figure 3 and Figure 4, the shape of the ruin probability $\psi(u)$ in terms of the initial wealth u is not necessarily exponential. It exhibits a rich class of behaviors as we vary the parameter β. Therefore, the state-dependent dual risk model we have built is much more flexible and robust than many of the classical dual risk models in the literature.

$\psi(u)$	$u = 1$	$u = 2$	$u = 3$	$u = 4$	$u = 5$
$\beta = 0.0$	0.3679	0.1353	0.0498	0.0183	0.0067
$\beta = 0.5$	0.4325	0.1184	0.0233	0.0035	0.0004
$\beta = 1.0$	0.4867	0.0981	0.0076	0.0002	0.0000
$\beta = 1.5$	0.5343	0.0747	0.0013	0.0000	0.0000
$\beta = 2.0$	0.5756	0.0506	0.0001	0.0000	0.0000

TABLE 3. Illustration of the ruin probability $\psi(u)$ when $\lambda(u) = (\alpha u^\beta + \gamma)\eta(u)$ for fixed $\alpha = \gamma = 1$.

3. APPENDIX: PROOFS

Proof of Theorem 1. From (1.5), the infinitesimal generator of the wealth process U_t can be written as

$$\mathcal{A}f(u) = -\eta(u)\frac{\partial f}{\partial u} + \lambda(u) \int_0^\infty [f(u+c) - f(u)]\gamma e^{-\gamma c} dc. \tag{3.1}$$

Let us find a function f such that $\mathcal{A}f = 0$, which is equivalent to

$$-\eta(u)f'(u) - \lambda(u)f(u) + \lambda(u) \int_u^\infty f(c)\gamma e^{-\gamma(c-u)} dc = 0. \tag{3.2}$$

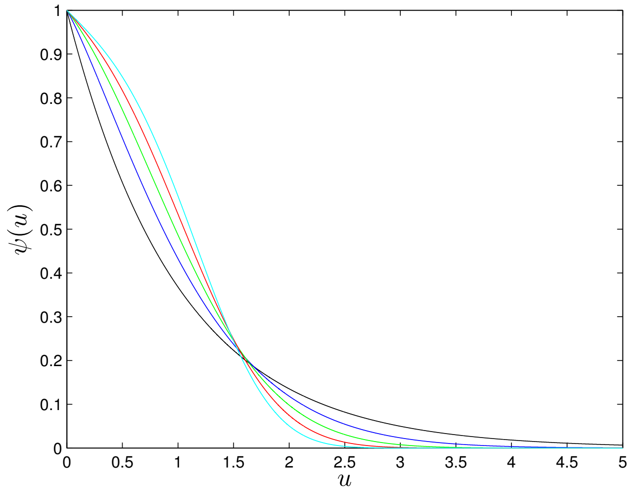

FIGURE 3. Illustration of the ruin probability $\psi(u)$ against the initial wealth u when $\lambda(u) = (\alpha u^{\beta} + \gamma)\eta(u)$. The black, blue, green, red and cion lines denote the cases when $\beta = 0.0$, 0.5, 1.0, 1.5 and 2.0. The α and γ are fixed to be 1.0. We can see from the plot that when $\beta = 0.0$, the ruin probability exponentially decays in the initial wealth. Otherwise, the shape of decay is not exponential.

$\psi(u)$	$u = 1$	$u = 2$	$u = 3$	$u = 4$	$u = 5$
$\beta = 1.5$	0.5893	0.4491	0.3750	0.3280	0.2948
$\beta = 2.0$	0.3750	0.2222	0.1562	0.1200	0.0972
$\beta = 2.5$	0.2475	0.1155	0.0688	0.0465	0.0340
$\beta = 3.0$	0.1667	0.0617	0.0312	0.0187	0.0123
$\beta = 3.5$	0.1136	0.0336	0.0145	0.0077	0.0046

TABLE 4. Illustration of the ruin probability $\psi(u)$ when $\lambda(u) = (\gamma + \frac{\beta}{1+u})\eta(u)$ for fixed $\gamma = 1$.

Differentiating (3.2) with respect to u, we get

(3.3)
$$\begin{aligned}&-\eta'(u)f'(u) - \eta(u)f''(u) - \lambda'(u)f(u) - \lambda(u)f'(u) + \lambda'(u)\int_u^\infty f(c)\gamma e^{-\gamma(c-u)}dc \\ &\qquad - \lambda(u)\gamma f(u) + \lambda(u)\gamma\int_u^\infty f(c)\gamma e^{-\gamma(c-u)}dc = 0.\end{aligned}$$

Substituting (3.2) into (3.3), we get

$$\eta(u)f''(u) = \left[\frac{\lambda'(u)}{\lambda(u)}\eta(u) + \gamma\eta(u) - \eta'(u) - \lambda(u)\right]f'(u). \tag{3.4}$$

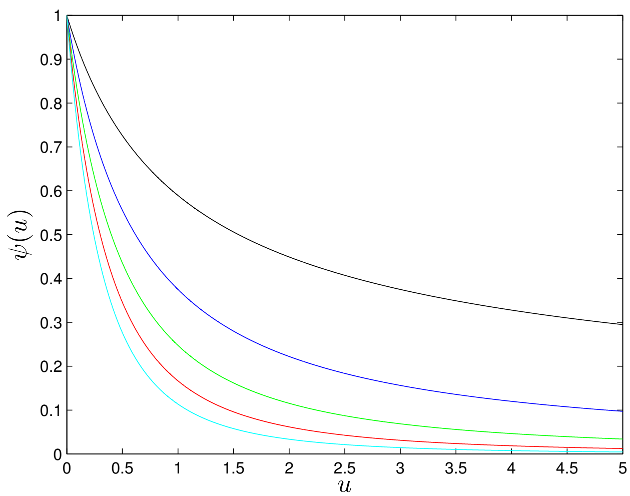

FIGURE 4. Illustration of the ruin probability $\psi(u)$ against the initial wealth u when $\lambda(u) = (\gamma + \frac{\beta}{1+u})\eta(u)$. The black, blue, green, red and cion lines denote the cases when $\beta = 1.5$, 2.0, 2.5, 3.0 and 3.5. The γ is fixed to be 1.0. The ruin probability decays polynomially against the initial wealth.

By letting $f'(u) = g(u)$, we have

$$\frac{dg}{g} = \left(\frac{\lambda'(u)}{\lambda(u)} + \gamma - \frac{\eta'(u)}{\eta(u)} - \frac{\lambda(u)}{\eta(u)} \right) du, \tag{3.5}$$

which implies that

$$g(u) = \frac{\lambda(u)}{\eta(u)} e^{\gamma u - \int_0^u \frac{\lambda(v)}{\eta(v)} dv} \tag{3.6}$$

is a particular solution to (3.5). Hence, $\mathcal{A}f(u) = 0$ for

$$f(u) := \int_0^u \frac{\lambda(v)}{\eta(v)} e^{\gamma v - \int_0^v \frac{\lambda(w)}{\eta(w)} dw} dv. \tag{3.7}$$

By our assumption, $f(\infty)$ exists and is finite and it is also clear that for any $0 \leq u \leq \infty$, $0 \leq f(u) \leq f(\infty) < \infty$. Hence, by optional stopping theorem,

$$f(u) = \mathbb{E}_u[f(U_\tau)] = f(0)\mathbb{P}(\tau < \infty) + f(\infty)\mathbb{P}(\tau = \infty), \tag{3.8}$$

which implies that

$$\psi(u) = \mathbb{P}(\tau < \infty) = \frac{\int_u^\infty \frac{\lambda(v)}{\eta(v)} e^{\gamma v - \int_0^v \frac{\lambda(w)}{\eta(w)} dw} dv}{\int_0^\infty \frac{\lambda(v)}{\eta(v)} e^{\gamma v - \int_0^v \frac{\lambda(w)}{\eta(w)} dw} dv}. \tag{3.9}$$

□

Proof of Theorem 9. Let us recall that for

$$f(u) = \int_0^u \frac{\lambda(v)}{\eta(v)} e^{\gamma v - \int_0^v \frac{\lambda(w)}{\eta(w)} dw} dv, \tag{3.10}$$

we have $\mathcal{A}f = 0$ and by our assumption f is uniformly bounded. By optional stopping theorem,

$$\begin{aligned} f(u) &= \mathbb{E}_u[f(U_{\tau\wedge\tau_b})] \\ &= f(0)\mathbb{P}(\tau < \tau_b) + \int_0^\infty f(b+c)\gamma e^{-\gamma c} dc \mathbb{P}(\tau_b < \tau), \end{aligned} \tag{3.11}$$

which implies that

$$\begin{aligned} \mathbb{E}[D 1_{\tau_b<\tau}] &= \frac{1}{\gamma}\mathbb{P}(\tau_b < \tau) \\ &= \frac{1}{\gamma}\frac{f(u)-f(0)}{\int_0^\infty f(b+c)\gamma e^{-\gamma c}dc - f(0)} \\ &= \frac{1}{\gamma}\frac{\int_0^u \frac{\lambda(v)}{\eta(v)} e^{\gamma v - \int_0^v \frac{\lambda(w)}{\eta(w)}dw} dv}{\int_0^\infty \gamma e^{-\gamma c}\int_0^{b+c}\frac{\lambda(v)}{\eta(v)} e^{\gamma v - \int_0^v \frac{\lambda(w)}{\eta(w)}dw} dv dc}. \end{aligned} \tag{3.12}$$

□

Proof of Proposition 14. The infinitesimal generator of U_t process is given by

$$\mathcal{A}f(u) = -(\rho+\mu u)f'(u) + (\alpha+\beta u)\int_0^\infty [f(u+c)-f(u)]Q(dc). \tag{3.13}$$

Let $f(u) = u$, we get $\mathcal{A}u = -\rho - \mu u + \mathbb{E}^Q[c](\alpha+\beta u)$. By Dynkin's formula,

$$\begin{aligned} \mathbb{E}[U_t] &= u + \int_0^t \mathbb{E}\left[-\rho - \mu U_s + \mathbb{E}^Q[c](\alpha+\beta U_s)\right] ds \\ &= u + (-\rho + \alpha\mathbb{E}^Q[c])t + (\beta\mathbb{E}^Q[c]-\mu)\int_0^t \mathbb{E}[U_s]ds, \end{aligned} \tag{3.14}$$

which yields that

$$\mathbb{E}[U_t] = \left(\frac{\rho-\alpha\mathbb{E}^Q[c]}{\mu-\beta\mathbb{E}^Q[c]} + u\right)e^{-(\mu-\beta\mathbb{E}^Q[c])t} - \frac{\rho-\alpha\mathbb{E}^Q[c]}{\mu-\beta\mathbb{E}^Q[c]}. \tag{3.15}$$

Let $f(u) = u^2$, we get $\mathcal{A}u^2 = \alpha\mathbb{E}^Q[c^2] + (2(\alpha\mathbb{E}^Q[c]-\rho)+\beta\mathbb{E}[c^2])u + 2(\beta\mathbb{E}[c]-\mu)u^2$. By Dynkin's formula,

$$\mathbb{E}[U_t^2] = u^2 + \alpha\mathbb{E}^Q[c^2]t + (2(\alpha\mathbb{E}^Q[c]-\rho)+\beta\mathbb{E}[c^2])\int_0^t \mathbb{E}[U_s]ds + 2(\beta\mathbb{E}[c]-\mu)\int_0^t \mathbb{E}[U_s^2]ds, \tag{3.16}$$

which implies the ODE:

$$\frac{d}{dt}\mathbb{E}[U_t^2] = \alpha\mathbb{E}^Q[c^2] + (2(\alpha\mathbb{E}^Q[c]-\rho)+\beta\mathbb{E}[c^2])\mathbb{E}[U_t] + 2(\beta\mathbb{E}[c]-\mu)\mathbb{E}[U_t^2]. \tag{3.17}$$

This is a first-order linear ODE. Together with (3.15), we get

(3.18)

$$\begin{aligned}\mathbb{E}[U_t^2] &= ue^{-2\beta(\mathbb{E}^Q[c]-\mu)t} + \int_0^t e^{-2\beta(\mathbb{E}^Q[c]-\mu)(t-s)}\alpha\mathbb{E}^Q[c^2]ds \\ &\quad + (2(\alpha\mathbb{E}^Q[c]-\rho)+\beta\mathbb{E}[c^2])\int_0^t e^{-2\beta(\mathbb{E}^Q[c]-\mu)(t-s)}\left(\frac{\rho-\alpha\mathbb{E}^Q[c]}{\mu-\beta\mathbb{E}^Q[c]}+u\right)e^{-(\mu-\beta\mathbb{E}^Q[c])s}ds \\ &\quad - (2(\alpha\mathbb{E}^Q[c]-\rho)+\beta\mathbb{E}[c^2])\int_0^t e^{-2\beta(\mathbb{E}^Q[c]-\mu)(t-s)}\frac{\rho-\alpha\mathbb{E}^Q[c]}{\mu-\beta\mathbb{E}^Q[c]}ds \\ &= ue^{-2\beta(\mathbb{E}^Q[c]-\mu)t} + \alpha\mathbb{E}^Q[c^2]\frac{1-e^{-2\beta(\mathbb{E}^Q[c]-\mu)t}}{2\beta(\mathbb{E}^Q[c]-\mu)} \\ &\quad - (2(\alpha\mathbb{E}^Q[c]-\rho)+\beta\mathbb{E}[c^2])\frac{\rho-\alpha\mathbb{E}^Q[c]}{\mu-\beta\mathbb{E}^Q[c]}\frac{1-e^{-2\beta(\mathbb{E}^Q[c]-\mu)t}}{2\beta(\mathbb{E}^Q[c]-\mu)} \\ &\quad + (2(\alpha\mathbb{E}^Q[c]-\rho)+\beta\mathbb{E}[c^2])\left(\frac{\rho-\alpha\mathbb{E}^Q[c]}{\mu-\beta\mathbb{E}^Q[c]}+u\right)\frac{e^{-\beta(\mathbb{E}^Q[c]-\mu)t}-e^{-2\beta(\mathbb{E}^Q[c]-\mu)t}}{\beta(\mathbb{E}^Q[c]-\mu)}.\end{aligned}$$

□

Proof of Theorem 15. Assume that we have a uniformly bounded positive function f such that $\mathcal{A}f=\delta f$. Note that $\frac{f(U_t)}{f(U_0)}e^{-\int_0^t\frac{\mathcal{A}f}{f}(U_s)ds}=\frac{f(U_t)}{f(U_0)}e^{-\delta t}$ is a martingale. By optional stopping theorem,

$$1=\mathbb{E}\left[\frac{f(U_\tau)}{f(U_0)}e^{-\int_0^\tau\frac{\mathcal{A}f}{f}(U_s)ds}\right]=\frac{f(0)}{f(u)}\mathbb{E}\left[e^{-\delta\tau}1_{\tau<\infty}\right]. \tag{3.19}$$

Therefore,

$$\psi(u,\delta)=f(u)/f(0). \tag{3.20}$$

Let us now try to find a function f such that $\mathcal{A}f=\delta f$. Note that $\mathcal{A}f=\delta f$ is equivalent to

$$-\eta(u)f'(u)-\lambda(u)f(u)+\lambda(u)\int_0^\infty f(u+c)\gamma e^{-\gamma c}dc=\delta f, \tag{3.21}$$

which implies that

(3.22)

$$\begin{aligned}\lambda(u)\eta(u)f''(u)+[\lambda(u)\eta'(u)+\lambda^2(u)-\gamma\eta(u)\lambda(u)-\lambda'(u)\eta(u)+\delta\lambda(u)]f'(u) \\ -(\gamma\lambda(u)+\lambda'(u))\delta f(u)=0.\end{aligned}$$

□

Proof of Theorem 20. Recall that the infinitesimal generator of U_t process is given by

$$\mathcal{A}f(u)=-\eta(u)\frac{\partial f}{\partial u}+\lambda(u)\int_0^\infty[f(u+c)-f(u)]\gamma e^{-\gamma c}dc. \tag{3.23}$$

Let us find a function f such that $\mathcal{A}f=-1$. That is,

$$-\eta(u)f'(u)-\lambda(u)f(u)+\lambda(u)\int_u^\infty f(c)\gamma e^{-\gamma(c-u)}dc=-1. \tag{3.24}$$

Differentiating the equation (3.24) with respect to u, we get

(3.25)

$$-\eta'(u)f'(u)-\eta(u)f''(u)-\lambda'(u)f(u)-\lambda(u)f'(u)+\lambda'(u)\int_u^\infty f(c)\gamma e^{-\gamma(c-u)}dc$$
$$-\lambda(u)\gamma f(u)+\lambda(u)\gamma\int_u^\infty f(c)\gamma e^{-\gamma(c-u)}dc=0.$$

Substituting (3.24) into (3.25), we get

$$f''(u)+\left[\frac{\eta'(u)}{\eta(u)}+\frac{\lambda(u)}{\eta(u)}-\frac{\lambda'(u)}{\lambda(u)}-\gamma\right]f'(u)=-\left[\frac{\lambda'(u)}{\lambda(u)}+\gamma\right]\frac{1}{\eta(u)}. \tag{3.26}$$

Let $g(u)=f'(u)$, then $g(u)$ satisfies a first-order linear ODE:

$$g'(u)+\left[\frac{\eta'(u)}{\eta(u)}+\frac{\lambda(u)}{\eta(u)}-\frac{\lambda'(u)}{\lambda(u)}-\gamma\right]g(u)=-\left[\frac{\lambda'(u)}{\lambda(u)}+\gamma\right]\frac{1}{\eta(u)}, \tag{3.27}$$

which yields the general solution

$$\begin{aligned}g(u)&=\int_0^u\left[-\frac{\lambda'(v)}{\lambda(v)}-\gamma\right]\frac{1}{\eta(v)}e^{-\int_v^u\left[\frac{\eta'(w)}{\eta(w)}+\frac{\lambda(w)}{\eta(w)}-\frac{\lambda'(w)}{\lambda(w)}-\gamma\right]dw}dv\\&\qquad+g(0)e^{-\int_0^u\left[\frac{\eta'(v)}{\eta(v)}+\frac{\lambda(v)}{\eta(v)}-\frac{\lambda'(v)}{\lambda(v)}-\gamma\right]dv}\\&=\frac{\lambda(u)}{\eta(u)}\int_0^u[-\lambda'(v)-\gamma\lambda(v)]\frac{1}{\lambda(v)^2}e^{\gamma(u-v)-\int_v^u\frac{\lambda(w)}{\eta(w)}dw}dv\\&\qquad+g(0)\frac{\eta(0)}{\eta(u)}\frac{\lambda(u)}{\lambda(0)}e^{\gamma u}e^{-\int_0^u\frac{\lambda(v)}{\eta(v)}dv}.\end{aligned} \tag{3.28}$$

Hence, we can choose $f(u)=\int_0^u g(v)dv$, which gives

$$\begin{aligned}f(u)=&\int_0^u\frac{\lambda(v)}{\eta(v)}\int_0^v[-\lambda'(w)-\gamma\lambda(w)]\frac{1}{\lambda(w)^2}e^{\gamma(v-w)-\int_w^v\frac{\lambda(r)}{\eta(r)}dr}dwdv\\&+g(0)\int_0^u\frac{\eta(0)}{\eta(v)}\frac{\lambda(v)}{\lambda(0)}e^{\gamma v}e^{-\int_0^v\frac{\lambda(w)}{\eta(w)}dw}dv.\end{aligned} \tag{3.29}$$

Next, let us determine $g(0)$. Recall that $g(0)=f'(0)$ and also notice that $f(0)=0$. Note that by letting $u=0$ in (3.24), we get

$$-\eta(0)g(0)+\lambda(0)\int_0^\infty f(c)\gamma e^{-\gamma(c-u)}dc=-1, \tag{3.30}$$

which implies that

$$\begin{aligned}-1=-\eta(0)g(0)+\lambda(0)&\int_0^\infty\int_0^c\frac{\lambda(v)}{\eta(v)}\int_0^v[-\lambda'(w)-\gamma\lambda(w)]\frac{1}{\lambda(w)^2}\\&\cdot e^{\gamma(v-w)-\int_w^v\frac{\lambda(r)}{\eta(r)}dr}\gamma e^{-\gamma(c-u)}dwdvdc\\&+g(0)\lambda(0)\int_0^\infty\int_0^c\frac{\eta(0)}{\eta(v)}\frac{\lambda(v)}{\lambda(0)}e^{\gamma v}e^{-\int_0^v\frac{\lambda(w)}{\eta(w)}dw}\gamma e^{-\gamma(c-u)}dvdc.\end{aligned} \tag{3.31}$$

Therefore,

(3.32)

$$g(0)=\frac{1+\lambda(0)\int_0^\infty\int_0^c\frac{\lambda(v)}{\eta(v)}\int_0^v[-\lambda'(w)-\gamma\lambda(w)]\frac{1}{\lambda(w)^2}e^{\gamma(v-w)-\int_w^v\frac{\lambda(r)}{\eta(r)}dr}\gamma e^{-\gamma(c-u)}dwdvdc}{\eta(0)-\lambda(0)\int_0^\infty\int_0^c\frac{\eta(0)}{\eta(v)}\frac{\lambda(v)}{\lambda(0)}e^{\gamma v}e^{-\int_0^v\frac{\lambda(w)}{\eta(w)}dw}\gamma e^{-\gamma(c-u)}dvdc}.$$

By Dynkin's formula, for any $K > 0$,

$$\mathbb{E}[f(U_{\tau\wedge K})] = f(u) + \mathbb{E}\left[\int_0^{\tau\wedge K} \mathcal{A}f(U_s)ds\right] = f(u) - \mathbb{E}[\tau \wedge K]. \tag{3.33}$$

By our assumption $\sup_{0<u<\infty} f(u) < \infty$ and $\tau < \infty$ a.s. Hence as $K \to \infty$, from bounded convergence theorem, we have $\mathbb{E}[f(U_{\tau\wedge K})] \to \mathbb{E}[f(U_\tau)]$ and by monotone convergence theorem, $\mathbb{E}[\tau \wedge K] \to \mathbb{E}[\tau]$. Therefore, $\mathbb{E}[\tau] = f(u) - \mathbb{E}[f(U_\tau)]$, which implies that

$$\begin{aligned}\mathbb{E}[\tau] = &\int_0^u \frac{\lambda(v)}{\eta(v)} \int_0^v [-\lambda'(w) - \gamma\lambda(w)] \frac{1}{\lambda(w)^2} e^{\gamma(v-w) - \int_w^v \frac{\lambda(r)}{\eta(r)}dr} dwdv \\ &+ g(0) \int_0^u \frac{\eta(0)}{\eta(v)} \frac{\lambda(v)}{\lambda(0)} e^{\gamma v} e^{-\int_0^v \frac{\lambda(w)}{\eta(w)}dw} dv.\end{aligned} \tag{3.34}$$

where $g(0)$ was defined in (2.51). □

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