Does Black-Scholes framework for Option Pricing use Constant Volatilities and Interest Rates ?

New Solution for a New Problem

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1. Abstract

It is a common belief that for using the famous Black-Scholes framework for Option Pricing, we need to assume that Stock Volatility and Risk Free Interest Rate have to be constant. We prove that this belief is only partially true and there are work arounds, in which the volatility and interest rates can be held as non-constant parameters.

Key Words

Black Scholes, PDE, Finite Difference Methods

Paper Type

Mathematical Finance

2. Overview

It is a common belief that for using the famous Black-Scholes framework for Option Pricing, we need to assume that Stock Volatility and Risk Free Interest Rate have to be constant. We prove that this belief is only partially true and there are work arounds, in which the volatility and interest rates can be held as non-constant parameters.

In this white paper, we derive the famous Black-Scholes PDE (partial differential equation) first, from the first principles of no arbitrage. Thereafter we stop at a point where we take a de-tour and apply a famous numerical technique for solving the PDEs, i.e. FDM (Finite Difference Method) to prove that indeed volatilities and rates do not have to be constant, to solve the Black-Scholes PDE.

3. Deriving the Black-Scholes PDE

We start by considering the following Geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dX_t$$

We also set up a special portfolio, called Π, which consists of one long option and one short stock. We short only Δ quantities of this stock. So our special portfolio Π becomes:

$$\pi_t = V(S,t) - \Delta S_t$$

Now as the time passes, we need to consider the change in our portfolio. We consider the change in our portfolio in a very small time step i.e. $t \rightarrow t + dt$. Therefore the change in our portfolio is:

$$d\pi_t = dV(S,t) - \Delta dS_t$$

Now we consider the Taylor Series expansion for dV, i.e.:

$$dV(S,t) = \frac{\partial V(S,t)}{\partial t} dt + \frac{\partial V(S,t)}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V(S,t)}{\partial S^2} dS^2$$

We know that:

$$dS^2 = \mu^2 S_t^2 dt^2 + \sigma^2 S_t^2 dX_t^2 + 2 S_t^2 \mu\sigma dt dX_t$$

However, since the time step dt is a very small time step, so:

$$dt^2 \to 0$$

$$dX_t^2 \to dt$$

$$dt dX_t \to 0$$

So, we have:

$$dS^2 = \sigma^2 S_t^2 dt$$

Substituting this form of dS^2 and dS in the Taylor Series expansion for the Option Price yields:

$$dV(S,t) = \left(\frac{\partial V(S,t)}{\partial t} + \mu S_t \frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S,t)}{\partial S^2} \right) dt + \sigma S_t \frac{dV(S,t)}{dS} dX_t$$

Substituting the above expression in our expression for change in portfolio i.e. dΠ, we have:

$$d\pi = \left(\frac{\partial V(S,t)}{\partial t} + \mu S_t \frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S,t)}{\partial S^2} \right) dt + \sigma S_t \frac{dV(S,t)}{dS} dX_t - \Delta(\mu S_t dt + \sigma S_t dX_t)$$

Note that the above expression contains terms in dX, which is an increment in Brownian Motion. Therefore any risk present in our portfolio is because of this random Brownian increment. So we need to make the coefficients of dX equal to zero, to make our Portfolio Risk less. So, we must have the following, to achieve a risk free portfolio:

$$\sigma S_t \frac{dV(S,t)}{dS} dX_t - \Delta \sigma S_t dX_t = 0$$

This gives:

$$\Delta = \frac{dV(S,t)}{dS}$$

Therefore the above choice of Δ, makes our portfolio completely risk less. The above term is also called Delta. It determines the quantity of stock to be shorted.

Now we will substitute the expression for Δ in our expression for change in portfolio, to get:

$$d\pi = \left(\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S,t)}{\partial S^2}\right) dt$$

This change in portfolio is completely riskless, as we do the Dynamic Hedging using Δ. Now we know that if we manage to create a completely risk less portfolio, then its growth should be same as of a growth rate of an amount which is put in a risk free interest bearing account.

Hence we should have:

$$d\pi = r\pi dt$$

$$\left(\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S,t)}{\partial S^2}\right) dt = r(V(S,t) - \Delta S_t)dt$$

$$\left(\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S,t)}{\partial S^2}\right) dt = r\left(V(S,t) - \frac{dV(S,t)}{dS} S_t\right) dt$$

On dividing by dt and rearranging, we get the famous Black-Scholes equation:

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS_t \frac{dV(S,t)}{dS} - rV(S,t) = 0$$

The Black-Scholes equation is a Linear Parabolic Partial Differential Equation.

In the subsequent sections, we examine if we need to keep the volatility constant, to solve the above PDE. We also solve this equation using a famous numerical method, i.e. FDM (Finite Difference Method).

4. A brief note about constant volatilities and interest rates

In the derivation of the above Black-Scholes PDE we have not assumed the volatilities to be constant anywhere. However, simply having the Black-Scholes PDE is no good for practical purposes. We need a solution of the above PDE, given the particular Final Conditions and Boundary Conditions.

There are two broad methods to find the solution to the above PDE, analytical methods and numerical methods. Analytical methods are more popular than the numerical methods, as they give very elegant closed form solutions to the PDE. For e.g. the following is the closed form solution to value a Call Option on a Stock:

$$Call\ option\ value\ C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$where \quad d_1 = \frac{\log\left({}^{S}/_{E}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log\left({}^{S}/_{E}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

However, the problem in the analytical methods to solve the Black-Scholes PDE is that, they are not very simple if we assume the volatility to change with time. At most we can use some kind of average volatility, but the above formula really needs one hard number for the volatility. Further, the above formulae are not valid for the American Options, where the user has the right to exercise the option anytime till its maturity.

Worst still, if one has to value American Options, using a volatility which is changing with time, then the closed form solutions do not remain very elegant and approachable. These arguments remain true for interest rates as well, i.e. if we were to make interest rates, a function of time, then analytical methods do not remain very approachable.

To overcome this limitation the trading industry has resorted to numerical methods to solve the above PDE. In the next section, we explain one of the basic numerical techniques to solve the Black-Scholes PDE and show that we can use this powerful method to value American Options as well and change the volatility and interest rates throughout the process as we wish.

5. Finite Difference Method

Recall that we have to solve the following Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

The above PDE is a Linear Parabolic PDE and given the particular final and boundary conditions, we need to find the value of "V" in this PDE.

In this section we explain how to solve this PDE using a robust numerical technique called Finite Difference Method. First we introduce some notation. We let the Stock price vary from 0 to N and we divide the Stock price into small increments of δS. Similarly we vary the time from 0 to M and we divide the time into small increments of δt. So, we have,

$$S = n\delta S \qquad 0 \le n \le N,$$

$$t = m\delta t \qquad 0 \le m \le M$$

Now we consider the different terms of our PDE. We first consider the first term i.e. the term dV/ dt. It is called the Theta. This is the derivative of the Option price w.r.t time. It follows that:

$$\frac{dV}{dt} = \lim_{n\to\infty} \frac{V(S, t+h) - V(S,t)}{h}$$

$$\frac{\delta V}{\delta t} \approx \frac{V_n^m - V_n^{m-1}}{\delta t}$$

To see how accurate we are in the above approximation, we consider the Taylor Series of Option value at the asset value S and time t, as follows:

$$V(S, t+\delta t) = V(S,t) + \delta t \frac{\partial V(S,t)}{\partial S} + O(\partial S^2)$$

We translate the above differential equation into the following difference equation:

$$V_n^m = V_n^{m-1} + \delta t \frac{\partial V}{\partial t} + O(\delta t^2)$$

$$\frac{\partial V}{\partial t} = \frac{V_n^m - V_n^{m-1}}{\delta t} + O(\delta t)$$

So now that we have the difference equation for the first term i.e. Theta of our PDE, let us consider the next term i.e. Delta dV/ dS. Consider the following Taylor Series first,

$$V(S + \delta S, t) = V(S,t) + \delta S \frac{\partial V(S,t)}{\partial S} + \frac{1}{2} \partial S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + O(\partial S^3)$$

Similarly,

$$V(S - \delta S, t) = V(S,t) - \delta S \frac{\partial V(S,t)}{\partial S} + \frac{1}{2} \partial S^2 \frac{\partial^2 V(S,t)}{\partial S^2} - O(\partial S^3)$$

Subtracting from one another, dividing by 2δS and rearranging gives:

$$\frac{\partial V}{\partial S} = \frac{V_{n+1}^m - V_{n-1}^m}{2\delta S} + O(\delta S^2)$$

The above is the second term of our BS PDE, i.e. Delta. Finally let us consider the third term of our PDE i.e. Gamma d^2V/ d^2S. Consider the Forward difference and Backward difference forms of the Delta, i.e.

$$Forward\ Difference = \frac{V_{n+1}^m - V_n^m}{\delta S}$$

$$Backward\ Difference = \frac{V_n^m - V_{n-1}^m}{\delta S}$$

Take the difference between the forward difference and backward difference and divide it by δS, to see how this difference is changing w.r.t change in underlying stock price. So, we have:

$$\frac{\partial^2 V}{\partial S^2} = \frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{\delta S^2} + O(\delta S^2)$$

So now we finally write all the above results at one place:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0\ \ ----1$$

$$\frac{\partial V}{\partial t}=\frac{V_n^m-V_n^{m-1}}{\delta t}+O(\delta t)----2$$

$$\frac{\partial V}{\partial S}=\frac{V_{n+1}^m-V_{n-1}^m}{2\delta S}+O(\delta S^2)\ \ ----3$$

$$\frac{\partial^2 V}{\partial S^2}=\frac{V_{n+1}^m-2V_n^m+V_{n-1}^m}{\delta S^2}+O(\delta S^2)\ \ ----4$$

$$where$$

$$S=n\delta S\quad 0\le n\le N$$

$$t=m\delta t\quad 0\le m\le M$$

We write 1 as:

$$\frac{\partial V}{\partial t}+a\frac{\partial^2 V}{\partial S^2}+b\frac{\partial V}{\partial S}+cV=0\ \ ----5$$

$$a=\ \frac{1}{2}\sigma^2 S^2, b=rS, c=-r$$

Now substitute 2, 3 and 4 into 5 to get

$$\left[\frac{V_n^m-V_n^{m-1}}{\delta t}\right]+a\left[\frac{V_{n+1}^m-2V_n^m+V_{n-1}^m}{\delta S^2}\right]+b\left[\frac{V_{n+1}^m-V_{n-1}^m}{2\delta S}\right]+cV_n^m=O(\delta t,\delta S^2)$$

Multiply by δt and take V_n^{m-1} on LHS

$$V_n^{m-1}=\ V_n^m+\frac{\delta t}{\delta S^2}a[V_{n+1}^m-2V_n^m+V_{n-1}^m]+\frac{\delta t}{2\delta S}b[V_{n+1}^m-V_{n-1}^m]+\delta tcV_n^m+O(\delta t^2,\delta t\delta S^2)$$

Collect the terms in V_{n-1}^m, V_n^m and V_{n+1}^m

$$V_n^{m-1} = [\frac{\delta t}{\delta S^2} a - \frac{\delta t}{2\delta S} b]V_{n-1}^m + [1 - \frac{2\delta t}{\delta S^2} a + \delta tc]V_n^m + [\frac{\delta t}{\delta S^2} a + \frac{\delta t}{2\delta S} b]V_{n+1}^m + O(\delta t^2, \delta t\delta S^2)$$

Substitute $V_1 = \frac{\delta t}{\delta S^2} and\ V_2 = \frac{\delta t}{\delta S}$, so

$$V_n^{m-1} = [V_1 a - \frac{1}{2} V_2 b]V_{n-1}^m + [1 - 2V_1 a + \delta tc]V_n^m + [V_1 a + \frac{1}{2} V_2 b]V_{n+1}^m + O(\delta t^2, \delta S^2)$$

$$V_n^{m-1} = A_n^m V_{n-1}^m + [1 + B_n^m]V_n^m + C_n^m V_{n+1}^m - - - - - - -6$$

Where

$$A_n^m = V_1 a - \frac{1}{2} V_2 b$$

$$B_n^m = -2V_1 a + \partial tc$$

$$C_n^m = V_1 a + \frac{1}{2} V_2 b$$

Eqn 6 is of the form

$$V_n^{m-1} = f[V_{n-1}^m, V_n^m, V_{n+1}^m]$$

This equation has an error of $O(\delta t^2, \delta t\delta S^2)$

5.a Boundary conditions

Boundary Condition 1:

For a call option when S = 0 then V = 0, so $V_0^0 = 0$.

Boundary Condition 2:

We know that the price of the call option is

$$Call\ option\ value\ C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

$$where \quad d_1 = \frac{\log\left({S}/{E}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log\left({S}/{E}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Now when $S \to \infty, d_1 \to \infty, N(d_1) \to 1, N(d_2) \to 1$

So, we have,

$$C(S,t) = S - Ee^{-r(T-t)}$$

So boundary condition is

$$V_N^m = N\delta S - Ee^{-rm\delta t}$$

Boundary Condition 3:

As S = 0, the BSE becomes

$$\frac{dV}{dt} - rV = 0$$

Numerically this becomes

$$\frac{V_0^{m-1} - V_0^m}{\delta t} - rV_0^m = 0$$

$$V_0^{m-1} - V_0^m = r\delta t V_0^m$$

$$V_0^m = V_0^{m-1}(1 - r\delta t)$$

Boundary Condition 4:

From BC 2, we know that when

$$S \to \infty, d_1 \to \infty, N(d_1) \to 1, N(d_2) \to 1$$

So, we have,

$$C(S,t) = S - Ee^{-r(T-t)}$$

This is linear payoff. So,

$$\frac{\partial V}{\partial S} \to 1 \; and \; \frac{\partial^2 V}{\partial S^2} \to 0$$

This means,

$$V_{N-1}^m = \frac{V_N^m + V_{N-2}^m}{2}$$

Or,

$$V_N^m = 2V_{N-1}^m - V_{N-2}^m$$

This is a boundary condition for vanishing gamma.

5.b The method to change the volatilities and interest rates

The expression for $A_n^m, B_n^m \; and \; C_n^m$ can be further simplified. Recall that,

$$A_n^m = V_1 a - \frac{1}{2} V_2 b$$

$$B_n^m = -2V_1 a + \partial t c$$

$$C_n^m = V_1 a + \frac{1}{2} V_2 b$$

Substitute the values for V_1 ,V_2, a, b and c

$$V_1 = \frac{\partial t}{\partial S^2}, V_2 = \frac{\partial t}{\partial S}, a = \frac{1}{2}\sigma^2 S^2, b = rS, c = -r$$

Also we use $n = \frac{S}{\delta S}$

So we have

$$A_n^m = \frac{\delta t}{\delta S^2}\frac{1}{2}\sigma^2 S^2 - \frac{1}{2}\frac{\delta t}{\delta S} rS$$

$$A_n^m = \frac{1}{2}(\sigma^2 n^2 - rn)\delta t$$

$$B_n^m = -2\frac{\delta t}{\delta S^2}\frac{1}{2}\sigma^2 S^2 + \delta t(-r)$$

$$B_n^m = -(\sigma^2 n^2 + r)\delta t$$

$$C_n^m = \frac{\delta t}{\delta S^2}\frac{1}{2}\sigma^2 S^2 + \frac{1}{2}\frac{\delta t}{\delta S} rS$$

$$C_n^m = \frac{1}{2}(\sigma^2 n^2 + rn)\delta t$$

Now that we have the expressions for $A_n^m, B_n^m \; and \; C_n^m$, examine our earlier difference equation,

$$V_n^{m-1} = \; A_n^m V_{n-1}^m + [1 + B_n^m] V_n^m + C_n^m V_{n+1}^m$$

The above difference is of the form,

$$V_n^{m-1} = \; f[V_{n-1}^m, V_n^m, V_{n+1}^m]$$

Note that the above difference equation implies that, we can calculate the option value one step back in time, from three option values which are one step ahead in time, but at three different points of the underlying Stock price. This is really the crux of the Finite Difference Methods. We can travel back in time, using option values at different points, which are one step ahead in time.

Now examine the expressions for $A_n^m, B_n^m \; and \; C_n^m$

$$A_n^m = \frac{1}{2}(\sigma^2 n^2 - rn)\delta t$$

$$B_n^m = -(\sigma^2 n^2 + r)\delta t$$

$$C_n^m = \frac{1}{2}(\sigma^2 n^2 + rn)\delta t$$

Note that only $n = \frac{S}{\delta S}$ and $\delta t,$ need to be held constant (because we have to define these from the start of our algorithm). Rest of the parameters in the above equations are volatility(σ) and interest rate(r). So we can vary these as we like while we are moving back in time, calculating option values.

In this framework, we can even make very realistic assumptions that volatilities are some functions of the underlying stock price and time, and we can implement some external econometric processes like GARCH/ ARIMA etc., which can give us volatilities and interest rates at different points of underlying stock price and time.

Our objective in this paper was to show that Black-Scholes framework does not really imply that volatilities and interest rates have to be held constant to compute the prices of the derivatives. We hope that we are able to demystify this common misbelief.