THE QUANT CLASSROOM by ATTILIO MEUCCI

“The Prayer”: The 10 Steps of Advanced Risk and Portfolio Management – Part 2

The second of a two-part article on the path from data analysis to optimal execution across all asset classes and investment styles.

Here, we continue our journey through “The Prayer,” a blueprint of 10 sequential steps for quants across the board to achieve their common goal. Steps 1-4 were discussed in the Quant Classroom published in the April issue of Risk Professional (also available at symmys.com). In this article, we discuss the remaining Steps 5-10. Please refer to Figure 1 (left) for a map of The Prayer.

5 Aggregation

The Pricing Step 4 yields the projected P&Ls of the single securities. The Aggregation Step generates the P&L distribution for a portfolio with multiple securities.

Consider a market of N securities, whose P&Ls $\Pi = \Pi_1, \ldots, \Pi_N$ are obtained from the Pricing Step 4. We drop from the notation of the P&L the subscript $T$ to simplify, because it is understood that from now on The Prayer focuses on the projected P&L between now and the future investment horizon.

Consider a portfolio, which is defined by the holdings in each position at the beginning of the period $h = (h_1, \ldots, h_N)$. The holdings are the number of shares for stocks, the number of standardized-face-value contracts for swaps, the number of standardized-face-value bonds for bonds, etc.

The portfolio P&L is determined by the “conservation law of money,” the total portfolio P&L is the sum of the holdings in each security times the P&L, generated by each security, as follows:

$$\Pi_n = \sum_{i=1}^{N} h_i \Pi_i,$$

where we have assumed no rebalancing during the period.

Key concept. The Aggregation Step is the process of computing the distribution of the portfolio P&L, $\Pi$, by aggregating the joint distribution of the securities P&Ls with the given holdings $f_n : h \mapsto f_n(h)$.

Given one single scenario for the risk drivers $\Gamma_n$, and thus for the securities P&Ls $\Pi_1, \ldots, \Pi_N$, the computation of the portfolio P&L is immediately determined by the conservation law of money (1) as the sum of the single-security P&Ls in that scenario.

However, to arrive at the whole continuous distribution of the portfolio P&L, $f_n$, we must compute multiple integrals, as follows:

$$f_n(h) \, dh = \int_{\Delta \Gamma_n} f_n(x_1, \ldots, x_N) \, dx_1 \ldots dx_N,$$

which is in general a very challenging operation.

On the other hand, the computation of the aggregate P&L distribution becomes trivial when the market is represented in terms of scenarios, as the conservation law of money (1) is simply repeated in a discrete way, scenario-by-scenario.

Illustration. In our example with a stock and a call option, whose P&Ls are normally distributed, suppose we hold a positive or negative number $h$ of shares of the stock and a positive or negative number $\tau$ of the call. Then the total P&L follows from applying the aggregation rule (1) to the stock P&L and the option P&L, which were obtained in the first part of this article in terms of their delta-vega pricing approximation, i.e.:

$$\Pi_n = \sum_{i=1}^{N} h_i \Pi_i + \sum_{i=1}^{N} \tau_i \Pi_i + \sum_{i=1}^{N} \rho_{i,j} \Pi_i + \sum_{i=1}^{N} \delta_{i,j} \Pi_i,$$

where

$$\rho_{i,j} = (\nu_i - \nu_j + \delta_{i,j}) \sigma_{i,j} \Pi_i + \delta_{i,j} \Pi_i,$$

and

$$\delta_{i,j} = 2 \rho_{i,j} \sigma_{i,j} \Pi_i,$$

where $\Pi_i$ is the single-security P&L, $h_i$ the number of shares for stocks, $\tau_i$ the number of standardized-face-value contracts for swaps, and $\delta_{i,j}$ the number of standardized-face-value bonds for bonds, etc.

Thus, from the joint normal assumption for $\ln\Pi_i$ and $\ln\Pi_j$, and from the fact that sums of jointly normal variables are normal, the total portfolio is normally distributed.

Isolating the horizon $h$, we obtain

$$\Pi_n \sim N(\mu_n, \sigma_n^2),$$

where

$$\mu_n = (h_1 \nu_1 + h_2 \nu_2 + \cdots + h_N \nu_N) \Pi_1 + \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{i,j} \Pi_i$$

$$\sigma_n^2 = (h_1^2 \sigma_1^2 + h_2^2 \sigma_2^2 + \cdots + h_N^2 \sigma_N^2) \Pi_1 + 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{i,j} \sigma_{i,j} \Pi_i.$$

Previously, we described in full the Aggregation Step. However, this topic is not complete without comparing the aggregation of the P&L with an equivalent, more popular — yet more error-prone — formulation in terms of returns.

The reader is probably familiar with the notion of returns, which allow for performance comparisons across different securities and portfolio weights. The return is the ratio of the P&L over the current price $R_{T,T} = \Pi_T / \Pi_T$. The weight of a security is its relative market value within the portfolio $w_i = \Pi_i / \sum \Pi_i$ and satisfies the “pie-chart” rule $\sum w_i = 1$.

The conservation law of money (1) becomes easier to interpret in terms of returns and weights, as the total portfolio return is the following weighted average of the single-security returns:

$$R_n = \sum_{i=1}^{N} w_i \Pi_i,$$

where we dropped the horizon subscript for simplicity.

In “The Prayer,” we refrain from conceptualizing the aggregation and the subsequent steps in terms of returns, and we rely on returns only for interpretation purposes, for the following reasons.

First, P&L and holdings are always unequivocal, whereas returns and weights are subjective. Indeed, for leveraged securities, such as swaps and futures, the definition of returns and weights is not straightforward. In these circumstances, we need to introduce a subjective “basis” denominator $\Delta$ known at the beginning of the return period, such that both the return $R_{\Delta,T}$ and the weight (see Meucci, 2010a) are always defined.

Second, returns are often confused with the invariants, and thus incorrectly used for estimation.

Third, the linear returns $R_{T,T}$, $R_{T,T-1}$, $R_{T-1,T}$, which appear in the aggregation rule (3), are often confused with the compounded returns $\ln(1 + R_{T,T})$ which do not satisfy the aggregation rule.

Pitfall. “...Returns are invariants. Therefore we can estimate their distribution from their past realizations and aggregate this distribution in the portfolio level using the weights.” Only in asset classes such as stocks do the concepts of invariant and return dangerously overlap. On the other hand, the computation of the aggregate P&L distribution becomes trivial when the market is represented in terms of scenarios, as the conservation law of money (1) is simply repeated in a discrete way, scenario-by-scenario.

6 Attribution

With the Aggregation Step 5, we have arrived at the projected portfolio P&L distribution. In order to assess, manage and hedge a portfolio with $h = (h_1, \ldots, h_N)$ holdings, it is important to

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RISK PROFESSIONAL JUNE 2011

35
ascertain the sources of risk that affect it. Given the distribution of the projected portfolio P&L, we would like to identify a parsimonious set of relevant factors \( \zeta \equiv (\zeta_1, \ldots, \zeta_n) \) that drive the portfolio P&L and that have a known joint distribution with the portfolio P&L, \( \zeta P = \zeta P \).

More specifically, because the identification of the factors should be actionable and easy to interpret, the attribution should be linear. Thus, the attribution is defined by coefficients \( b_h \equiv (b_{h1}, \ldots, b_{hn}) \), as follows:

\[
\Pi_t = \sum_h b_h \zeta_h Z_h ,
\]

(9)

Note that the attribution to arbitrary factors in general gives rise to a portfolio-specific residual. The formulation (9) covers this case, by setting such a residual as one of the factors, \( \zeta_n \), with attribution coefficient \( b_n = 1 \).

Key concept. The Attribution Step decomposes the projected portfolio P&L linearly into a set of \( K \) relevant risk factors \( \zeta \), yielding the \( K \)-portfolio specific exposures \( b_h \):

\[
f_{bh} \to = \Pi_{bh} Z_h ,
\]

(10)

The relevant question is which attribution factors \( Z_k \) to use. Naturally, different intentions of the trader or portfolio manager can apply for different choices of attribution factors.

The most trivial attribution assigns the projected portfolio P&L back to the contributions from each security: i.e., \( Z_t = \Pi = \Pi \) is the projected P&L from the generic \( h \)-security; \( b_h = h \) is the holdings of the \( h \)-security in the portfolio, and the number of factors is \( k = n \) the number of securities.) Then the attribution equation (9) becomes the conservation law of the P&L.

If on the other hand the trader wishes to hedge a given risk — say, volatility risk — then he or she will choose as a factor \( Z \) the projected P&L of a truly actionable instrument, such as a variance swap, that might or might not have been part of the original portfolio.

Alternatively, the portfolio manager might wish to monitor the exposure to a given risk factor, without the need to hedge it. If, for instance, the manager is interested in the total “vagga” of its portfolio, then he or she will use changes in implied volatility as one of the risk factors.

Furthermore, in case too many possible factors or hedging instruments exist, the manager will want to express his or her portfolio as a function of only those few factors that truly affect the P&L.

Notice that (9) is a portfolio-specific, top-down linear factor model. The flexible choice of the optimal attribution factors \( Z \) and optimal exposures \( b \) with flexible constraints that define this linear factor model — along with its connections with the linear factor models introduced in the Estimation Step P2 — is the spirit of the “Factors on Demand” approach in Meucci (2010a).

Illustration. In our stock and option example, we look at a simple attribution (9) to the original sources of risk. According to this attribution, the stock compensated return \( Z = \ln S_t - r T \) and the implied volatility log-change \( \ln(\sigma_t) \) are the only risk factors. Thus, we have \( K = 2 \) factors. From the expression of the portfolio P&L (4), we immediately obtain

\[
b_h = b_h Z_h + b_h Z_s ,
\]

(11)

where the total exposures to \( Z \) and \( Z_s \) read, respectively,

\[
b_h = b_h \Delta T + b_h \Delta R , \quad b_s = b_s \Delta V .
\]

(12)

Pinball — “If I use a factor model to estimate the returns distribution of some stock and I want my portfolio to be neutral to a given factor, I simply make sure that the exposure to that factor is zero in my portfolio...” Ensuring a null-exposure coefficient for one factor does not guarantee immunization, because the given factor is in general correlated with other factors. To provide full immunization, we must resort to Factors on Demand.

E.7 Evaluation

Up to this step, we have obtained the projected distribution of the portfolio P&L of a generic portfolio with holdings \( b \) and attributed it to relevant risk factors \( Z \). In the evaluation step, the goal is to compare the P&L distribution of the current portfolio \( \Pi \) with the P&L distribution of a different portfolio \( \Pi' \). Evaluation is one of the risk and portfolio manager’s primary tasks.

Since each portfolio is represented by the whole projected P&L and \( \Pi' \), it is not possible to compare two portfolios in terms of which P&L is higher. To obviate this problem, typically practitioners rely on one or more summary statistics for the projected P&L distribution.

The most standard attributes are the expected value, the standard deviation and the Sharpe ratio — also known, respectively, as expected outperformance, tracking error and information ratio in the case of benchmarked portfolio management. Other measures include the value-at-risk (VaR), the expected shortfall (ES or CVaR), skewness, kurtosis, etc. More innovative statistics include coherent measures of risk aversion (see Artzner, Delbaen, Eber and Heath, 1997); spectral measures of risk aversion (see Acerbi, 2002); and measures of diversification, such as the “effective number of bets” (see Meucci, 2009). We emphasize that, in this context, all the above are ex-ante measures of risk for the projected portfolio P&L, \( \Pi' \), rather than ex-post measures of performance.

Key concept. The Evaluation Step consists of two sub-steps. The first sub-step is the computation of one or more summary statistics \( S \), for the projected distribution of the given portfolio P&L, \( \Pi' \), with holdings \( b \):

\[
f_{bh} \to \to = S \left(h\right) .
\]

(13)

The second (optional) sub-step is the attribution of the summary statistics \( S \) to the fully flexible attribution factors \( Z \) utilized in the Attribution Step

\[
f_{bh} \to \to \to = S \left(h\right) = \sum_k b_h \delta k \sigma \eta_k .
\]

(14)

where \( \delta k \) represents the “amount” of the factor \( Z_k \) in the portfolio-projected P&L and \( S \) represents the “per-unit” contribution to the statistic \( S \) from the factor \( Z_k \).

Illustration. In our simple, normal market of one stock and one option, any portfolio is determined by the holdings \( b_{S0} b_{V0} \). Let us focus on the first sub-step (13) and let us compute the most basic summary statistics of the P&L: namely, its expected value. Then, from the distribution of a generic portfolio \( \Pi \), we obtain

\[
S_{bh} = E \left(h\right) = \gamma_h = b_{S0} \gamma_S + b_{V0} \gamma_V + \gamma_T \left(b_{S0} \gamma_{SV} + b_{V0} \gamma_{VV} \right) .
\]

(15)

Similarly, if the manager cares about a measure of volatility, a suitable measure is the standard deviation

\[
S_{bh} = \sigma_h = \sqrt{\delta \Pi} .
\]

(16)

where \( \sigma_h \) is defined in (7).

For the optional summary statistics attribution sub-step (14), a simple linear decomposition that mirrors the attribution equation (9) is feasible. For instance, for the standard deviation, it is well known that

\[
\sigma_h = \sum_k b_h \delta k \sigma_k \eta_k .
\]

(17)

However, notice that numerous summary statistics, such as expectation, standard deviation, VaR, ES and spectral measures, display an interesting feature: they are homogeneous — i.e., by doubling all the holdings in the portfolio, those summary statistics also double. As proved by Euler, for homogeneous statistics, the following identity holds true:

\[
S \left(h\right) = \sum_k b_h \delta k \sigma \eta_k .
\]

(18)

Therefore, if the summary statistics are homogeneous, we can take advantage of Euler’s identity (17) to perform the summary statistics attribution sub-step (14), which becomes identity (12).

In particular, for the VaR, the decomposition (17) amounts to the classical definition of marginal contributions to VaR (see, e.g., Garman, 1997), and for the standard deviation, the decomposition (17) amounts to the “hot-spot” (see Litterman, 1996).

We recall that the simplest case of the flexible, top-down, Factors on Demand attribution of the portfolio P&L (9) is the bottom-up attribution to the individual securities through the conservation law of money (1). Similarly, the simplest case of attribution of the summary statistics (14) is the attribution of the summary statistics \( S \) to the individual securities

\[
S \left(h\right) = \sum_k b_h \delta k \sigma \eta_k .
\]

(19)

Illustration. To illustrate the attribution to a summary statistic of the portfolio projected P&L, we rely on our example of a stock and a call option. We focus on the standard deviation (16). The exposure \( b_{S0} \) of the projected portfolio P&L (9) to the stock factor \( Z = \ln S_t - r T \) and the exposure \( b_{V0} \) to the implied volatility factor \( Z = \ln(\sigma_t) \) were calculated in (12). Then the attribution (17) to each of the two risk drivers of the standard deviation of the projected portfolio P&L becomes

\[
\sigma_h = \sum_k \delta k \sigma_k \eta_k = \left(b_{S0} \delta_k \sigma_S + b_{V0} \delta_k \sigma_V \right) \left(b_{S0} \gamma_{SV} + b_{V0} \gamma_{VV} \right) .
\]

(19)

Where \( \sigma_k \) is defined in (7) (see the proof in the technical appendix available at http://symmys.com/node/62). The total contributions to risk from the factor \( Z \) in the portfolio projected P&L and

\[
S \left(h\right) = \sum_k b_h \delta k \sigma \eta_k .
\]

(17)

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Therefore, if the summary statistics are homogeneous, we can take advantage of Euler’s identity
\[ E(\delta h) = E(\delta h, \sigma) \]
where
\[ \delta h = E(\delta h, \sigma) \]
see the proof in the technical appendix available at http://symmys.com/node/62. The total contributions to risk from the stock and the call option follow by multiplying the entries on the left-hand side of (20) by the respective holdings \( h \) and \( \lambda \).

The computation of the summary statistics \( S(h) \) is hard to perform in practice, unless the market is normal as in our example (16), because complex multivariate integrals are involved. For instance, using the same notation as in (3), the VaR with confidence \( c \) is defined by
\[ 1 - c = \int_{\delta h > 0} f_\delta(h, \sigma) \, dh \cdot \, d\sigma . \]  
(23)

To address this problem, one can rely on approximation methods such as the Cornish-Fisher expansion or the elliptical assumption (see Meucci, 2005a) for a review. The computation of the partial derivatives for the decomposition \( \Omega(h) = E(h) - \Pi(h) \) is easier. The first order condition on the P&L standard deviation, computed in the Estimation Step \( \Omega(h) \) is
\[ h^* = \text{argmax } \{ \delta h, \sigma \} \]  
(24)

Clearly, the optimal allocation should not violate a set of hard constraints, such as the budget constraint, or soft constraints, such as constraints on leverage, risk, etc. We denote by \( C \) the set of all such constraints and by \( h \in C \) the condition that the allocation \( h \) satisfies the given constraints.

Key concept. The Optimization Step is the process of computing the holdings that maximize satisfaction, while not violating a set of investment constraints, as follows:
\[ h^* = \text{argmax } \{ \delta h, \lambda \} \]  
(25)

We emphasize that the choice of the most suitable index of satisfaction \( S(h) \) as well as the specific constraints \( C \), vary widely depending on the profile of the securities P&L distribution, the investment horizon and other features of the market and the investor.

Illustration. In our stock and option example, we can compute the best hedge for one call option. In this context, the general framework (24) becomes
\[ (h_1, h_2)^* = \text{argmax } \{ \delta h_1, \lambda \} \]  
(26)

Then the first order condition on the P&L standard deviation, computed in (24), yields
\[ h_1^* = \frac{\partial \Pi(h_1)}{\partial \delta h_1} \]  
(27)

If the correlation \( \rho \) between implied volatility and underlying was greater, the best hedge would consist in shorting a "sticky" amount of underlying. In general, \( \rho \) is substantially negative: for instance, the sample correlation between VIX and S&P 500 is \( \rho = -0.7 \). Therefore, a correction to the simplistic delta hedge must be applied.

In general, the numerical optimization (24) is a challenging task. To address this issue, one can resort to the two-step mean-variance heuristic (27)-(28). These advanced approaches include robust optimization, which relies on more complex models for returns; the Bayesian inspired methodology by Michaud, 1998; robust Bayesian allocation by Meucci (2005a); and resampling (see Michaud, 1998). We refer to Meucci (2005a) for an in-depth review.

Since estimation is imperfect, tactical portfolio construction enhances performance by blending market views and predictive signals into the estimated market distribution. Well-known techniques to perform tactical portfolio construction are the approach by Grinold and Kahn (1999), which mixes signals based on linear factor models for returns; the Bayesian inspired methodology by Black and Litterman (1990); and the generalized Bayesian approach "Entropy Pooling" in Meucci (2005).

Due to the rapid decay of the quality of predictive tactical signals, managers separate tactical portfolio construction from strategic rebalancing, which takes into account shortfall and drawdown control and is optimized based on techniques that range from dynamic programming to heuristics (see, e.g., Moreton, 1992; Grossman and Zhou, 1993, and Browne and Kosovski, 2010). Refer to Meucci (2010b, for a review and code).

Finally, liquidity risk, discussed in the Pricing Step \( \Pi(h) \), impacts the Optimization Step: transaction costs must be paid to reallocate capital, and the process of executing a transaction impacts the execution price. Therefore, market impact models must be embedded in the portfolio optimization process. The standard approach in this direction is a power-law impact

In the second step of the mean-variance heuristic, the optimal portfolio is selected by a one-dimensional search
\[ h^* = \text{argmax } \{ S(h) \} \]  
(28)

The optimization (28) can be performed by a simple grid-search. As it was the case for the Evaluation Step \( \Pi(h) \), we must address estimation risk, introduced in the Estimation Step \( \Omega(h) \) and liquidity risk, introduced in the Pricing Step \( \Pi(h) \).

The Optimization Step is the process of computing the holdings that maximize satisfaction, while not violating a set of investment constraints, as follows:
\[ h^* = \text{argmax } \{ \delta h, \lambda \} \]  
(25)
model (e.g., Keim and Madhavan, 1998).

Pitfall: “...Mean-variance exames normality...” The mean-variance approach does not assume normality; any market distribution can be fed into the two-step process (Taj, 2006).

9 Execution

The Optimization Step generates an allocation $\pi^{*}$ to a linear combination of strategies. The Execution Step relaxes the constraints (e.g., $\pi^{*}$) to attain the target portfolio $\pi^{*}$ by a sequence of transactions at given prices $p = \{p_{i}\}$. Note that often the execution step is aggregated across different books. This aggregation is particularly useful, as it allows for netting of the current allocation $\pi^{*}$ and the investment horizon $T$. Upon reaching the horizon, we must evaluate the P&L realized over the horizon by the allocation, where the lower-case notation emphasizes that the P&L is no longer a random variable, but rather a number that we observe ex-post.

Pitfall: “…The Execution Step $\pi^{*}$ should be embedded into the Optimization Step $\pi^{*}$. In practice, it is not possible to process simultaneously real-time information and all the previous steps of The Prayer. Furthermore, execution works best across all books, whereas optimization is specific to each individual manager.

REFERENCES


*This is an abbreviated list. To read all of the references, please go to http://symm.com/risk/63/


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