Integrating the Stressing of LGD Parameters into the Basel AIRB Capital Supervisory Formula: A Closed-Form Solution

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Abstract

Current Basel minimum capital requirements for credit risk under the Advanced Internal Rating Based Method (AIRB) use both probability of default (PD) and loss given default (LGD) as input parameters into a supervisory formula (Supervisory Formula). However, the Supervisory Formula stresses the PD and LGD parameters differently.

For PD, the AIRB Supervisory Formula takes the PD parameter as input and transforms it into a stressed PD. On the other hand, for LGD, the Supervisory Formula directly takes the **stressed** LGD as an exogenous input that should reflect the economic downturn conditions. **Under the AIRB Supervisory Formula, the stressed LGD and stressed PD are not necessarily tied to each other.**

This disconnection in the stressing of PD and LGD undermines the theoretical consistency of the AIRB framework and leaves room for potential manipulation. **Basel Committee recognizes this issue, and thus encourages "the development of appropriate approaches to this issue."(http://www.bis.org/publ/bcbs128.pdf**)

This paper proposes a closed-form solution that mathematically ties the stressing of LGD and PD together, and therefore expands the current AIRB framework and establishes a simple methodology to estimate and stress test LGD.

Disclaimer

This paper was prepared by Hank Zhi Yang in his personal capacity. The views expressed in this paper are the author's own and do not necessarily reflect the view of the Office of Superintendent of Financial Institutions Canada or Government of Canada.

Section 1 Background and rationale

1.1 How PD is stressed in the Current AIRB Supervisory Formula

The financial and economic crisis that started in mid-2007 placed the capital base of many financial institutions (FIs) under severe stress, despite the fact that they had been in the process of implementing the Basel AIRB minimum capital requirements framework. This prompted both national financial regulators and the Basel Committee to consider how to strengthen the current capital requirement rules to make the banking sector more adequately capitalized.

In this paper, we focus on the enhancement of the Basel minimum capital requirement for credit risk under the AIRB framework. The AIRB capital requirement prescribes a supervisory formula (Supervisory Formula) that calculates the credit risk capital requirement (K) as a function of the long-run PD, confidence level (CL) to which the single general systematic risk factor is stressed, an average correlation factor among obligors (ρ) and an LGD input parameter that is calibrated outside of the AIRB Supervisory Formula. This LGD input parameter should reflect the downturn LGD in stressed conditions.

Equation 1 summarizes the AIRB Supervisory Formula. We ignore the maturity adjustment for the purpose of this paper.

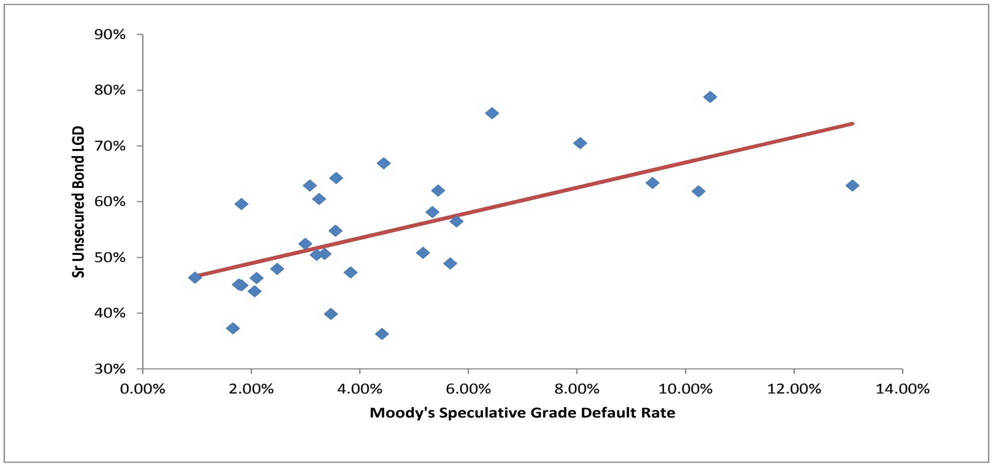
$$Capital\,\mathrm{Re}\,quirement(K) = LGD \times \phi\left(\frac{\phi^{-1}(PD) + \sqrt{\rho}\,\phi^{-1}(CL)}{\sqrt{(1-\rho)}}\right) - LGD \times PD \qquad (1)$$

The kernel component $\phi\left\{\frac{\phi^{-1}(PD) + \sqrt{\rho}\,\phi^{-1}(CL)}{\sqrt{(1-\rho)}}\right\}$ is a closed-form formula based on the asymptotic single risk factor (ASRF) model (see Gordy). Under the Basel AIRB framework, FIs estimate average long-run PD that reflects expected default rates of their credit portfolios over the long term. This long-run PD is plugged into Equation 1, which transforms the long-run PD into a stressed PD (SPD) that corresponds to a given confidence level of the stress (CL) applied to the single general systematic risk factor. Usually CL is set at 99.9%. All else being equal, the CL in Equations 1 is the key driver that drives how much the long-run PD will be stressed.

1.2 How LGD is stressed in the Current AIRB Supervisory Formula

Unlike the stressing of PD, the current AIRB Supervisory Formula does not include a kernel component that transforms a long-run average LGD into a stressed LGD. Instead, the LGD parameter in Equation 1 above should already be a stressed LGD (SLGD) as an exogenous input that "must take into account the potential for the LGD of the facility to be higher than the default-weighted average during a period when credit losses are substantially higher than average" (**http://www.bis.org/publ/bcbs128.pdf**). Therefore, under the current AIRB framework, the stressing of LGD is not mathematically tied to the stressing of PD in the Supervisory Formula.

However, it has been observed by many studies that default rate and LGD in reality do have positive correlation (that is, negative correlation between default rate and recovery rate). The chart below shows the relationship between the Moody's speculative grade corporate annual default rates and the average senior unsecured debt LGDs (which is equal to 1 - recovery rate) from 1982 to 2011: (Moody’s (2012) Annual Default Study: Corporate Default and Recovery Rates, 1920-2011)



1.3 Implication of the disconnection between stressing PD and LGD

The disconnection between stressing PD and LGD in the current AIRB Supervisory Formula may undermine the theoretical consistency of the Basel AIRB capital framework as it standardizes the calculation of the stressed PD but leaves the stressed LGD to be determined by the FIs. As a result, the SLGDs calculated by the FIs may not be tied to the SPDs that are calculated by the AIRB Supervisory Formula.

Moreover, the disconnection leaves room for potential manipulation and capital arbitrage by FIs that are required to implement Basel AIRB capital framework. For portfolios of similar credit quality, FIs may use different methodology in calibrating their SLGD. For example, FIs may choose different downturn periods for calibrating SLGD, or they may incorporate different levels of margin of conservatism in the estimation of SLGD.

Basel Committee recognizes this issue of disconnection and continues to monitor and encourage the development of appropriate approaches to quantifying downturn LGDs.

We will propose in the following sections a closed-form solution that links the stressing of LGD with the stressing of PD in the AIRB Supervisory Formula. The proposed model expands the AIRB Supervisory Formula, enhances the consistency between stressing PD and LGD, and provides a simple-to-use methodology to stress-test LGD given different confidence levels.

Section 2 Put LGD, SLGD, PD and SPD into Perspective

Assuming that a firm’s asset value follows lognormal distribution, the asset value as of the estimation time is V_0, the debt level is D (which we assume constant over the estimation horizon, normally one year) and the annual asset return volatility (which is the standard deviation of the natural logarithm of the asset values) is σ, we have the following relationship according to the Merton Model (On the pricing of corporate debt: The risk structure of interest rates):

$$PD = 1 - \Phi\left[\frac{Ln\left(\frac{V_0}{D}\right) + \left(r - \frac{\sigma^2}{2}\right)}{\sigma}\right] = \Phi\left[\frac{-Ln\left(\frac{V_0}{D}\right) - \left(r - \frac{\sigma^2}{2}\right)}{\sigma}\right] \qquad (2)$$

Where:

Φ = standard normal cumulative distribution function

Ln = natural logarithm;

PD = long - run average PD

r = expected asset return under the normal economic condition.

When the PD is stressed into SPD via the stressing of a single systemic risk factor as per Equation 1, Equation 2 should continue to hold except that PD is now replaced by SPD and r is replaced by r_s referring to the expected asset return under the stressed economic condition, if we assume that asset volatility σ stays the same. Then we have the equation below:

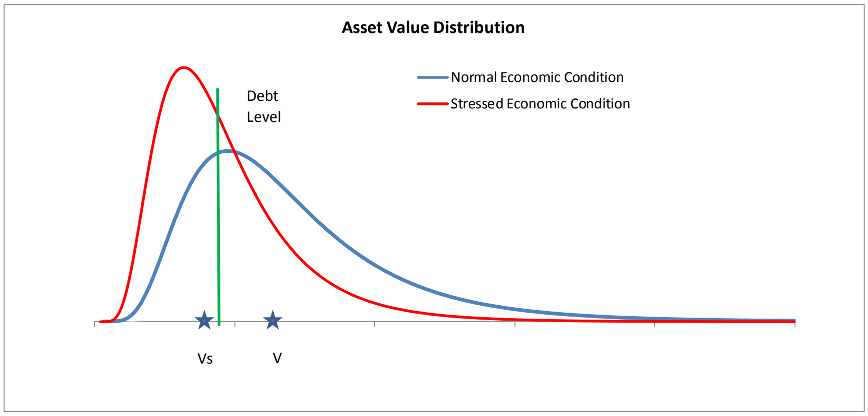
$$SPD = 1 - \Phi\left[\frac{Ln\left(\frac{V_0}{D}\right) + \left(r_s - \frac{\sigma^2}{2}\right)}{\sigma}\right] = \Phi\left[\frac{-Ln\left(\frac{V_0}{D}\right) - \left(r_s - \frac{\sigma^2}{2}\right)}{\sigma}\right]$$

Where:

$$SPD = \phi\left\{\frac{\phi^{-1}(PD) + \sqrt{\rho}\,\phi^{-1}(CL)}{\sqrt{(1-\rho)}}\right\} \qquad (3)$$

$SPD > PD$

Comparing Equations 2 and 3, we can see that the expected asset return under the stressed economic condition (r_s) must be smaller than the expected asset return under the normal economic condition (r) in order to satisfy the condition that $SPD > PD$ as all other parameters are not changed. In other words, the expected asset value at the end of one year under the stressed economic condition (V_s) must be smaller than the expected asset value under the normal economic condition (V) as illustrated in the graph below:



Note that the graph is for illustrative purpose only

The blue curve and red curve represent the asset value distributions of the same firm under the normal and stressed economic conditions respectively. The area to the left of the green bar under the blue curve represents the PD of the firm under the normal economic condition while the area to the left of the green bar under the red curve represents the SPD of the firm under the stressed economic condition.

Instead of looking at LGD directly, we focus on the recovery rate which is measured by the ratio of the expected asset value given default over the debt. Let V_d be the expected asset value given default and D is the debt value. We use v to represent the stochastic variable that represents the asset values at the year-end. Note that v is assumed to be log-normally distributed. Then V_d is the expected value of v conditional upon $v \le D$. Using the property of lognormal distribution we can find the conditional expected value V_d as follows:

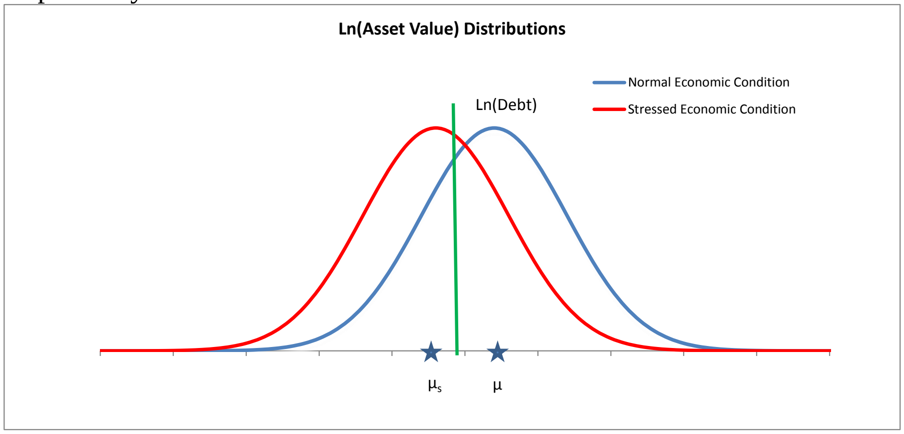
$$V_d = \frac{1}{PD}\int_0^D v \bullet \frac{1}{v\sigma\sqrt{2\pi}} e^{-\frac{(\ln v-\mu)^2}{2\sigma^2}} dv \tag{4}$$

$Where:$

$\mu = mean\ of\ Ln(v)$

The mean recovery rate ("R") is then $\frac{V_d}{D}$.

Since we assume V is log-normally distributed, we know that Ln(v) must be normally distributed as shown in the graph below where μ and μ_s are the mean of Ln(v) under the blue and red curves respectively.



Since μ is the mean of Ln(v) from Equations 2 and 3 we can derive:

$$\begin{aligned}
\mu &= LnD - \sigma\Phi^{-1}(PD) \\
\mu_S &= LnD - \sigma\Phi^{-1}(SPD) \\
\mu &= Ln(V_0) + r - \frac{\sigma^2}{2} \\
\mu_S &= Ln(V_0) + r_S - \frac{\sigma^2}{2}
\end{aligned} \tag{5}$$

Section 3 The closed-form solution for SLGD

In this section we will derive SLGD as a function of PD, LGD, σ and CL that will tie to the stressing of PD via the same single systemic risk factor.

Since v is log-normally distributed, we derive the following result from Equation 4 (see Appendix A for detailed derivation):

$$R = \frac{1}{D \bullet PD} e^{\mu + \frac{\sigma^2}{2}} \Phi(\frac{LnD - \mu - \sigma^2}{\sigma}) \tag{6}$$

$Where:$

$R =$ expected recovery rate under the normal economic condition

Substituting Equation 5 into the equation above, we have the following:

$$R = \frac{1}{D \bullet PD} e^{\mu + \frac{\sigma^2}{2}} \Phi[\Phi^{-1}(PD) - \sigma)] \tag{7}$$

By the same token, we can derive the expected recovery rate in the stressed time:

$$R_s = \frac{1}{D \bullet SPD} e^{\mu_s + \frac{\sigma^2}{2}} \Phi[\Phi^{-1}(SPD) - \sigma)] \tag{8}$$

$Where:$

$$SPD = \phi\left\{\frac{\phi^{-1}(PD) + \sqrt{\rho}\phi^{-1}(CL)}{\sqrt{(1-\rho)}}\right\}$$

Dividing Equation 8 by Equation 7, we have the following:

$$\frac{R_s}{R} = e^{\mu_s - \mu} \bullet \frac{PD}{SPD} \frac{\Phi[\Phi^{-1}(SPD) - \sigma)]}{\Phi[\Phi^{-1}(PD) - \sigma)]}$$

Substituting Equation 5 into the equation above, we derive the following:

$$R_s = R \bullet e^{\sigma\Phi^{-1}(PD) - \sigma\Phi^{-1}(SPD)} \bullet \frac{PD}{SPD} \frac{\Phi[\Phi^{-1}(SPD) - \sigma)]}{\Phi[\Phi^{-1}(PD) - \sigma)]}$$

$Where:$

$$SPD = \phi\left\{\frac{\phi^{-1}(PD) + \sqrt{\rho}\phi^{-1}(CL)}{\sqrt{(1-\rho)}}\right\}$$

PD = long-run average default probability **(9)**

R = long-run average recovery rate

σ = the standard deviation or volatility of asset return

CL = confidence level to which the single systemic risk factor will be stressed for both PD and LGD

Equation 9 is a closed-form solution that ties the stressing of the expected recovery rate (and hence the LGD) to the stressing of PD through the stressing of the same single systemic risk factor (CL). We name Equation 9 Recovery Mapping Function (RMF).

Note that the RMF can also be expressed in terms of LGD:

$$SLGD = 1-(1-LGD)\bullet e^{\sigma\Phi^{-1}(PD)-\sigma\Phi^{-1}(SPD)}\bullet\frac{PD}{SPD}\frac{\Phi[\Phi^{-1}(SPD)-\sigma)]}{\Phi[\Phi^{-1}(PD)-\sigma)]}$$

Section 4 A potential simplified form of RMF

The RMF shown in Section 4 implies that the stressing of the recovery rate or LGD depends on two multiplier components:

(1) $e^{\sigma\Phi^{-1}(PD)-\sigma\Phi^{-1}(SPD)}$

(2) $\frac{PD}{SPD}\frac{\Phi[\Phi^{-1}(SPD)-\sigma)]}{\Phi[\Phi^{-1}(PD)-\sigma)]}$

Now let's try to make further sense out of these two multiplier components.

We have assumed that the expected asset return is r as shown below:

$$V_1 = V_0 e^r$$

Where:

V1 = expected asset value at the end of one year following log - normal distribution

V0 = asset value at present

The above equation can be approximated by

$$V_1 = V_0(1+r)$$

Where: r is small

We then assume r_d is the expected asset return given default and r_{nd} is the expected asset return given non-default and ε is the spread between r_{nd} and r_d. We have the following equations:

$$V_1 = V_0(1+r) = V_0(1+r_d)PD + V_0(1+r_{nd})(1-PD)$$
$$r = r_d PD + r_{nd}(1-PD)$$
$$r = r_d + \varepsilon(1-PD)$$

Similarly we would have the following relationship for the expected asset return under the stressed economic condition:

$$r_s = r_{sd} + \varepsilon_s(1 - SPD)$$
Where:

r_s = expected asset return under stressed economic condition

ε_s = spread between expected asset returns given survival and default under stressed economic condition

SPD = the stressed PD determined by the AIRB Supervisory Formula

Comparing r_s and r, we will have the following relationship:

$$\frac{e^{r_s}}{e^{r}} = \frac{e^{r_{sd}}}{e^{r_d}} \frac{e^{\varepsilon_s(1-SPD)}}{e^{\varepsilon(1-PD)}} \qquad \text{Hence} \qquad \frac{e^{r_{sd}}}{e^{r_d}} = \frac{e^{r_s}}{e^{r}} \bullet \frac{e^{\varepsilon(1-PD)}}{e^{\varepsilon_s(1-SPD)}} \tag{10}$$

We know $\frac{R_s}{R} = \frac{e^{r_{sd}}}{e^{r_d}}$. Substituting Equation 5 and Equation 12 into Equation 10, we have the RMF (Equation 9) previously derived in Section 3.

$$R_s = R \bullet e^{\sigma\Phi^{-1}(PD) - \sigma\Phi^{-1}(SPD)} \bullet \frac{PD}{SPD} \frac{\Phi[\Phi^{-1}(SPD) - \sigma)]}{\Phi[\Phi^{-1}(PD) - \sigma)]}$$

Equation 10 shows that how much the recovery rate will deteriorate under the stressed economic condition depend on two components:

(1) How much will the (overall) expected asset return deteriorate
(2) How will the product of the expected asset return spreads between default and survival modes and the survival probabilities change

This intuitively makes sense if we re-arrange the relationship into the following:

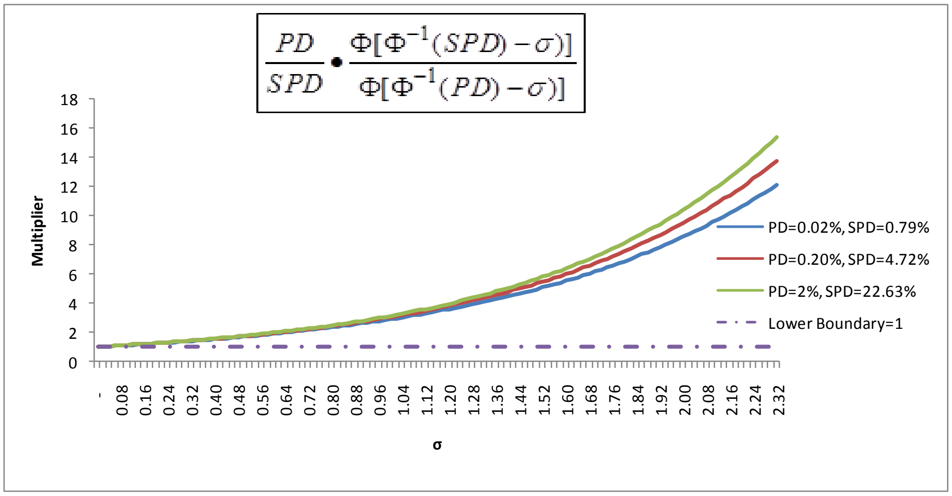
$$r_s - r = [r_{sd} - r_d] + [\varepsilon_s(1 - SPD) - \varepsilon(1 - PD)]$$

Here we can more clearly see that the deterioration in the (overall) expected asset return is determined by the deterioration in the expected asset return given default and the deterioration in the spreads in the asset returns between survival and default modes weighted by the survival probability.

It can be proven that (see Appendix B for proof):

$$\frac{e^{\varepsilon(1-PD)}}{e^{\varepsilon_s(1-SPD)}} = \frac{PD}{SPD} \frac{\Phi[\Phi^{-1}(SPD) - \sigma)]}{\Phi[\Phi^{-1}(PD) - \sigma)]} \tag{12}$$

Also Equation 12 will always be larger than 1 if $SPD > PD$ and $\sigma > 0$ (see Appendix C for proof).



And hence: $\varepsilon_S(1-SPD) < \varepsilon(1-PD)$, when $SPD > PD$ and σ > 0

This means that the deterioration in the overall expected asset return (r) cannot be fully allocated to the deterioration in the expected asset return given default (r_d). The deterioration in the survival probability-weighted spread $\varepsilon(1-PD)$ between expected return given survival and expected return given default also contributes to the reduction of overall expected return r. In other words, the difference between r_{sd} and r_d should be smaller than the difference between r_s and r. *The RMF function we derived in Section 3 fully reflects this effect via the two multiplier components-* $e^{\sigma\Phi^{-1}(PD)-\sigma\Phi^{-1}(SPD)}$ *and* $\frac{PD}{SPD}\frac{\Phi[\Phi^{-1}(SPD)-\sigma)]}{\Phi[\Phi^{-1}(PD)-\sigma)]}$.

However, if sometimes we want to have a more conservative stressing of recovery rate, we may assume that all the deterioration in the (overall) expected asset return from r to r_s is attributable to the deterioration in the expected asses return given default (from r_d to r_{sd}). We do this by setting $\varepsilon_S(1-SPD) = \varepsilon(1-PD)$ thus $\frac{PD}{SPD}\frac{\Phi[\Phi^{-1}(SPD)-\sigma)]}{\Phi[\Phi^{-1}(PD)-\sigma)]}$ to be 1 so that we in effect assume $r_{sd} - r_d = r_s - r$.

This way we obtained a simplified form of the RMF.

$$R_s = R \bullet e^{\sigma\Phi^{-1}(PD)-\sigma\Phi^{-1}(SPD)} \qquad \textbf{(13)}$$

We name Equation 13 Simplified Recovery Mapping Function (SRMF).

All else being equal, the SRMF produces more conservative stressed recovery rates than the RMF in Section 3 by allocating the deterioration in overall expected asset return entirely to the expected return given default.

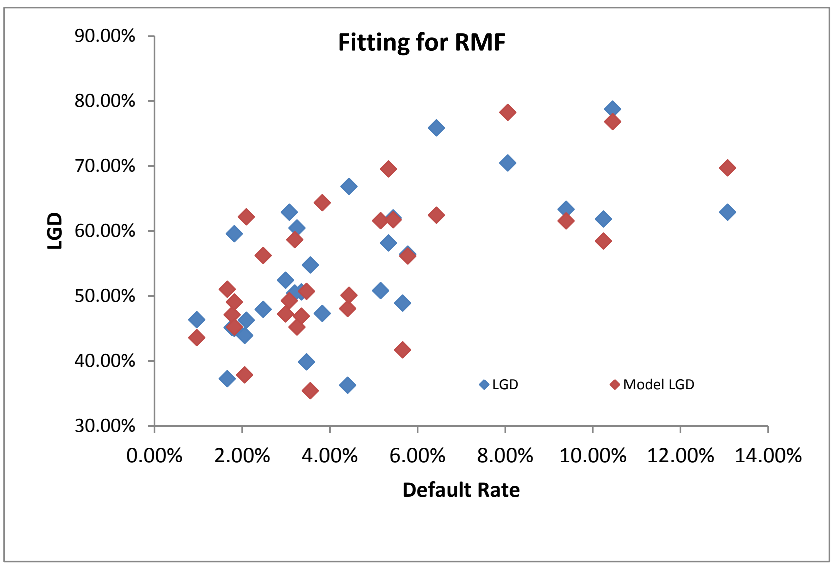
Section 5 Calibration of asset return volatility σ

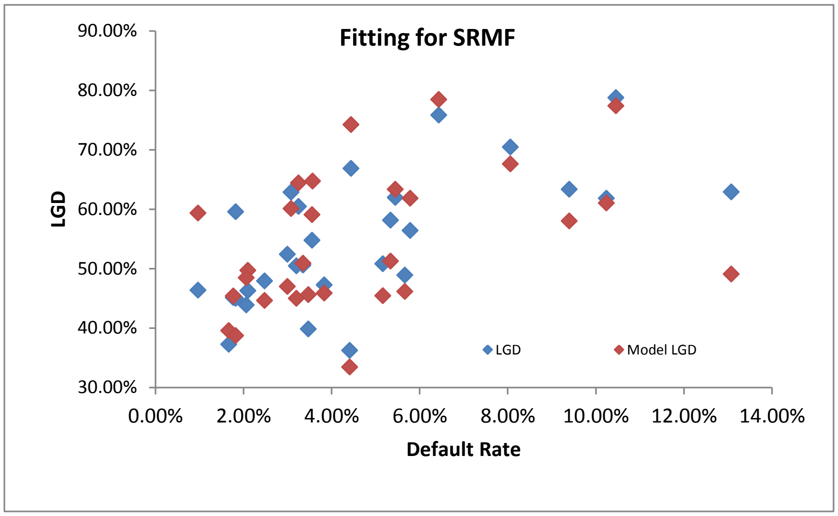
The key parameter to calibrate in RMF and SRMF is the asset return volatility σ since other parameters such as long-run PD and long-run default weighted average LGD can be readily determined using FIs' internal data or external information.

There could be several approaches to calibrate σ. One approach could be to find the implied σ that fits RMF and SRMF the best into the historical long-run average PD, the long-run default weighted average LGD, the downturn PD and the downturn LGD. In this paper, we demonstrate another approach by fitting the RMF and SRMF into the historical default rate and LGD time series and finding the implied σ that minimizes the fitting errors. The historical time series we used are Moody's speculative grade corporate annual default rates and the average senior unsecured debt recovery rates from 1982 to 2011 as shown in Section 1.2.

We calibrated σ to be 75% for RMF and 43% for SRMF (note that σ is the standard deviation of the Ln(v) and LGD = 1 – Recovery Rate).

The graphs below show the fitting of the model-fitted LGDs under RMF and SRMF with the actual historical LGDs and default rates published by Moody's.





The difference in the calibrated volatility parameter σ reflects the fact that SRMF assumes that the change in the expected asset value given default (hence recovery rate) equals to the change in the (overall) expected asset return. As a result, the implied asset return volatility σ under SRMF is much smaller than under RMF given the same set of calibration data.

Section 5 Application of the Recovery Mapping Functions

The Recovery Mapping Functions (RMF and SRMF) can be used in several ways to enhance financial risk management.

5.1 Using RMFs to stress test LGD under different systemic stress level

FIs can use the RMFs to stress-test LGD by changing the CL – the confidence level to which the general systemic risk factor is stressed. This CL will drive the stressing of both PD and LGD.

Assuming the long-run average PD and long-run default-weighted average LGD are 4.63% and 59.08%. Using the RMFs we can stress test LGD by changing the confidence level CL. The SLGD will change accordingly as shown in the tables below.

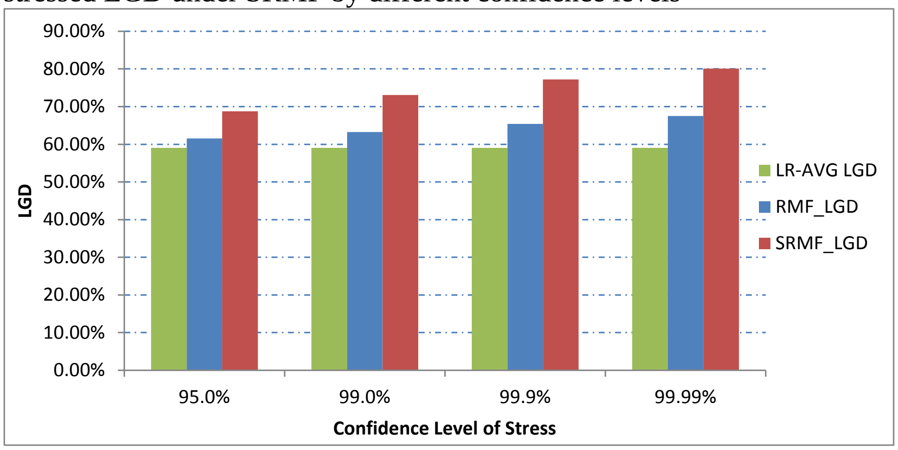
Stress testing LGD using RMF (σ=75%)

PD	LGD	rho	CL	SPD	SLGD
4.63%	59.08%	0.20	95.0%	14.50%	**61.54%**
4.63%	59.08%	0.20	99.0%	23.66%	**63.20%**
4.63%	59.08%	0.20	99.9%	36.87%	**65.39%**
4.63%	59.08%	0.20	99.99%	49.17%	**67.47%**

Stress testing LGD using SRMF (σ=43%)

PD	LGD	rho	CL	SPD	SLGD
4.63%	59.08%	0.20	95.00%	14.50%	**68.71%**
4.63%	59.08%	0.20	99.00%	23.66%	**72.97%**
4.63%	59.08%	0.20	99.90%	36.87%	**77.06%**
4.63%	59.08%	0.20	99.99%	49.17%	**79.97%**

The chart below compares the long-run average LGD, the stressed LGD under RMF and the stressed LGD under SRMF by different confidence levels



If we increase the asset return volatilities, we will see higher stressing of LGD given the same confidence levels.

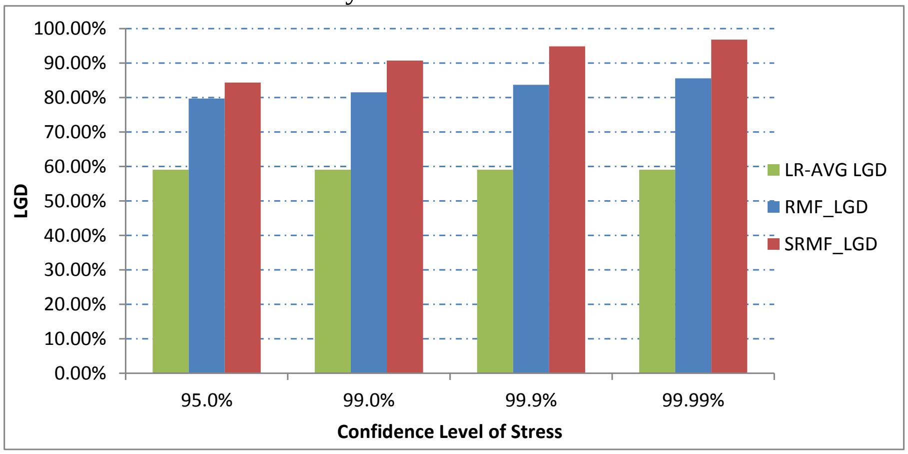
Stress testing LGD using RMF (σ=218%)

PD	LGD	rho	CL	SPD	SLGD
4.63%	59.08%	0.20	95.0%	14.50%	**79.72%**
4.63%	59.08%	0.20	99.0%	23.66%	**81.51%**
4.63%	59.08%	0.20	99.9%	36.87%	**83.69%**
4.63%	59.08%	0.20	99.99%	49.17%	**85.63%**

Stress testing LGD using SRMF (σ=154%)

PD	LGD	rho	CL	SPD	SLGD
4.63%	59.08%	0.20	95.00%	14.50%	**84.34%**
4.63%	59.08%	0.20	99.00%	23.66%	**90.73%**
4.63%	59.08%	0.20	99.90%	36.87%	**94.85%**
4.63%	59.08%	0.20	99.99%	49.17%	**96.83%**

The chart below compares the long-run average LGD, the stressed LGD under RMF and the stressed LGD under SRMF by different confidence levels



5.2 Incorporating the stressed LGD into Basel AIRB capital Supervisory Formula

Substituting RMF or SRMF into Equation 1, we obtain the following expanded AIRB capital Supervisory Formula combining the stressing of PD and LGD into one consistent model:

$$Capital\,\mathrm{Re}\,quirement(K) = SLGD \times \left(SPFD - PD\right) \qquad (14)$$

Where:

$$SPD = \phi\left\{\frac{\phi^{-1}(PD) + \sqrt{\rho}\,\phi^{-1}(CL)}{\sqrt{(1-\rho)}}\right\}$$

If RMF is used, then $SLGD = 1-(1-LGD)\bullet e^{\sigma\Phi^{-1}(PD)-\sigma\Phi^{-1}(SPD)}\bullet\dfrac{PD}{SPD}\dfrac{\Phi[\Phi^{-1}(SPD)-\sigma)]}{\Phi[\Phi^{-1}(PD)-\sigma)]}$

If SRMF is used, then $SLGD = 1-(1-LGD)\bullet e^{\sigma\Phi^{-1}(PD)-\sigma\Phi^{-1}(SPD)}$

Moody's (2012) Annual Default Study: Corporate Default and Recovery Rates, 1920-2011 shows the following Average Sr. Unsecured Bond Recovery Rates by Year Prior to Default (1982-2011) and by credit rating.

	Year 1	Year 2	Year 3	Year 4	Year 5
Aaa	n.a.	3.33%**	3.33%**	61.88%	75.58%
Aa	37.24%	39.02%	38.08%	43.95%	42.27%
A	31.77%	42.68%	44.28%	42.72%	42.29%
Baa	41.39%	42.25%	42.41%	42.96%	42.90%
Ba	47.11%	45.53%	44.39%	43.35%	42.89%
B	37.88%	36.85%	36.83%	37.16%	37.76%
Caa-C	35.72%	35.55%	35.29%	35.34%	35.31%

To illustrate the application of RMF and SRMF, we focus on the recovery rates by Year 1.

We apply Equation 14 by substituting RMF (σ=75%) for SLGD to calculate the SPD, SLGD and K as follows:

Rating	PD	R	SPD	LGD	rho	CL	SLGD	K
Aaa	0.01%	40.00%	0.45%	60.00%	0.20	99.9%	**61.85%**	0.27%
Aa	0.02%	37.24%	0.89%	62.76%	0.20	99.9%	**64.69%**	0.56%
A	0.07%	31.77%	2.11%	68.23%	0.20	99.9%	**70.18%**	1.43%
Baa	0.20%	41.39%	4.74%	58.61%	0.20	99.9%	**61.67%**	2.80%
Ba	1.15%	47.11%	15.94%	52.89%	0.20	99.9%	**57.89%**	8.56%
B	4.21%	37.88%	34.98%	62.12%	0.20	99.9%	**67.80%**	20.86%
Caa	14.36%	35.72%	63.89%	64.28%	0.20	99.9%	**72.44%**	35.88%

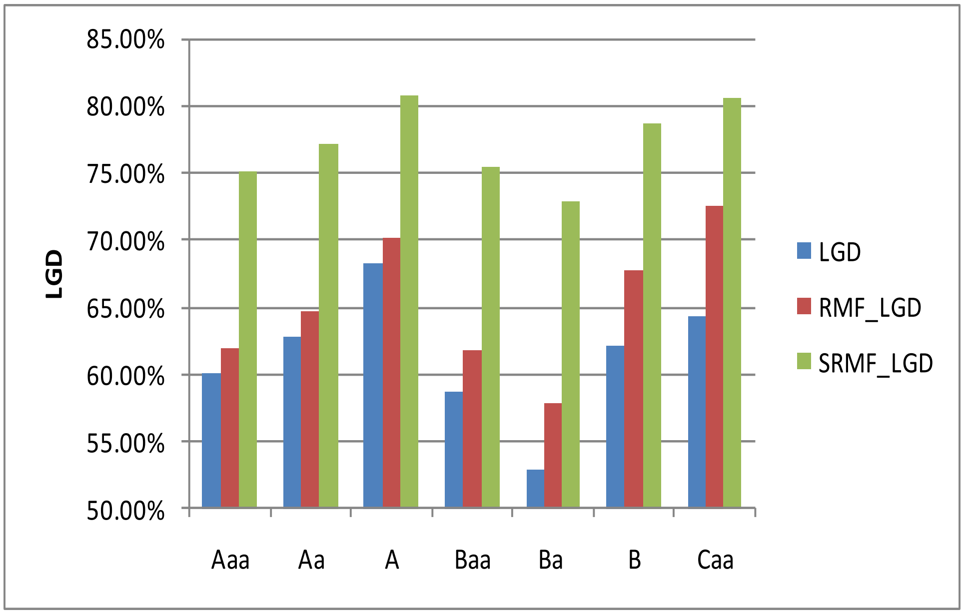
Note: For "Aaa" rating, we assume its PD = 0.01% and LGD = 40%

Similarly, we apply Equation 14 by substituting SRMF (σ=43%) for SLGD to calculate the SPD, SLGD and K as follows:

Rating	PD	R	SPD	LGD	rho	CL	SLGD	K
Aaa	0.01%	40.00%	0.45%	60.00%	0.20	99.9%	**75.14%**	0.33%
Aa	0.02%	37.24%	0.89%	62.76%	0.20	99.9%	**77.11%**	0.67%
A	0.07%	31.77%	2.11%	68.23%	0.20	99.9%	**80.77%**	1.65%
Baa	0.20%	41.39%	4.74%	58.61%	0.20	99.9%	**75.35%**	3.42%
Ba	1.15%	47.11%	15.94%	52.89%	0.20	99.9%	**72.79%**	10.76%
B	4.21%	37.88%	34.98%	62.12%	0.20	99.9%	**78.72%**	24.23%
Caa	14.36%	35.72%	63.89%	64.28%	0.20	99.9%	**80.60%**	39.91%

Note: For "Aaa" rating, we assume its PD = 0.01% and LGD = 40%

The following chart compares the Moody's-published LGD, stressed LGD under RMF and stressed LGD under SRMF by credit ratings.



5.3 Estimating the expected LGD from observed default rates

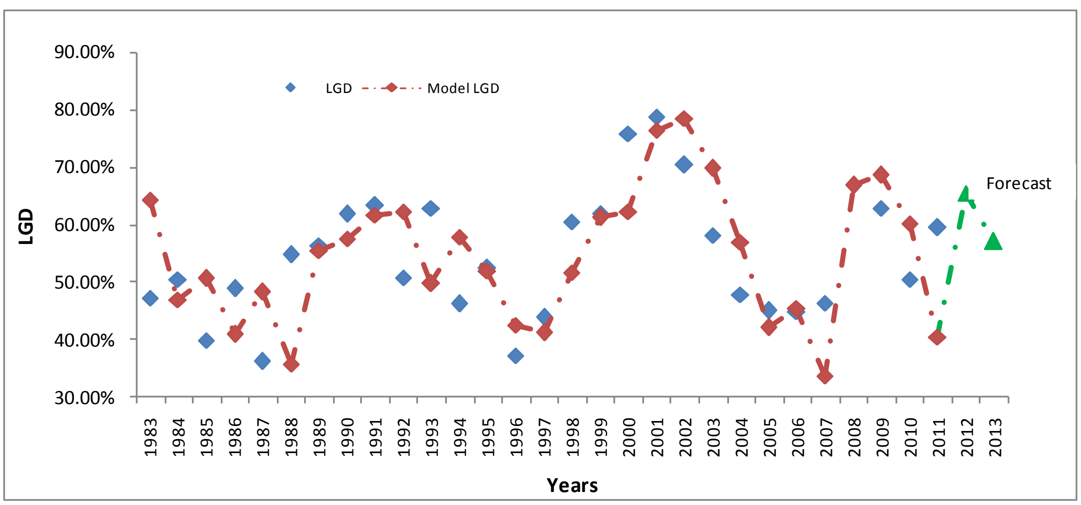
In practice, LGD can only be accurately calculated after the end of the recovery period which is usually two to three years after the default. So an FI often runs into situations where the FI observed a certain default rate for a given year but the average LGD for these facilities that defaulted within this given year will still not be known yet until a couple of years later.

What shall the FI do if they would like to have a quick estimation of the average LGD for these defaulted accounts?

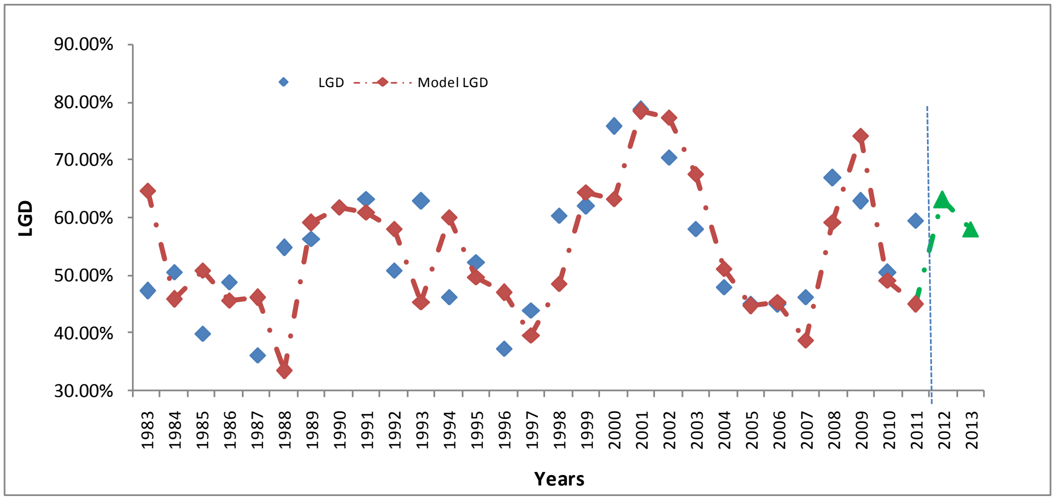
RMF can help the FI calculate an expected LGD for a specific year using the observed default rate observed in that year.

Assuming the FI observed default rates in 2012 and 2013 as 3% and 1.5% respectively. However, since the work-out period for the defaulted accounts are two years. So the FI needs to estimate the expected LGD for 2012 and 2013 without knowing the actual realized LGD for these two years. Using RMF the FI can calculate the estimated LGDs for 2012 and 2013 (in green) as 65.53% and 57.12%. If SRMF is used, the estimated LGDs for 2012 and 2013 are 63.11% and 58.18% respectively.

Using RMF to estimate LGD in 2012 and 2013



Using SRMF to estimate LGD in 2012 and 2013



Section 6 Conclusion

In this paper, we discussed the inherent disconnection and inconsistency in the current Basel AIRB Supervisory Formula for credit risk and proposed two closed-form solutions (RMF and its simplified form SRMF) that tie the stressing of both PD and LGD with the stressing of the single general systemic risk factor implied by the current AIRB Supervisory Formula. The proposed solutions expand the current AIRB capital framework, enhance its inherent consistency and provide an easy-to-use methodology to estimate and stress-test the LGD.

Recovery Mapping Function (RMF)

$$R_s = R \bullet e^{\sigma\Phi^{-1}(PD) - \sigma\Phi^{-1}(SPD)} \bullet \frac{PD}{SPD} \frac{\Phi[\Phi^{-1}(SPD) - \sigma)]}{\Phi[\Phi^{-1}(PD) - \sigma)]}$$

Where:

$$SPD = \phi\left\{\frac{\phi^{-1}(PD) + \sqrt{\rho}\phi^{-1}(CL)}{\sqrt{(1-\rho)}}\right\}$$

Simplified Recovery Mapping Function (SRMF)

$$R_s = R \bullet e^{\sigma\Phi^{-1}(PD) - \sigma\Phi^{-1}(SPD)}$$

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Appendix A - Derivation of Equation 6

By definition of expected asset value given default, we have Equation 4 as follows:

$$V_d = \frac{1}{PD}\int_0^D v \bullet \frac{1}{v\sigma\sqrt{2\pi}} e^{-\frac{(\ln v-\mu)^2}{2\sigma^2}} dv$$

Let $y = \frac{\ln v - \mu}{\sigma}$, then we have:

$$V_d = \frac{1}{PD}\int_{-\infty}^{\frac{\ln D-\mu}{\sigma}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2}} de^{\sigma y+\mu}$$

$$V_d = \frac{1}{PD}\int_{-\infty}^{\frac{\ln D-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}+\sigma y+\mu} dy$$

$$V_d = \frac{1}{PD}\int_{-\infty}^{\frac{\ln D-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2-2\sigma y+\sigma^2-2\mu-\sigma^2}{2}} dy$$

$$V_d = \frac{1}{PD}\int_{-\infty}^{\frac{\ln D-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sigma)^2}{2}} e^{\mu+\frac{\sigma^2}{2}} dy$$

$$V_d = \frac{1}{PD} e^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{\frac{\ln D-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sigma)^2}{2}} dy$$

Let $z = y - \sigma$ then we have:

$$V_d = \frac{1}{PD} e^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{\frac{\ln D-\mu-\sigma^2}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$V_d = \frac{1}{PD} e^{\mu+\frac{\sigma^2}{2}} \Phi\left(\frac{\ln D - \mu - \sigma^2}{\sigma}\right)$$

Where Φ is the standard normal distribution function

Since expected recovery rate R = V_d / D, we can then derive the following Equation 6:

$$R = \frac{1}{D \bullet PD} e^{\mu+\frac{\sigma^2}{2}} \Phi(\frac{LnD - \mu - \sigma^2}{\sigma})$$

Appendix B – Proof of Equation 12

Assuming α is the expected asset value return conditional upon default and β is the expected asset value return conditional upon non-default, we have the following:

$$V_d = V_0 e^{\alpha} = \frac{1}{PD} e^{\mu + \frac{\sigma^2}{2}} \Phi[\Phi^{-1}(PD) - \sigma)]$$

$$V_{nd} = V_0 e^{\beta} = \frac{1}{(1-PD)} e^{\mu + \frac{\sigma^2}{2}} \left\langle 1 - \Phi[\Phi^{-1}(PD) - \sigma)] \right\rangle$$

Let ε = β - α, we have:

$$e^{\varepsilon} = e^{\beta - \alpha} = \frac{\left[1 - \Phi[\Phi^{-1}(PD) - \sigma)]\right] \bullet PD}{\Phi[\Phi^{-1}(PD) - \sigma)] \bullet (1 - PD)}$$

When α and β are small, we can approximate:

$$1 + \varepsilon = \frac{\left[1 - \Phi[\Phi^{-1}(PD) - \sigma)]\right] \bullet PD}{\Phi[\Phi^{-1}(PD) - \sigma)] \bullet (1 - PD)}$$

$$1 - PD + \varepsilon(1 - PD) = \frac{\left[1 - \Phi[\Phi^{-1}(PD) - \sigma)]\right] \bullet PD}{\Phi[\Phi^{-1}(PD) - \sigma)]}$$

$$1 + \varepsilon(1 - PD) = \frac{\left[1 - \Phi[\Phi^{-1}(PD) - \sigma)]\right] \bullet PD}{\Phi[\Phi^{-1}(PD) - \sigma)]} + PD$$

$$1 + \varepsilon(1 - PD) = \frac{PD - \Phi[\Phi^{-1}(PD) - \sigma)] \bullet PD + \Phi[\Phi^{-1}(PD) - \sigma)] \bullet PD}{\Phi[\Phi^{-1}(PD) - \sigma)]}$$

$$1 + \varepsilon(1 - PD) = \frac{PD}{\Phi[\Phi^{-1}(PD) - \sigma)]}$$

When ε is small, we can approximate:

$$e^{\varepsilon(1-PD)} = \frac{PD}{\Phi[\Phi^{-1}(PD) - \sigma)]} \qquad \text{(B1)}$$

Similarly, we have:

$$e^{\varepsilon_s(1-SPD)} = \frac{SPD}{\Phi[\Phi^{-1}(SPD) - \sigma)]} \qquad \text{(B2)}$$

Dividing (B1) by (B2), we can prove Equation 12

$$\frac{e^{\varepsilon(1-PD)}}{e^{\varepsilon_s(1-SPD)}} = \frac{PD}{SPD} \frac{\Phi[\Phi^{-1}(SPD) - \sigma)]}{\Phi[\Phi^{-1}(PD) - \sigma)]}$$

Appendix C – Proof of Equation 12 must be > 1

To prove that $\frac{PD}{SPD}\frac{\Phi[\Phi^{-1}(SPD)-\sigma)]}{\Phi[\Phi^{-1}(PD)-\sigma)]}>1$ when $SPD>PD$ and $\sigma>0$, we first let

$G(\sigma)=PD\Phi[\Phi^{-1}(SPD)-\sigma)]-SPD\Phi[\Phi^{-1}(PD)-\sigma)]$

1) If σ = 0, it is obvious that G(σ) = 0, and hence $\frac{PD}{SPD}\frac{\Phi[\Phi^{-1}(SPD)-\sigma)]}{\Phi[\Phi^{-1}(PD)-\sigma)]}=1$
2) If σ > 0, we take the first derivative of G(σ) as:

$$G'(\sigma)=-PD\Phi'[\Phi^{-1}(SPD)-\sigma)]+SPD\Phi'[\Phi^{-1}(PD)-\sigma)]$$

$$G'(\sigma)=SPD\Phi'[\Phi^{-1}(PD)-\sigma)]-PD\Phi'[\Phi^{-1}(SPD)-\sigma)]$$

$$G'(\sigma)=SPD\frac{1}{\sqrt{2\pi}}e^{-\frac{[\Phi^{-1}(PD)-\sigma]^2}{2}}-PD\frac{1}{\sqrt{2\pi}}e^{-\frac{[\Phi^{-1}(SPD)-\sigma]^2}{2}}$$

$$G'(\sigma)=\frac{1}{\sqrt{2\pi}}e^{-\frac{[\Phi^{-1}(SPD)-\sigma]^2}{2}}\left(SPDe^{\frac{[\Phi^{-1}(SPD)-\sigma]^2-[\Phi^{-1}(PD)-\sigma]^2}{2}}-PD\right)$$

We then let $F(\sigma)=SPDe^{\frac{[\Phi^{-1}(SPD)-\sigma]^2-[\Phi^{-1}(PD)-\sigma]^2}{2}}-PD$

We take the first derivative of F(σ):

$$F'(\sigma)=SPDe^{\frac{[\Phi^{-1}(SPD)-\sigma]^2-[\Phi^{-1}(PD)-\sigma]^2}{2}}\bullet(\frac{[\Phi^{-1}(SPD)-\sigma]^2-[\Phi^{-1}(PD)-\sigma]^2}{2})'$$

$$F'(\sigma)=SPDe^{\frac{[\Phi^{-1}(SPD)-\sigma]^2-[\Phi^{-1}(PD)-\sigma]^2}{2}}\bullet(\frac{[\Phi^{-1}(SPD)-\Phi^{-1}(PD)][\Phi^{-1}(SPD)+\Phi^{-1}(PD)-2\sigma]}{2})'$$

$$F'(\sigma)=SPDe^{\frac{[\Phi^{-1}(SPD)-\sigma]^2-[\Phi^{-1}(PD)-\sigma]^2}{2}}\bullet[\Phi^{-1}(PD)-\Phi^{-1}(SPD)]$$

Since $\Phi^{-1}(PD)<\Phi^{-1}(SPD)]<0$, we must have $F'(\sigma)<0$. This means that F(σ) must be a monotonically decreasing function of σ.

When $\sigma\to\infty$, we have $F(\sigma)=SPDe^{\frac{[-\infty]^2-[-\infty]^2}{2}}-PD=SPDe^0-PD=SPD-PD>0$
Given that $\sigma\to\infty$ we have $F(\sigma)>0$, and F(σ) is a monotonically decreasing function of σ, we conclude that $F(\sigma)>0$ for $\sigma>0$.

Hence we have $G'(\sigma)=\frac{1}{\sqrt{2\pi}}e^{-\frac{[\Phi^{-1}(SPD)-\sigma]^2}{2}}F(\sigma)>0$. This means that G(σ) must be a monotonically increasing function of σ. Given that $G(\sigma)=0$ when $\sigma=0$, we conclude that G(σ) must be >0 where $\sigma>0$.

Hence we proved that $\frac{PD}{SPD}\frac{\Phi[\Phi^{-1}(SPD)-\sigma)]}{\Phi[\Phi^{-1}(PD)-\sigma)]}>1$ when $\sigma>0$ and $SPD>PD$.