Collective Risk Models with Hierarchical Archimedean Copulas

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Abstract

In the finance and insurance industries, collective risk models, where the aggregate claim amount of a portfolio is defined in terms of random sums, play a crucial rule. In these models, it is common to assume that the number of claims and their amounts are independent, even if this might not always be the case. This paper uses Archimedean and hierarchical Archimedean copulas in collective risk models, to model the dependence between claim counts and the amounts involved in the random sum. Such dependence structures allow us to derive a computational methodology for the assessment of the aggregate claim amount. While being very flexible, this methodology is easy to implement, and can easily fit more complicated hierarchical structures. Using specific distributions for the number and the amounts of claims, we also derive explicit expressions for the aggregate claim amount and its related quantities.

Keywords: Random sums; Collective Risk Models; Archimedean Copulas; Hierarchical Archimedean Copulas

1 Introduction

Random sums are often used to model the aggregated losses of an insurance company. For a given portfolio of policyholders, the total amount S paid on all claims over a fixed period of time is defined as

$$S = \sum_{i=1}^{N} X_i,$$

where $\underline{X} = \{X_i, i \in \mathbb{N}\}$ is a sequence of non-negative random variables (rvs), and N is a positive counting rv. The rv N represents the number of claims and X_i corresponds to the amount of the i^{th} claim.

In the classical collective risk model, $X_1, X_2, \ldots$ are assumed to be independent of N, and also independent and identically distributed (iid) rvs (see e.g. [Rolski et al., 1999] and [Klugman et al., 2009]). However in practice, these assumptions are not always verified. For example, while analyzing a car insurance data-set, [Gschlößl and Czado, 2007] found that the number and the size of claims are significantly dependent. See also, e.g. [Kousky and Cooke, 2009] for other related examples, such as the highlighted dependency between flood damage and wind damage, using a catastrophic loss data.

While several papers proposed models that only account for the dependence between claim amounts (see e.g., [Denuit et al., 2006]), few others considered an extra dependency between claim amounts and claim counts as well. For example, claim counts can be considered as predictors for the claim amounts, see e.g., [Gschlößl and Czado, 2007], [Frees et al., 2011] and [Garrido et al., 2016]. Among others, [Czado et al., 2012] and [Krämer et al., 2013] propose to use families of bivariate copulas to model the dependency relationship between the number of claims and the average claim amount. In another setting, inter-claim times and claim sizes are assumed to be dependent in a compound Poisson process, see e.g., [Albrecher et al., 2014], [Boudreault et al., 2006], [Cossette et al., 2008], and [Landriault et al., 2014].

In this paper, we look at collective risk models incorporating dependent components. The underlying dependence structure is induced via an Archimedean or a hierarchical Archimedean copula. Here, a distinction is made between two types of dependency relationships: the one between the components of $\underline{X}$ and the one between N and $\underline{X}$. The risk model considered in this article is an extension of the one studied in Section 4 of [Cossette et al., 2018], in which $\underline{X} = \{X_j, j \in \mathbb{N}\}$ forms a sequence of exchangeable rvs independent of the counting positive discrete rv N. In addition to considering a dependence relationship between claim amounts, the proposed model here links N and the exchangeable sequence $\underline{X}$ with an Archimedean or a hierarchical Archimedean copula. Based on stochastic orderings, dependence properties are studied and links with [Liu and Wang, 2017] are established. Similarly to [Cossette et al., 2018], a computational methodology is proposed to analyze the distribution of the aggregate claim amount rv S, which is defined as a random sum.

The outline of the paper is as follows. In Section 2, the model is presented, and increasing convex ordering inequalities on S are derived in different settings. Section 3 presents the computational methodology for the distribution of S and its related quantities using either an Archimedean or a hierarchical Archimedean copula. In the last section, an extension of the risk model described and studied in Section 2 of [Albrecher et al., 2011] is proposed, and explicit formulas are derived in different setups.

2 Collective risk models with dependence

Let the aggregate claim amount be defined as $S = \sum_{k=1}^{N} X_k$, where N is a counting rv (i.e., $N \in \mathbb{N}_0$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), and $\underline{X} = (X_1, X_2, \dots)$ be a vector of exchangeable rvs, i.e., $X_k \sim X$ for $k \in \mathbb{N}$. Note that $\sum_{k=1}^{0} X_k = 0$ by convention. Let $(N, \underline{X}) = (N, X_1, X_2, \dots)$ be a vector of rvs for which the multivariate cumulative distribution function (cdf) (or its survival function) is defined with the copula C and the univariate cdfs $F_N, F_{X_1}, F_{X_2}, \dots,$ (or the univariate survival functions $\overline{F}_N, \overline{F}_{X_1}, \overline{F}_{X_2}, \dots$) of $N, X_1, X_2, \dots$, i.e.,

$$F_{N,\underline{X}}(k, x_1, \dots, x_k) = C\left(F_N(k), F_{X_1}(x_1), \dots, F_{X_k}(x_k)\right), \; k \in \mathbb{N}, \tag{1}$$

or

$$\overline{F}_{N,\underline{X}}(k, x_1, \dots, x_k) = C\left(\overline{F}_N(k), \overline{F}_{X_1}(x_1), \dots, \overline{F}_{X_k}(x_k)\right), \; k \in \mathbb{N}, \tag{2}$$

where C is an Archimedean or a hierarchical Archimedean copula. This model allows to incorporate a dependence structure between the claim number and the claim amounts.

Archimedean copulas could be good candidates to model such a dependence structure due to their flexibility, simple construction procedure, multivariate generalization, and their ability to capture different tail dependencies. However, the inherent exchangeability in Archimedean copulas implies that the dependence between the number of claims N and their amounts X_i for $i \in \mathbb{N}$. is the same as the dependence between the components of $\underline{X}$. In practice, this exchangeability is a very strong assumption. A more realistic dependence structure could be a hierarchical one. For example, we can consider a one level hierarchical Archimedean copula allowing to have different dependency relationships between N and X_i and between the rvs X_i for $i = 1, 2, \dots$ Such a dependence structure can be illustrated with a tree representation as shown in Figure 1. In this paper, we consider nested Archimedean copulas and the hierarchical Archimedean copulas through compounding proposed in [Cossette et al., 2017], to model the dependence structure depicted in Figure 1.

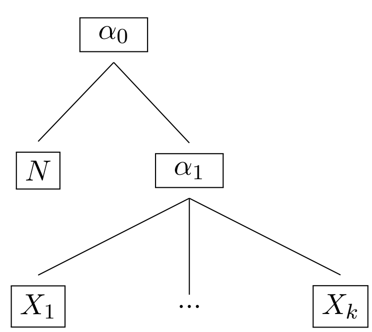

Figure 1: One level hierarchical tree structure.

Nested Archimedean copulas, first introduced by [Joe, 1997], are obtained by plugging-in

Archimedean copulas into each other. The resulting copula can capture hierarchical Archimedean dependence structures (i.e., different dependencies between and within groups), which can be more appropriate and helpful for practical applications as discussed in, e.g., [Joe, 1997] and [Hofert, 2010].

A copula C is said to be a nested Archimedean copula if at least one of its arguments is an Archimedean copula. Therefore, there are infinitely many ways to nest Archimedean copulas. For instance, a one level d-dimensional nested Archimedean copula, that can fit the dependence structure depicted in Figure 1, can be written as

$$C(u_1, u_2, \ldots, u_d) = C(u_1, C(u_2, \ldots, u_d)) = \psi_0\left(\psi_0^{-1}(u_1) + \psi_0^{-1} \circ \psi_1\left(\sum_{i=2}^{d} \psi_1^{-1}(u_i)\right)\right), \tag{3}$$

where ψ_0 and ψ_1 are respectively the generators of the outer copula (also named the mother copula) and the inner copula (or the child copula). Note that ψ_0 and ψ_1 are Laplace-Stieltjes transforms (LSTs) of positive rvs Θ_0 and Θ_1 respectively. A sufficient condition for (3) to be a proper copula is that $\psi_0^{-1} \circ \psi_1$ (or equivalently $\mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_1}$) must have completely monotone derivatives (see e.g. [Joe, 1997] and [McNeil, 2008] for more details). Such a condition implies that the function $\psi_{0,1}(t;\theta) = \exp\left\{-\theta\mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_1}(t)\right\}$ is a LST of positive a rv $\Theta_{0,1}$ (see e.g. [Joe, 2014] for details).

In general, for a hierarchical copula with several nesting levels, [Joe, 1997] and [McNeil, 2008] show that for any node with parent i and child j, the complete monotonicity of the function $\mathcal{L}_i^{-1} \circ \mathcal{L}_j$, for $(i < j)$, is a sufficient condition for a nested Archimedean copula to be a proper copula. Such a condition implies that the function $\psi_{i,j}(t;\theta) = \exp\left\{-\theta\mathcal{L}_{\Theta_i}^{-1} \circ \mathcal{L}_{\Theta_j}(t)\right\}$ is a LST of a positive rv (see e.g. [Joe, 2014] for details). In this paper, we only consider nested Archimedean copulas of the form (3) for which the nesting condition is verified.

To bypass the constraints related to the nesting condition of the nested Archimedean copulas, several research papers proposed other hierarchical Archimedean structures (see e.g. [Hering et al., 2010], [Brechmann, 2014], [Bedford and Cooke, 2002] and [Joe, 1997]). The approach recently proposed in [Cossette et al., 2017] consists of constructing a hierarchical copula from a multivariate mixed exponential distribution.

Let us consider a risk portfolio with d different sectors of activity (subgroups), such that every subgroup $i = 1, \ldots, d$, is influenced by a mixing rv Θ_i, and the dependence between subgroups is induced by the dependence linking the components of the vector $\underline{\Theta} = (\Theta_1, \ldots, \Theta_d)$. In this case, the associated copula C can be written as

$$C(\underline{u}) = \mathcal{L}_{\underline{\Theta}}\left(\sum_{j=1}^{n_1} \mathcal{L}_{\Theta_1}^{-1}(u_{1,j}), \ldots, \sum_{j=1}^{n_d} \mathcal{L}_{\Theta_d}^{-1}(u_{d,j})\right), \tag{4}$$

where $\underline{u} = (\underline{u_1}, \ldots, \underline{u_d})$ with $\underline{u_i} = (u_{i,1}, \ldots, u_{i,n_i})$ for $i = 1, \ldots, d$.

Further, the dependence structure of the mixing vector $\underline{\Theta}$ is modelled with a compound distribution such that every mixing rv Θ_i, $i = 1, \ldots, d$, can be represented as a random sum, i.e. $\Theta_i = \sum_{j=1}^{M} B_{i,j}$. The rv M is a positive discrete rv and the elements of the sequence $\underline{B_i} = \{B_{i,j}, j = 1, 2, \ldots\}$ are assumed to be iid and strictly positive rvs. Also, the sequences $\underline{B_i}$, $i = 1, \ldots, d$, are independent from each other and from the rv M. With this assumption, the copula in (4) becomes

$$C(\underline{u}) = \mathcal{L}_M\left(\sum_{i=1}^{d} - \ln\left(\mathcal{L}_{B_i}\left(\sum_{j=1}^{n_i} \mathcal{L}_{\Theta_i}^{-1}(u_{i,j})\right)\right)\right). \tag{5}$$

We can adapt this hierarchical Archimedean structure to fit the dependence model presented in this paper as follows

$$C(u_1, \ldots, u_d) = \mathcal{L}_M\left(\mathcal{L}_M^{-1}(u_1) - \ln\left(\mathcal{L}_{B_i}\left(\sum_{i=2}^{d} \mathcal{L}_{\Theta}^{-1}(u_i)\right)\right)\right). \tag{6}$$

Let $(N, \underline{X}) = (N, X_1, X_2, \ldots)$, $\left(N^{(+)}, \underline{X}^{(+,+)}\right) = \left(N^{(+)}, X_1^{(+,+)}, X_2^{(+,+)}, \ldots\right)$, and $\left(N^{(\perp)}, \underline{X}^{(\perp,\perp)}\right) = \left(N^{(\perp)}, X_1^{(\perp,\perp)}, X_2^{(\perp,\perp)}, \ldots\right)$, where $N^{(+)} \sim N^{(\perp)} \sim N$ and $X^{(+,+)} \sim X^{(\perp,\perp)} \sim X$, be, respectively, a vector of dependent rvs whose joint cdf is defined as in (1) with a hierarchical Archimedean copula as in (3) or in (6), with dependence parameters α_0 and α_1, a vector of comonotonic rvs, and a vector of independent rvs. We define the rvs S, $S^{(+,+)}$, and $S^{(\perp,\perp)}$ respectively by

$$S = \sum_{i=1}^{N} X_i,$$

$$S^{(+,+)} = \sum_{i=1}^{N^{(+)}} X_i^{(+,+)},$$

and

$$S^{(\perp,\perp)} = \sum_{i=1}^{N^{(\perp,\perp)}} X_i^{(\perp,\perp)}.$$

The expression of $E[S]$ is given by

$$E[S] = E_{\Theta_0}[E[S|\Theta_0]] = \int E[S|\Theta_0 = \theta_0] \, \mathrm{d}F_{\Theta_0}(\theta_0). \tag{7}$$

Given $\Theta_0 = \theta_0$, $(N|\Theta_0 = \theta_0)$ and $(\underline{X}|\Theta_0 = \theta_0)$ are conditionally independent, which implies that

$$E[S|\Theta_0 = \theta_0] = E[N|\Theta_0 = \theta_0] \, E[X|\Theta_0 = \theta_0]. \tag{8}$$

Inserting (8), (7) becomes

$$E[S] = \int E[N|\Theta_0 = \theta_0] \, E[X|\Theta_0 = \theta_0] \, \mathrm{d}F_{\Theta_0}(\theta_0). \tag{9}$$

Note that (9) is obtained by only conditioning on Θ_0. Hence, the expectation of S solely depends on α_0 (and not α_1). Also, $E\left[S^{(+,+)}\right] = E\left[N^{(+)} X^{(+,+)}\right]$ (see e.g., [Liu and Wang, 2017]), and $E\left[S^{(\perp,\perp)}\right] = E\left[N^{(\perp)}\right] E\left[X^{(\perp,\perp)}\right]$ (see e.g., [Klugman et al., 2009]).

The popular risk measures Value-at-Risk (VaR) and Tail-Value-at-Risk (TVaR) are used namely to determine the capital of the insurance portfolio. The VaR at the confidence level $\kappa \in (0,1)$ of the rv S is defined as $VaR_\kappa(S) = F_S^{-1}(\kappa)$, where $F_S^{-1}(\kappa) = \inf\{x \in \mathbb{R}, F_S(x) \geq \kappa\}$, and the TVaR at the confidence level $\kappa \in (0,1)$ of the rv S is given by

$$TVaR_\kappa(S) = \frac{1}{1-\kappa}\int_\kappa^1 VaR_u(S)\,du.$$

In the example just below, we consider a simple case for which the probability mass function (pmf) of the random sums rvs S, $S^{(+,+)}$, and $S^{(\perp,\perp)}$ can be easily and directly calculated even when using a complicated dependence structure.

Example 1 *Let $N \in \{0,1,2,3\}$ and $X_i \in A = \{1,2,3,\ldots\}$ for $i = 1,2,\ldots,k$. The joint distribution of $(N, \underline{X})$ is defined as in (1) with a hierarchical Archimedean copula as in (3) or in (6), with dependence parameters α_0 and α_1.*

Note that the joint pmf of $(N, \underline{X})$ can be derived from its joint cdf as follows

$$f_{N,\underline{X}}(\underline{k}) = \sum_{i_1=0,1} \cdots \sum_{i_d=0,1} (-1)^{i_1+\ldots+i_d} F_{N,\underline{X}}\left((k_1-i_1),\ldots,(k_d-i_d)\right), \tag{10}$$

where $\underline{k} = (k_1,\ldots,k_d)$, for $k_1 \in \{0,1,2,3\}$, $d = k_1+1$, and $k_i \in \mathbb{N}$, for $i = 2,\ldots,d$.

The pmf of $S = \sum_{i=1}^N X_i$ is given by

$$\Pr(S=k) = \begin{cases} \Pr(N=0) & ,k=0 \\ \Pr(N=1, X_1=1) & ,k=1 \\ \Pr(N=1, X_1=2) + \Pr(N=2, X_1=1, X_2=1) & ,k=2 \\ \Pr(N=1, X_1=k) + \sum_{j=1}^k \Pr(N=2, X_1=j, X_2=k-j) & \\ +\sum_{j=1}^k \sum_{i=1}^{k-j} \Pr(N=2, X_1=j, X_2=i, X_3=k-i-j) & ,k\geq 3 \end{cases}. \tag{11}$$

A numerical illustration of Example 1 is provided in the following example.

Example 2 *Let $N \sim Binomial(3, 0.1)$ and $X_i = aY_i$, where $a = 10000$ and $Y_i - 1 \sim Negative\,Binomial(10, 0.2)$, for $i = 1,2,\ldots$ The joint distribution of $(N, \underline{X})$ is assumed to be defined with a hierarchical Archimedean copula through compounding as given in (6), with $M \sim Logarithmic(q = 1-\mathrm{e}^{-\alpha_0})$ and $B_i \sim B \sim Gamma(\lambda = 1/\alpha_1, 1)$. We consider the values 0.5 and 0.9 for q, and the values 0.04 and 0.2 for λ, which results in the following dependence parameters of the copula $\alpha_0 \simeq 0.7, 2$, and $\alpha_1 = 25, 5$, respectively. Values of F_s, $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$ are given in Table 1. For comparison purposes, the same quantities are also provided for the case of comonotonic $N^{(+)}$ and $X_i^{(+,+)}$ using (11) with the comonotonic copula $C(u_1,\ldots,u_n) = \min(u_1,\ldots,u_n)$, for $u_1,\ldots,u_n \in (0,1)$, and for the independence case, i.e., $X_i^{(\perp,\perp)}$ are iid and independent of $N^{(\perp)}$, for $i = 1,2,\ldots$. From this example, we can see that as the dependence parameter α (α being the parameter of either the outer or the inner copula) increases,*

(α_0, α_1)	$(0.7, 5)$	$(2, 5)$	$(2, 25)$	$(\perp, \perp)$	$(+, +)$
$F_S(0)$	0.7290000	0.7290000	0.7290000	0.7290000	0.7290000
$F_S(40\,a)$	0.8435889	0.8112038	0.8112225	0.8575668	0.7290000
$F_S(160\,a)$	0.9996532	0.9994003	0.9992956	0.9999112	0.9902371
$E\,[S]\,/a$	12.91612	14.25007	14.25007	12.30000	18.37708
$Var\,(S)\,/a^2$	575.0697	689.7329	692.2178	513.87000	1230.54
$VaR_{0.9}\,(S)$	490000	530000	530000	470000	600000
$VaR_{0.999}\,(S)$	1420000	1510000	1540000	1270000	1960000
$TVaR_{0.9}\,(S)$	684865.8	744200.1	744234.9	653169.6	923295
$TVaR_{0.999}\,(S)$	1597225	1698044	1736602	1408195	3137555

Table 1: Values of the cdf, expectation, variance, VaR and TVaR of S as defined in Example 2.

the expectation, the variance and the TVaR increase as well. Also, if the outer parameter is fixed, the expectation $E[S]$ does not change when the inner dependence parameter changes, which is to be expected. Moreover, we can see that the values of the TVaR always fall between the TVaR in the independence case and the ones in the comonotonic case. In general, for collective risk models, the comonotonic case is considered as the worst case dependence scenario.

These last observations made on Example 2 drove us to further investigate the dependence relationship connecting the components of $(N, \underline{X})$. Further, since the computation of (10) becomes more cumbersome, and even impossible for dimensions larger than 5, we need to find a calculation method that is more suitable in large dimensions.

In the following two subsections, we will discuss different properties of the proposed model.

2.1 Impact of dependence on the aggregate claim amount

Stochastic orders are used to compare risks according to how risky and dangerous they are. They have many applications in e.g. actuarial science, applied probability, reliability, and economics. See [Müller and Stoyan, 2002], [Denuit et al., 2006], [Bäuerle and Müller, 2006], and [Shaked and Shanthikumar, 2007] for a review on stochastic orders.

Definition 3 *Let X and X^* be two rvs with finite expectations. Then, X is said to be smaller than X^* according to the convex order (increasing convex order), denoted $X \preceq_{cx} X^*$ ($X \preceq_{icx} X^*$), if $E\,[\phi(X)] \leq E\,[\phi(X^*)]$ for all (increasing) convex function ϕ, when the expectations exist.*

The convex and increasing convex orders are variability orders. Note that, if $X \preceq_{icx} X^*$ and $E[X] = E[X^*]$, then $X \preceq_{cx} X^*$. Proposition 3.4.8 of [Denuit et al., 2006] provides an important application of the increasing convex order in actuarial science and risk management:

$$X \preceq_{icx} X^* \text{ if and only if } TVaR_\kappa(X) \leq TVaR_\kappa(X^*), \text{ for all } \kappa \in (0,1). \tag{12}$$

For properties and more details on the convex and the increasing convex orders, see e.g. [Müller and Stoyan, 2002], and [Shaked and Shanthikumar, 2007]. Our objective is to examine the impact of the degrees of dependence between the components of $(N, \underline{X})$ on the aggregate claim amount S. In the case of the collective risk model with dependence, [Liu and Wang, 2017] provide the following result on increasing convex order and comonotonicity.

Proposition 4 *Assume the multivariate distribution of $(N, \underline{X})$ to be defined with an Archimedean or a hierarchical Archimedean copula C with dependence parameters α_0 and α_1, using either (1) or (2). Then,*

$$S = \sum_{i=1}^{N} X_i \preceq_{icx} \sum_{i=1}^{N^{(+)}} X_i^{(+,+)} = S^{(+,+)},$$

Also, equivalently, $TVaR_\kappa(S) \leq TVaR_\kappa\left(S^{(+,+)}\right)$.

Proof. See [Liu and Wang, 2017]. ■

In Proposition 4, we compare our proposed dependence structure for $(N, \underline{X})$ with the case of perfect positive dependence. What happens if we slightly increase the dependence? Can we compare two random sums with the same dependence structure but different dependence parameters? In order to address these questions, we resort to the supermodular dependence order.

Definition 5 *Let $f : \mathbb{R}^n \to \mathbb{R}$. The function f is supermodular if the following inequality is true:*

$$\begin{aligned} & f(x_1, \ldots, x_i + \epsilon, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i + \epsilon, \ldots, x_j, \ldots, x_n) \\ \geq \ & f(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n), \end{aligned}$$

$\forall (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\forall \epsilon, \delta > 0$, and $1 \leq i \leq j \leq n$.

Definition 6 *Let $\underline{X} = (X_1, \ldots, X_n)$ and $\underline{X}^* = (X_1^*, \ldots, X_n^*)$ be two random vectors such that, for $i = 1, \ldots, n$, X_i and X_i^* have the same marginal distribution. Then, $\underline{X}^*$ is greater than $\underline{X}$ according to the supermodular order, denoted $\underline{X} \preceq_{sm} \underline{X}^*$, if $E[f(\underline{X})] \leq E[f(\underline{X}^*)]$, for any supermodular function f, when the expectations exist.*

The following two properties aim to compare, according to the supermodular order, two random vectors $(N, \underline{X})$ and $(N^*, \underline{X^*})$ for which the multivariate distribution is defined with either an Archimedean copula or a hierarchical Archimedean copula.

Proposition 7 *Let the multivariate distributions of $(N, \underline{X})$ and $(N^*, \underline{X^*})$ be defined using either (1) or (2), using an Archimedean copula C with dependence parameter α and α^* respectively. If $\alpha \leq \alpha^*$, then,*

$$C_\alpha \preceq_{sm} C_{\alpha^*},$$

$$(N, X_1, \ldots, X_k) \preceq_{sm} (N^*, X_1^*, \ldots, X_k^*),$$

and

$$S = \sum_{i=1}^{N} X_i \preceq_{icx} \sum_{i=1}^{N^*} X_i^* = S^*. \tag{13}$$

Proof. The proof for $C_\alpha \preceq_{sm} C_{\alpha^*}$ is given in [Wei and Hu, 2002]. Using the property of closure under all increasing (or decreasing) transforms of the supermodular order (see, e.g. Theorem 9.A.9.(a) of [Shaked and Shanthikumar, 2007]), we can conclude that $(N, X_1, \ldots, X_k) \preceq_{sm} (N^*, X_1^*, \ldots, X_k^*)$, for all $k \in \mathbb{N}$. Also, we have that

$$S = \sum_{j=1}^{N} X_j = \sum_{j=1}^{\infty} X_j 1_{\{N>j\}}.$$

We define S_k as

$$S_k = \sum_{j=1}^{k} X_j 1_{\{N>j\}} = \phi(N, X_1, ..., X_k),$$

where

$$\phi(x_0, x_1, ..., x_k) = \sum_{j=1}^{k} x_j 1_{\{x_0>j\}} \tag{14}$$

is a supermodular function. Note that

$$S_\infty = \lim_{k\to\infty} S_k = S.$$

Since $(N, X_1, \ldots, X_k) \preceq_{sm} (N^*, X_1^*, \ldots, X_k^*)$, for all $k \in \mathbb{N}$, and given that the function ϕ defined in (14) is supermodular, it implies that

$$S_k \preceq_{icx} S_k^*, \tag{15}$$

for all $k \in \mathbb{N}$ (see Theorem 9.A.16 on page 399 of [Shaked and Shanthikumar, 2007] for more details). Letting $k \to \infty$ in (15), we obtain the result in (13). ■

Proposition 8 *Let the multivariate distributions of $(N, \underline{X})$ and $(N^*, \underline{X^*})$ be defined using either (1) or (2), using a one level hierarchical Archimedean copula C as illustrated in Figure 1, with parameters α_0, α_1 and α_0^*, α_1^* respectively. Then, we have*

1. *if $\alpha_0 \le \alpha_0^*$ and $\alpha_1 = \alpha_1^*$, then $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1}$;*
2. *if $\alpha_0 = \alpha_0^*$ and $\alpha_1 \le \alpha_1^*$, then $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0,\alpha_1^*}$;*
3. *if $\alpha_0 \le \alpha_0^*$ and $\alpha_1 \le \alpha_1^*$, then $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1^*}$.*

Therefore, for all three cases, we have

$$(N, X_1, \ldots, X_k) \preceq_{sm} (N^*, X_1^*, \ldots, X_k^*), \forall k \in \mathbb{N}.$$

Then,

$$S = \sum_{i=1}^{N} X_i \preceq_{icx} \sum_{i=1}^{N^*} X_i^* = S^*. \tag{16}$$

Equivalently, by (12)

$$TVaR_\kappa(S) \leq TVaR_\kappa(S^*),$$

for $\kappa \in (0,1)$.

Proof. The proofs for 8.1 and 8.2 can be found in [Wei and Hu, 2002], for nested Archimedean copulas. Given the links between nested Archimedean copulas and hierarchical Archimedean copulas through compounding (see [Cossette et al., 2017] for details), this result also holds here for the one-level hierarchical Archimedean copula defined in (6). If $\alpha_0 \leq \alpha_0^*$ and $\alpha_1 \leq \alpha_1^*$, then, using 8.1 and 8.2, we have $C_{\alpha_0,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1} \preceq_{sm} C_{\alpha_0^*,\alpha_1^*}$.

Once again the property of closure under all increasing (or decreasing) transforms of the supermodular order can be used to show that $(N, X_1, \ldots, X_k) \preceq_{sm} (N^*, X_1^*, \ldots, X_k^*)$, for all $k \in \mathbb{N}$.

Also, using the same steps as the ones in the proof of Proposition 7, we obtain the desired relation in (16). ■

Remark 9 *Note that if the dependence structures of both* $(N, \underline{X})$ *and* $(N^*, \underline{X}^*)$ *are modelled with the same hierarchical Archimedean copula with identical outer dependence parameter but different inner dependence parameters, then the expectations of* S *and* S^* *are equal. In this case, the result given in (16) of Proposition 8 is given in terms of convex order instead of increasing convex order, i.e.,* $S \preceq_{cx} S^*$.

2.2 Sampling Algorithm

Now that we better understand the dependence relationship linking different components of the proposed model, we propose an efficient algorithm to generate samples of S. Inspired by the sampling algorithms of both nested Archimedean copulas and hierarchical copulas through compounding (see e.g., [Marshall and Olkin, 1988], [Hofert, 2008], and [Cossette et al., 2017]), we derive Algorithm 10, which is a general sampling algorithm that generates samples of a random sum S incorporating the dependence structure defined earlier.

Let Θ_0 and $\Theta_{0,1}$ denote the mixing rvs such that given $\Theta_0 = \theta_0$ and $\Theta_{0,1} = \theta_{0,1}$, $(X_1|\Theta_0 = \theta_0, \Theta_{0,1} = \theta_{0,1}), \ldots, (X_k|\Theta_0 = \theta_0, \Theta_{0,1} = \theta_{0,1})$ are conditionally iid and independent of $(N|\Theta_0 = \theta_0)$. Note that if C is a nested Archimedean copula as in (3), Θ_0 represents the mixing rv related to the outer copula C and $\Theta_{0,1}$ is such that $\mathcal{L}_{\Theta_{0,1}}(t;\theta) = \exp\left\{-\theta \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_1}(t)\right\}$. As for the case where C is defined as in (6), the rv Θ_0 plays the same role as the rv M, and $\Theta_{0,1} = \sum_{j=1}^{M} B_{i,j}$.

Algorithm 10 *Let C be a one level hierarchical Archimedean copula with generators $\mathcal{L}_{\Theta_0}$ and $\mathcal{L}_{\Theta_{0,1}}$ allowing to fit the dependence structure depicted in Figure 1.*

1. *Sample Θ_0;*
2. *Sample $R \sim Exp(1)$;*
3. *Calculate $U = \mathcal{L}_{\Theta_0}\left(\frac{R}{\Theta_0}\right)$;*
4. *Return $N = F_N^{-1}(U)$;*
5. *If $N = 0$ return $S = 0$; else*
   - 5.1. *Sample $\Theta_{0,1}$;*
   - 5.2. *Sample $R_i \sim Exp(1)$ for $i = 1, \ldots, N$;*
   - 5.3. *Calculate $U_i = \mathcal{L}_{\Theta_{0,1}}\left(\frac{R_i}{\Theta_{0,1}}\right)$ for $i = 1, \ldots, N$;*
   - 5.4. *Calculate $X_i = F_X^{-1}(U_i)$ for $i = 1, \ldots, N$;*
   - 5.5. *Return $S = \sum_{k=1}^{N} X_i$;*
6. *Return S.*

In the following example, we provide an application of Algorithm 10.

Example 11 *Let $N \sim Poisson(2)$ and $X_i \sim Pareto(3, 100)$, for $i = 1, 2, \ldots$ The joint distribution of $(N, \underline{X})$ is defined with the same copula and dependence parameters as in Example 2. Approximated values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$, using 10 million simulations, are given in Table 2. As we can see in Figure 2, the three curves, representing the cdf of S for different values of the dependence parameters, intersect multiple times which confirms the results of Proposition 8. Values of the TVaR in Table 2 and in Figure 3 also illustrate the results of Proposition 8.*

As discussed in Example 2, a computational methodology is needed to calculate the cdf of S in high dimensions. The sampling algorithm just presented, can be used to derive approximated values of the cdf of S via the Monte Carlo simulation method. This approach is efficient and very practical especially in the case of continuous mixing rvs and/or continuous marginals. Based on [Cossette et al., 2018], we derive, in the following section, another computational methodology allowing to compute the exact values of the cdf of S, for discrete rvs X_i, $i = 1, 2, \ldots$, even in high dimensions.

	$\alpha_0 = 0.7,\ \alpha_1 = 5$	$\alpha_0 = 2,\ \alpha_1 = 5$	$\alpha_0 = 2,\ \alpha_1 = 25$
$E[S]$	108.0424	125.4414	125.440
$Var(S)$	34426.7124	44913.8009	52347.687
$VaR_{0.9}(S)$	293.2429	348.5767	347.406
$VaR_{0.99}(S)$	852.9771	966.2973	1070.826
$VaR_{0.999}(S)$	1664.4522	1837.6631	2048.952
$TVaR_{0.9}(S)$	534.5209	616.3646	658.370
$TVaR_{0.99}(S)$	1211.5718	1351.9504	1500.564
$TVaR_{0.999}(S)$	2301.5431	2540.4982	2748.067

Table 2: Values of the expectation, variance, VaR and TVaR of S as defined in Example 11.

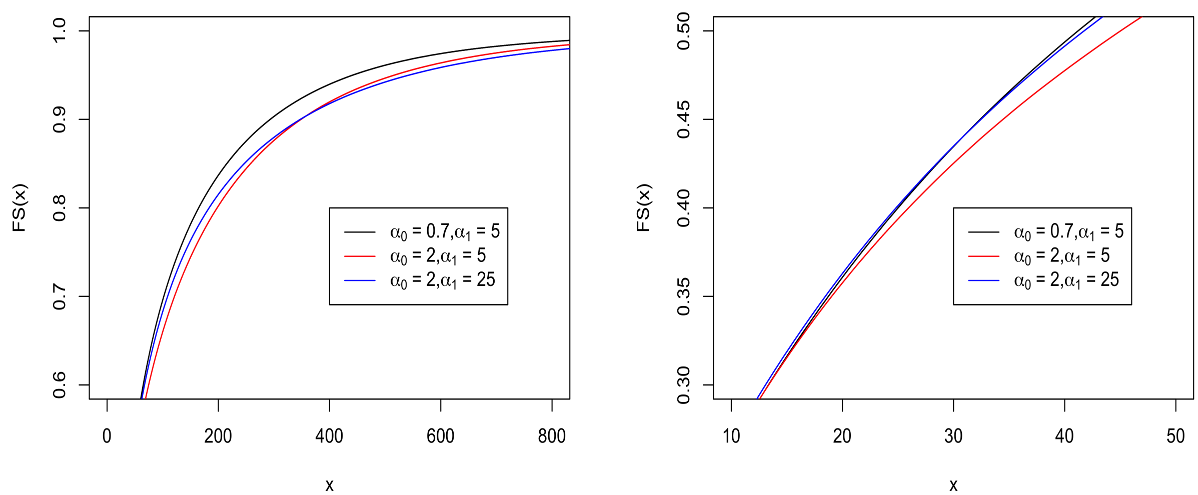

Figure 2: Cdf of S, for different values of α_0 and α_1, as defined in Example 11.

3 Computational Methodology

In this section, we adapt the computational methodology presented in [Cossette et al., 2018] to derive an algorithm to compute the cdf of the random sum S, incorporating a dependence relationship between the claim number rv N and the claim amounts rvs $X_1, X_2, \ldots$. This paper only looks at the case of discrete mixing rvs and discrete marginals, whilst the case of continuous marginals or continuous mixing rvs can be treated in the similarly as in [Cossette et al., 2018].

3.1 A Simple hierarchical structure

Let $(N, \underline{X}) = (N, X_1, X_2, \ldots)$ be a vector of rvs with a multivariate distribution defined in terms of a one level hierarchical Archimedean copula C as defined in (3) or (6). Let

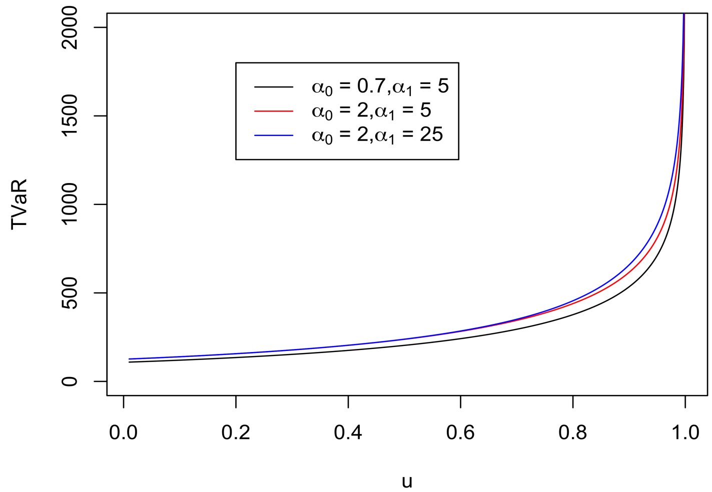

Figure 3: $TVaR_u(S)$ for different values of α_0 and α_1, as defined in Example 11.

Θ_0 and $\Theta_{0,1}$ be the underlying mixing rvs such that, given $\Theta_0 = \theta_0$ and $\Theta_{0,1} = \theta_{0,1}$, $(X_1|\Theta_0 = \theta_0, \Theta_{0,1} = \theta_{0,1}), \ldots, (X_k|\Theta_0 = \theta_0, \Theta_{0,1} = \theta_{0,1})$ are conditionally iid and independent of $(N|\Theta_0 = \theta_0)$. As explained in Section 2.2, such a dependence structure can fit both nested Archimedean copulas and hierarchical Archimedean copulas constructed through compounding.

If the multivariate cdf $F_{N,\underline{X}}$ of $(N, \underline{X})$ is defined with (1), using the copula C and univariate cdfs F_N, F_{X_1}, F_{X_2},..., then, with the common mixture representation of hierarchical Archimedean copulas, $F_{N,\underline{X}}$ can be written as

$$\begin{aligned} F_{N,\underline{X}}(n, \underline{x}) &= \int_0^\infty F_{N|\Theta_0=\theta_0}(n) \left(\int_0^\infty \prod_{i=1}^n F_{X_i|\Theta_0=\theta_0,\Theta_{0,1}=\theta_{0,1}}(x_i)\, \mathrm{d}F_{\Theta_{0,1}}(\theta_{0,1}) \right) \mathrm{d}F_{\Theta_0}(\theta_0) \\ &= \int_0^\infty \mathrm{e}^{-\theta_0 \mathcal{L}^{-1}_{\Theta_0}(F_N(n))} \left(\int_0^\infty \prod_{i=1}^n \mathrm{e}^{-\theta_{0,1}\mathcal{L}^{-1}_{\Theta_{0,1}}\left(F_{X_i}(x_i)\right)} \mathrm{d}F_{\Theta_{0,1}}(\theta_{0,1}) \right) \mathrm{d}F_{\Theta_0}(\theta_0), \end{aligned} \tag{17}$$

where $\underline{x} = (x_1, x_2, \ldots, x_n)$ for $n \in \mathbb{N}_0$,

$$F_{N|\Theta_0=\theta_0}(n) = \mathrm{e}^{-\theta_0 \mathcal{L}^{-1}_{\Theta_0}(F_N(n))},\ n \in \mathbb{N}_0, \tag{18}$$

and

$$F_{X_i|\Theta_0=\theta_0,\Theta_{0,1}=\theta_{0,1}}(x_i) = \mathrm{e}^{-\theta_{0,1}\mathcal{L}^{-1}_{\Theta_{0,1}}\left(F_{X_i}(x_i)\right)}\ (i = 1, 2, \ldots, n). \tag{19}$$

Similarly, we define the multivariate distribution of $(N, \underline{X})$ through its multivariate survival function with the copula C and the univariate survival functions $\overline{F}_N$, $\overline{F}_{X_1}$, $\overline{F}_{X_2}$,..., using (2). In this case, the common mixture representation of $\overline{F}_{N,\underline{X}}$ is given by

$$\begin{aligned}\overline{F}_{N,\underline{X}}\left(n,\underline{x}\right) &= \int_0^\infty \overline{F}_{N|\Theta_0=\theta_0}(n)\left(\int_0^\infty \prod_{i=1}^n \overline{F}_{X_i|\Theta_0=\theta_0,\Theta_{0,1}=\theta_{0,1}}\left(x_i\right)\mathrm{d}F_{\Theta_{0,1}}\left(\theta_{0,1}\right)\right)\mathrm{d}F_{\Theta_0}\left(\theta_0\right) \\ &= \int_0^\infty \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}\left(\overline{F}_N(n)\right)}\left(\int_0^\infty \prod_{i=1}^n \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_{0,1}}^{-1}\left(\overline{F}_{X_i}(x_i)\right)}\mathrm{d}F_{\Theta_{0,1}}\left(\theta_{0,1}\right)\right)\mathrm{d}F_{\Theta_0}\left(\theta_0\right), \quad (20)\end{aligned}$$

where $\underline{x} = (x_1, \ldots, x_n)$,

$$\overline{F}_{N|\Theta_0=\theta_0}(n) = \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}\left(\overline{F}_N(n)\right)},\ n \in \mathbb{N}_0, \quad (21)$$

and

$$\overline{F}_{X_i|\Theta_0=\theta_0,\Theta_{0,1}=\theta_{0,1}}\left(x_i\right) = \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_{0,1}}^{-1}\left(\overline{F}_{X_i}(x_i)\right)}\ (i = 1, 2, \ldots, n). \quad (22)$$

Since the collective risk model with dependence presented in this paper is an extension of the one presented in [Cossette et al., 2018], we will adapt its proposed computational strategy to our model. The idea behind it is to use the conditional independence assumption to identify the conditional distribution of N and X_i through (18) and (19) or (21) and (22). While this computational strategy works naturally for discrete rvs X_i, discretization methods can be used to approximate continuous rvs X_i, $i = 1, 2, \ldots$ (see details in [Cossette et al., 2018]; see also e.g. [Müller and Stoyan, 2002] and [Bargès et al., 2009] for a review of different discretization methods).

In this section, we assume discrete marginals for X_i, $i = 1, 2, \ldots$, and discrete mixing rvs Θ_0 and $\Theta_{0,1}$ with respective LSTs $\mathcal{L}_{\Theta_0}$ and $\mathcal{L}_{\Theta_{0,1}}$, respective pmfs $f_{\Theta_0}\left(\theta_0\right) = \Pr\left(\Theta_0 = \theta_0\right)$ and $f_{\Theta_{0,1}}\left(\theta_{0,1}\right) = \Pr\left(\Theta_{0,1} = \theta_{0,1}\right)$, and respective cdfs $F_{\Theta_0}\left(\theta_0\right) = \Pr\left(\Theta_0 \leq \theta_0\right) = \sum_{j=1}^{\theta_0} f_{\Theta_0}(j)$ and $F_{\Theta_{0,1}}\left(\theta_{0,1}\right) = \Pr\left(\Theta_{0,1} \leq \theta_{0,1}\right) = \sum_{j=1}^{\theta_{0,1}} f_{\Theta_{0,1}}(j)$, for $\theta_0, \theta_{0,1} \in \mathbb{N}$. Let $(N, \underline{X}) = (N, X_1, X_2, \ldots)$ be a vector of rvs, where N is a counting rv, and $\underline{X} = (X_1, X_2, \ldots)$ be a vector of discrete and exchangeable rvs, i.e., $X_n \sim X$ for $n \in \mathbb{N}$ and $X_i \in A = \{0, 1h, 2h, \ldots\}$ $(i = 1, \ldots, n)$. For $S = \sum_{n=1}^N X_n$, (17) becomes

$$F_{N,\underline{X}}\left(n, \underline{k}h\right) = \sum_{\theta_0=0}^\infty \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(F_N(n))} \sum_{\theta_{0,1}=0}^\infty \prod_{i=1}^n \mathrm{e}^{-\theta_{0,1}\mathcal{L}_{\Theta_{0,1}}^{-1}\left(F_{X_i}(k_i h)\right)} f_{\Theta_{0,1}}\left(\theta_{0,1}\right) f_{\Theta_0}\left(\theta_0\right), \quad (23)$$

where $\underline{k} = (k_1, k_2, \ldots, k_n)$, with $n \in \mathbb{N}$. From (23), we have

$$F_{N|\Theta_0=\theta_0}(n) = \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(F_N(n))},\ \forall n \in \mathbb{N}_0.$$

Let N_{θ_0} and $X_{i,\theta_0,\theta_{0,1}}$ denote, respectively, the conditional rvs $(N|\Theta_0 = \theta_0)$ and $(X_i|\Theta_0 = \theta_0, \Theta_{0,1} = \theta_{0,1})$, for $i = 1, 2, \ldots$ The pmf and the probability generating function (pgf) of N_{θ_0} are respectively given by

$$\Pr\left(N_{\theta_0} = n\right) = \begin{cases} \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(F_N(0))} & , n = 0 \\ \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(F_N(n))} - \mathrm{e}^{-\theta_0\mathcal{L}_{\Theta_0}^{-1}(F_N(n-1))} & , n \in \mathbb{N} \end{cases}, \quad (24)$$

and

$$\mathcal{P}_{N_{\theta_0}}(t) = E\left[t^{N_{\theta_0}}\right] = \sum_{k=0}^{\infty} t^k \Pr\left(N_{\theta_0} = k\right). \tag{25}$$

As for $X_{i,\theta_0,\theta_{0,1}}$, we have

$$F_{X_{i,\theta_0,\theta_{0,1}}}(k_i h) = \mathrm{e}^{-\theta_{0,1}\mathcal{L}^{-1}_{\Theta_{0,1}}\left(F_{X_i}(k_i h)\right)}, \tag{26}$$

for $k_i \in \mathbb{N}$, $i = 1, 2, \ldots$, and $\theta_{0,1} \in \mathbb{N}$. For $i = 1, 2, ...$ and for each $\theta_{0,1} \in \mathbb{N}$, we can find the values of $f_{X_{i,\theta_0,\theta_{0,1}}}(k_i h)$ with

$$f_{X_{i,\theta_0,\theta_{0,1}}}(k_i h) = \begin{cases} \mathrm{e}^{-\theta_{0,1}\mathcal{L}^{-1}_{\Theta_{0,1}}\left(F_{X_i}(0)\right)} & , k_i = 0 \\ \mathrm{e}^{-\theta_{0,1}\mathcal{L}^{-1}_{\Theta_{0,1}}\left(F_{X_i}(k_i h)\right)} - \mathrm{e}^{-\theta_{0,1}\mathcal{L}^{-1}_{\Theta_{0,1}}\left(F_{X_i}(k_i - 1)\right)} & , k_i \in \mathbb{N} \end{cases}. \tag{27}$$

Similar results are obtained when the dependence structure of $(N, \underline{X})$ is induced via the copula C and the survival functions as in (2). More precisely, the survival function of $(N, \underline{X})$ is given by

$$\overline{F}_{N,\underline{X}}(n, \underline{k}h) = \sum_{\theta_0=0}^{\infty} \mathrm{e}^{-\theta_0\mathcal{L}^{-1}_{\Theta_0}\left(\overline{F}_N(n)\right)} \sum_{\theta_{0,1}=0}^{\infty} \prod_{i=1}^{n} \mathrm{e}^{-\theta_{0,1}\mathcal{L}^{-1}_{\Theta_{0,1}}\left(\overline{F}_{X_i}(k_i h)\right)} f_{\Theta_{0,1}}(\theta_{0,1}) f_{\Theta_0}(\theta_0), \tag{28}$$

where $\underline{k} = (k_1, \ldots, k_n)$ and $n \in \mathbb{N}$. In this case, the pmf of N_{θ_0} is

$$\Pr\left(N_{\theta_0} = n\right) = \begin{cases} 1 - \mathrm{e}^{-\theta_0\mathcal{L}^{-1}_{\Theta_0}\left(\overline{F}_N(0)\right)} & , n = 0 \\ \mathrm{e}^{-\theta_0\mathcal{L}^{-1}_{\Theta_0}\left(\overline{F}_N(n-1)\right)} - \mathrm{e}^{-\theta_0\mathcal{L}^{-1}_{\Theta_0}\left(\overline{F}_N(n)\right)} & , n \in \mathbb{N} \end{cases}. \tag{29}$$

As for $X_{i,\theta_0,\theta_{0,1}}$, we have

$$\overline{F}_{X_{i,\theta_0,\theta_{0,1}}}(k_i h) = \mathrm{e}^{-\theta_{0,1}\mathcal{L}^{-1}_{\Theta_{0,1}}\left(\overline{F}_{X_i}(k_i h)\right)}, \tag{30}$$

for $k_i \in \mathbb{N}$, $i = 1, 2, \ldots$, and $\theta \in \mathbb{N}$.

The expression for $f_{N,\underline{X}}(n, k_1 h, \ldots, k_n h)$ in this case is consequently given by

$$f_{N,\underline{X}}(n, k_1 h, \ldots, k_n h) = \sum_{\theta_0=1}^{\infty} \Pr\left(N_{\theta_0} = n\right) \left\{ \sum_{\theta_{0,1}=1}^{\infty} \prod_{i=1}^{n} f_{X_{i,\theta_0,\theta_{0,1}}}(k_i h) f_{\Theta_{0,1}}(\theta_{0,1}) \right\} f_{\Theta_0}(\theta_0). \tag{31}$$

Let $S_{\theta_0,\theta_{0,1}} = \sum_{i=1}^{N_{\theta_0}} X_{i,\theta_0,\theta_{0,1}}$ be the sum of conditionally independent rvs and $f_{S_{\theta_0,\theta_{0,1}}}$ be its corresponding pmf. Let $\mathcal{L}_{X_{\theta_0,\theta_{0,1}}}$ be the LST of $X_{\theta_0,\theta_{0,1}}$, where $X_{i,\theta_0,\theta_{0,1}} \sim X_{\theta_0,\theta_{0,1}}$, for $i = 1, 2, \ldots$ Then, the LST of $S_{\theta_0,\theta_{0,1}}$ is given by

$$\mathcal{L}_{S_{\theta_0,\theta_{0,1}}}(t) = \mathcal{P}_{N_{\theta_0}}\left(\mathcal{L}_{X_{\theta_0,\theta_{0,1}}}(t)\right). \tag{32}$$

Then, using (32) and FFT, it is easy to compute the exact values of $f_{S_{\theta_0,\theta_{0,1}}}$ for each θ_0 and $\theta_{0,1}$.

Finally, due to the representation of $f_{N,\underline{X}}$ in (31), the unconditional pmf of S can be computed using

$$f_S(kh) = \sum_{\theta_0=1}^{\infty} \sum_{\theta_{0,1}=1}^{\infty} f_{S_{\theta_0,\theta_{0,1}}}(kh)\, f_{\Theta_{0,1}}(\theta_{0,1})\, f_{\Theta_0}(\theta_0)\,, \ k \in \mathbb{N}_0. \tag{33}$$

The computational methodology used to find the exact values of f_S is summarized in the following algorithm.

Algorithm 12 ***Computation of the values of*** f_S

1. *Fix* $\theta_0 = 1$*;*
2. *Fix* $\theta_{0,1} = 1$*;*
3. *Calculate either* $F_{X_{i,\theta_0,\theta_{0,1}}}(k_i h)$ *with (26) or* $\overline{F}_{X_{i,\theta_0,\theta_{0,1}}}(k_i h)$ *with (30), for* $k_i \in \mathbb{N}_0$*;*
4. *Calculate* $f_{X_{i,\theta_0,\theta_{0,1}}}(k_i h)$*, for* $k_i \in \mathbb{N}_0$*;*
5. *Use FFT to return the vector* $\underline{\tilde{f}}_{X_{i,\theta_0,\theta_{0,1}}}$*, where* $\underline{\tilde{f}}$ *denotes the vector of values of the characteristic function, also known as the Fourier transform (see e.g., [Klugman et al., 2009]);*
6. *For* $n = 0, 1, 2, \ldots,$ *calculate* $\Pr(N_{\theta_0} = n)$ *using either (24) or (29);*
7. *Use the pgf of* N_{θ_0} *given in (25) to calculate* $\underline{\tilde{f}}_{S_{\theta_0,\theta_{0,1}}}$ *such that*
$$\underline{\tilde{f}}_{S_{\theta_0,\theta_{0,1}}} = \mathcal{P}_{N_{\theta_0}}\left(\underline{\tilde{f}}_{X_{\theta_0,\theta_{0,1}}}\right);$$
8. *Use FFT (inverse) to compute* $f_{S_{\theta_0,\theta_{0,1}}}(kh)$ *for* $k \in \mathbb{N}_0$*;*
9. *Repeat steps 3-8 for* $\theta_{0,1} = 2, ..., \theta_{0,1}^*$ *where* $\theta_{0,1}^*$ *is chosen such that* $F_{\Theta_{0,1}}\left(\theta_{0,1}^*\right) \leq 1 - \varepsilon$ *where* ε *is fixed as small as desired (e.g.* $\varepsilon = 10^{-10}$*);*
10. *Compute* $f_{S|\Theta_0=\theta_0}(kh) = \sum_{\theta_{0,1}=1}^{\theta_{0,1}^*} f_{S_{\theta_0,\theta_{0,1}}}(kh)\, f_{\Theta_{0,1}}(\theta_{0,1})$*, for* $k \in \mathbb{N}_0$*;*
11. *Repeat steps 2-10 for* $\theta_0 = 2, ..., \theta_0^*$ *where* θ_0^* *is chosen such that* $F_{\Theta_0}(\theta_0^*) \leq 1 - \varepsilon$ *where* ε *is fixed as small as desired (e.g.,* $\varepsilon = 10^{-10}$*);*
12. *Compute* $f_S(kh) = \sum_{\theta_0=1}^{\theta_0^*} f_{S|\Theta_0=\theta_0}(kh)\, f_{\Theta_0}(\theta_0)$*, for* $k \in \mathbb{N}_0$*.*

An application of Algorithm 12 is provided in Example 13.

Example 13 *Let* $N \sim Poisson(2)$ *and* $X_i - 1 \sim Binomial(10, 0.1)$*, for* $i = 1, 2, \ldots$ *Also, assume that the joint distribution of* $(N, \underline{X})$ *is defined as in (1) with a nested Ali-Mikhail-Haq (AMH) copula*

	Exact values (Alg.12)	1M MC simulations (Alg.10)	Comonotonicity
$E[S]$	4.039336	4.040922	5.25118
$Var(S)$	10.482232	10.450744	35.58824
$VaR_{0.9}(S)$	8.000000	8.000000	12.00000
$VaR_{0.99}(S)$	14.000000	14.000000	30.00000
$VaR_{0.9999}(S)$	23.000000	23.000000	63.00000
$TVaR_{0.9}(S)$	10.809860	10.793440	18.79353
$TVaR_{0.99}(S)$	15.835839	15.781300	34.38692
$TVaR_{0.9999}(S)$	24.651145	24.630000	68.10100

Table 3: Values of the expectation, variance, VaR and TVaR of S as defined in Example 13.

with parameters $\alpha_0 = 0.1$ and $\alpha_1 = 0.2$, i.e., $\Theta_0 \sim Geometric(1-\alpha_0)$, $\Theta_1 \sim Geometric(1-\alpha_1)$, and $\Theta_{0,1} \sim Shifted\,Negative\,Binomial\left(\alpha_0, \frac{1-\alpha_1}{1-\alpha_0}\right)$. Using Algorithm 12, we compute the values of f_S allowing to derive the exact values of $E[S]$, $Var(S)$, $VaR_\kappa(S)$, and $TVaR_\kappa(S)$, provided in Table 3. The same quantities are also calculated using 1000000 Monte Carlo simulations (using Algorithm 10), and also for comonotonic rvs $N^{(+)}$ and $X_i^{(+,+)}$, for $i = 1, 2, \ldots$ As expected, the highest values for the TVaR of S are obtained for the perfect positive dependence case, which is in line with Proposition 4. Also, we can see that the results for both the Algorithms 12 and 10 are close, with comparable computation times.

Remark 14 *If C is an Archimedean copula with mixing rv Θ, meaning that $N, X_1, X_2, \ldots$ are conditionally independent given $\Theta = \theta$, the procedure of computation is nearly the same. One has only to replace $\Theta_{0,1}$ and Θ_0 by Θ and Algorithm 12 becomes the following:*

Algorithm 15 Computation of the values of f_S

1. *Fix $\theta = 1$;*
2. *Calculate either $F_{X|\Theta=\theta}(k_i h) = \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(F_X(k_i h))}$ or $\overline{F}_{X|\Theta=\theta}(k_i h) = \mathrm{e}^{-\theta\mathcal{L}_\Theta^{-1}(\overline{F}_X(k_i h))}$, for $k_i \in \mathbb{N}_0$;*
3. *Deduce $f_{X|\Theta=\theta}(k_i h)$ from step 2;*
4. *Use FFT to return the vector $\underline{\tilde{f}}_{X|\Theta=\theta}$;*
5. *For $n = 0, 1, 2, \ldots$, calculate $\Pr(N_\theta = n)$ using either (24) or (29);*
6. *Use the pgf of N_θ given in (25) to calculate $\underline{\tilde{f}}_{S|\Theta=\theta}$: $\underline{\tilde{f}}_{S|\Theta=\theta} = \mathcal{P}_{N_\theta}\left(\underline{\tilde{f}}_{X|\Theta=\theta}\right)$;*
7. *Use FFT (inverse) to compute $f_{S|\Theta=\theta}(kh)$ for $k \in \mathbb{N}_0$;*
8. *Repeat steps 2-7 for $\theta = 2, ..., \theta^*$ where θ^* is chosen such that $F_\Theta(\theta^*) \leq 1-\varepsilon$ where ε is fixed as small as desired (e.g. $\varepsilon = 10^{-10}$);*
9. *Compute $f_S(kh) = \sum_{\theta=1}^{\theta^*} f_{S|\Theta=\theta}(kh)\, f_\Theta(\theta)$, for $k \in \mathbb{N}_0$.*

3.2 General hierarchical structure

Until now, we have only considered one level hierarchical dependence structures. We can generalize the proposed model to a multi-level hierarchical structure by considering the dependence structure linking the rvs X_i, for $i = 1, 2, \ldots$, to be a hierarchical one. Using the mixture representation as in Section 3.1, Algorithm 12 can be easily modified to fit a multi-level hierarchical Archimedean copula (see Section 7 of [Cossette et al., 2018]).

Another interesting generalization would be to consider a portfolio of different classes of business. Let $(N_1, \ldots, N_d)$ be a vector of discrete and positive counting random variables, and $\underline{X}_1$,..., $\underline{X}_d$ sequences of positive rvs, where $\underline{X}_i = \{X_{i,j}, j \in \mathbb{N}\}$, for $i = 1, \ldots, d$. Consider the random sums

$$S_i = \sum_{j=1}^{N_i} X_{i,j},\ \forall i \in \{1, \ldots, d\}. \tag{34}$$

Note that, for $i = 1, \ldots, d$, S_i is defined exactly as in Section 3.1, i.e., the dependence structure linking N_i and $X_{i,j}$, for $j = 1, 2, \ldots$, is defined via a one level hierarchical Archimedean copula. To go for a more general case, we will consider an extra dependence between the counting rvs N_i, for $i = 1, \ldots, d$. More specifically, the rvs N_i are assumed to be exchangeable and linked via an Archimedean copula. An example of such a dependence structure is depicted in Figure 4.

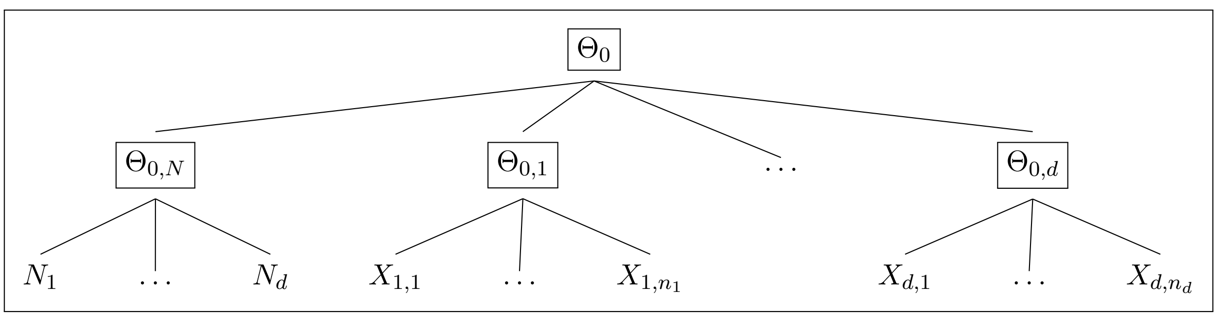

Figure 4: Example of a general hierarchical structure.

Let $S = \sum_{i=1}^{d} S_i$. In order to derive the exact values of quantities of interest related to S, we use the mixture representation of hierarchical Archimedean copulas as in Section 3.1 to adapt, once again, the computation methodology of [Cossette et al., 2018] to the proposed model.

To apply the methodology in question, we need to assume positive and discrete marginals and that all the mixing rvs $\Theta_0, \Theta_{0,N}, \Theta_{0,1}, \ldots, \Theta_{0,d}$, are strictly positive discrete rvs with known distributions. Let $\underline{\Theta} = (\Theta_0, \Theta_{0,N}, \Theta_{0,1}, \ldots, \Theta_{0,d})$. Then, given $\underline{\Theta} = \underline{\theta}$, where $\underline{\theta} = (\theta_0, \theta_{0,N}, \theta_{0,1}, \ldots, \theta_{0,d})$, it is assumed that

$$(N_1 \mid \underline{\Theta} = \underline{\theta}), \ldots, (N_d \mid \underline{\Theta} = \underline{\theta}),$$

and

$$(X_{1,1} \mid \underline{\Theta} = \underline{\theta}), \ldots, (X_{1,n_1} \mid \underline{\Theta} = \underline{\theta}), \ldots, (X_{d,1} \mid \underline{\Theta} = \underline{\theta}), \ldots, (X_{d,n_d} \mid \underline{\Theta} = \underline{\theta}),$$

are all conditionally independent. For $i = 1, \dots, d$, the conditional distribution of N_i is only influenced by both $\Theta_{0,N}$ and Θ_0. Also, the conditional distributions of the components of $\underline{X}_i$ are only influenced by $\Theta_{0,i}$ and Θ_0, for $i = 1, \dots, d$.

Assume that the joint cdf of $(\underline{N}, \underline{X}) = (N_1, \dots, N_d, X_{1,1}, \dots, X_{1,n_1}, \dots, X_{d,1}, \dots, X_{d,n_d})$ is defined in terms of a hierarchical Archimedean copula C related to the hierarchical structure depicted in Figure 4, as

$$\begin{aligned}
& F_{\underline{N},\underline{X}}\left(n_1, \dots, n_d, k_{1,1}h, \dots, k_{1,n_1}h, \dots, k_{d,1}h, \dots, k_{d,k_d}h\right) \\
= \; & C\left(F_{N_1}(n_1), \dots, F_{N_d}(n_d), F_{X_{1,1}}(k_{1,1}h), \dots, F_{X_{1,n_1}}(k_{1,n_1}h), \dots, F_{X_{d,1}}(k_{d,1}h), \dots, F_{X_{d,n_d}}(k_{d,n_d})h\right) \\
= \; & \sum_{\theta_0=1}^{\infty}\left(\sum_{\theta_{0,N}=1}^{\infty}\prod_{i=1}^{d} F_{N_i|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N}}\left(n_i\right) f_{\Theta_{0,N}}(\theta_{0,N})\right) \\
& \times \prod_{i=1}^{d}\left(\sum_{\theta_{0,i}}^{\infty}\left(\prod_{j=1}^{n_i} F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}h)\right) f_{\Theta_{0,i}}(\theta_{0,i})\right) f_{\Theta_0}(\theta_0), \qquad (35)
\end{aligned}$$

where $F_{N_i|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N}}\left(n_i\right) = \mathrm{e}^{-\theta_{0,N}\mathcal{L}^{-1}_{\Theta_{0,N}}\left(F_{N_i}(n_i)\right)}$, and $F_{X_{i,j}|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}h) = \mathrm{e}^{-\theta_{0,i}\mathcal{L}^{-1}_{\Theta_{0,i}}\left(F_{X_{i,j}}(k_{i,j}h)\right)}$, for $i = 1, \dots, d$, and $j = 1, \dots, n_i$. Also, we denote by f_{Θ_0}, $f_{\Theta_{0,N}}$, and $f_{\Theta_{0,i}}$, for $i = 1, \dots, d$, the pmfs of Θ_0, $\Theta_{0,N}$, and $\Theta_{0,i}$, for $i = 1, \dots, d$, respectively.

Remark 16 *The multivariate distribution of $(\underline{N}, \underline{X})$ can also be defined with its survival function as in Section 3.1. Since the procedure is the same, we simply consider the model given in (35).*

Since $(N_1|\Theta_0 = \theta_0, \Theta_{0,N} = \theta_{0,N})$, ..., $(N_d|\Theta_0 = \theta_0, \Theta_{0,N} = \theta_{0,N})$ are conditionally independent and within each class $i = 1, \dots, d$, $(X_{i,1}|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i})$, ..., $(X_{i,n_i}|\Theta_0 = \theta_0, \Theta_{0,i} = \theta_{0,i})$ are also conditionally independent, we can easily calculate the pmf of $(S_i|\Theta_0 = \theta_0, \Theta_{0,N} = \theta_{0,N})$ using the same procedure as in Section 3.1. It implies that

$$f_{S|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N}}(kh) = f_{S_1|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N}} * \dots * f_{S_d|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N}}(kh),$$

for $k \in \mathbb{N}_0$, where $*$ denotes the convolution product.

Finally, the unconditional pmf of S is given by

$$f_S(kh) = \sum_{\theta_0=1}^{\infty}\sum_{\theta_{0,N}=1}^{\infty} f_{S|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N}}(kh) f_{\Theta_{0,N}}(\theta_{0,N}) f_{\Theta_0}(\theta_0), \qquad (36)$$

for $k \in \mathbb{N}_0$.

We can summarize the computation methodology of f_S as follows:

Algorithm 17 ***Computation of the values of*** f_S.

1. *Fix* $\theta_0 = 1$;
2. *Fix* $\theta_{0,N} = 1$;
3. *For each class* $i = 1, \ldots, d$, *fix* $\theta_{0,i} = 1$;
   (a) *Calculate either* $F_{X_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_{i,j}h) = \mathrm{e}^{-\theta_{0,i}\mathcal{L}^{-1}_{\Theta_{0,i}}\left(F_{X_{i,j}}(k_{i,j}h)\right)}$, *for* $k_{i,j} \in \mathbb{N}_0$ ($X_{i,1} \sim X_{i,2} \sim X_i$);
   (b) *Deduce* $f_{X_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}(k_i h)$;
   (c) *Use FFT to return the vector* $\underline{\tilde{f}}_{X_i|\Theta_0=\theta_0,\Theta_{0,i}=\theta_{0,i}}$;
   (d) *For* $n_i = 0, 1, 2, \ldots$, *calculate the pmf of* $(N_i|\Theta_0 = \theta_0, \Theta_{0,N} = \theta_{0,N})$, *such that* $F_{N_i|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N}}(n_i) = \mathrm{e}^{-\theta_{0,N}\mathcal{L}^{-1}_{\Theta_{0,N}}\left(F_{N_i}(n_i)\right)}$;
   (e) *Use the pgf of* $(N_i|\Theta_0 = \theta_0, \Theta_{0,N} = \theta_{0,N})$ *to calculate* $\underline{\tilde{f}}_{S_i|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N},\Theta_{0,1}=\theta_{0,1}}$;
   (f) *Use FFT (inverse) to compute* $f_{S_i|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N},\Theta_{0,1}=\theta_{0,1}}(kh)$ *for* $k \in \mathbb{N}_0$;
   (g) *Repeat steps 3a-3f for* $\theta_{0,i} = 2, ..., \theta^*_{0,i}$ *where* $\theta^*_{0,i}$ *is chosen such that* $F_{\Theta_{0,i}}\left(\theta^*_{0,i}\right) \leq 1 - \varepsilon$ *where* ε *is fixed as small as desired (e.g.* $\varepsilon = 10^{-10}$*)*;
   (h) *Compute* $f_{S_i|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N}}(kh) = \sum_{\theta_{0,i}=1}^{\theta^*_{0,i}} f_{S_i|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N},\Theta_{0,1}=\theta_{0,1}}(kh)\, f_{\Theta_{0,i}}(\theta_{0,i})$, *for* $k \in \mathbb{N}_0$.
4. *Convolute all* $f_{S_i|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N}}$, *for* $i = 1, \ldots, d$, *to calculate* $f_{S|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N}}$
5. *Repeat steps 3-4 for* $\theta_{0,N} = 2, ..., \theta^*_{0,N}$ *where* $\theta^*_{0,N}$ *is chosen such that* $F_{\Theta_{0,N}}\left(\theta^*_{0,N}\right) \leq 1 - \varepsilon$ *where* ε *is fixed as small as desired (e.g.* $\varepsilon = 10^{-10}$*)*;
6. *Compute* $f_{S|\Theta_0=\theta_0}(kh) = \sum_{\theta_{0,N}=1}^{\theta^*_0} f_{S|\Theta_0=\theta_0,\Theta_{0,N}=\theta_{0,N}}(kh)\, f_{\Theta_{0,N}}(\theta_{0,N})$, *for* $k \in \mathbb{N}_0$;
7. *Repeat steps 2-6 for* $\theta_0 = 2, ..., \theta^*_0$ *where* θ^*_0 *is chosen such that* $F_{\Theta_0}(\theta^*_0) \leq 1 - \varepsilon$ *where* ε *is fixed as small as desired (e.g.* $\varepsilon = 10^{-10}$*)*;
8. *Compute* $f_S(kh) = \sum_{\theta_0=1}^{\theta^*_0} f_{S|\Theta_0=\theta_0}(kh)\, f_{\Theta_0}(\theta_0)$, *for* $k \in \mathbb{N}_0$.

4 Explicit formulas for collective risks models with dependence

In this section, we consider a specific class of collective risk models within the framework defined in Section 2. We focus on the family of multivariate mixed exponential distributions to derive explicit expressions for the survival function of the random sum S incorporating a dependence relationship between the underlying frequency and severity of the form (2).

4.1 Archimedean copulas

We consider the class of collective risk models for which the dependence structure is modelled via an Archimedean copula C using (2). This model is inspired from the one discussed in [Albrecher et al., 2011] in which the claim sizes $\underline{X}$ are considered to be completely monotone with a multivariate mixed exponential distribution. In our case, the class of collective risk models incorporates an extra Archimedean dependence between the number of claims N and the claim amounts.

Let Θ be a positive mixing rv (discrete or continuous) with cdf F_Θ. As in [Albrecher et al., 2011], consider that, given $\Theta = \theta$, the rvs $(X_i|\Theta = \theta)$, for $i = 1, 2, \ldots$, are conditionally independent and distributed as $Exp(\theta)$. See e.g. [Marshall and Olkin, 1988] for more details concerning such distributions. In order to obtain explicit formulas for quantities related to the random sum $S = \sum_{i=1}^{N} X_i$, assume also that, given $\Theta = \theta$, the rv $(N|\Theta = \theta)$ follows a geometric distribution with parameter $q_\theta = 1 - (1-q)^\theta$, and with pmf $p_n = q_\theta(1-q_\theta)^n$, for $n \in \mathbb{N}_0$, i.e., $(N|\Theta = \theta) \sim Geo\left(1-(1-q)^\theta\right)$, with $q \in (0,1)$.

In this case, the univariate survival functions $\overline{F}_N$, $\overline{F}_{X_1}, \ldots, \overline{F}_{X_n}$ are such that

$$\overline{F}_N(n) = \int (1-q)^{\theta(n+1)} \mathrm{d}F_\Theta(\theta) = \mathcal{L}_\Theta\left(-(n+1)\ln(1-q)\right), \, n \in \mathbb{N}_0, \tag{37}$$

and

$$\overline{F}_{X_i}(x_i) = \mathcal{L}_\Theta\left(x_i\right), \tag{38}$$

for $x_i \geq 0$, $i = 1, 2, \ldots, n$, and $n \in \mathbb{N}$.

Let C be an Archimedean copula with generator $\mathcal{L}_\Theta$. Then, using (2), we can write the multivariate survival function of $(N, \underline{X}) = (N, X_1, X_2, \ldots)$ as

$$\overline{F}_{N,\underline{X}}\left(n, x_1, \ldots, x_n\right) = C\left(\overline{F}_N(n), \overline{F}_{X_1}(x_1), \ldots, \overline{F}_{X_n}(x_n)\right), \tag{39}$$

for $x_i \geq 0$, $i = 1, 2, \ldots, n$, and $n \in \mathbb{N}$.

Combining (37), (38), and (39), the multivariate survival function of $(N, \underline{X})$ becomes

$$\begin{aligned}
\overline{F}_{N,\underline{X}}(n, x_1, \ldots, x_n) &= \mathcal{L}_\Theta\left(\mathcal{L}_\Theta^{-1}\left(\overline{F}_N(n)\right) + \sum_{i=1}^{n} \mathcal{L}_\Theta^{-1}\left(\overline{F}_{X_i}(x_i)\right)\right) \\
&= \mathcal{L}_\Theta\big(-(n+1)\ln(1-q) + x_1 + \ldots + x_n\big),
\end{aligned}$$

for $n \in \mathbb{N}$.

Then, using the compound mixture representation as explained in [Cossette et al., 2018], it is clear that the conditional distribution of $(S|\Theta = \theta)$ is a compound geometric distribution where the counting rv N follows a geometric distribution with parameter $q_\theta = 1 - (1-q)^\theta$ and the claim amount follows an exponential distribution with parameter θ. This means that

$$\begin{aligned}
\overline{F}_{S|\Theta=\theta}(x) &= \sum_{n=1}^{\infty} q_\theta \left(1-q_\theta\right)^n \overline{F}_{X_1+\ldots+X_n|\Theta=\theta}(x) \\
&= \sum_{n=1}^{\infty} \left(1-(1-q)^\theta\right) (1-q)^{\theta n} \,\mathrm{e}^{-\theta x} \sum_{j=0}^{n-1} \frac{(\theta x)^j}{j!} \\
&= \sum_{n=1}^{\infty} (1-q)^{\theta n} \,\mathrm{e}^{-\theta x} \sum_{j=0}^{n-1} \frac{(\theta x)^j}{j!} - \sum_{n=1}^{\infty} (1-q)^{\theta (n+1)} \,\mathrm{e}^{-\theta x} \sum_{j=0}^{n-1} \frac{(\theta x)^j}{j!}, \ x \geq 0. \qquad (40)
\end{aligned}$$

Finally, using (40), the unconditional survival function of S is given by

$$\begin{aligned}
\overline{F}_S(x) &= \int \overline{F}_{S|\Theta=\theta}(x) \mathrm{d}F_\Theta(\theta) \\
&= \int \left\{ \sum_{n=1}^{\infty} (1-q)^{\theta n} \,\mathrm{e}^{-\theta x} \sum_{j=0}^{n-1} \frac{(\theta x)^j}{j!} - \sum_{n=1}^{\infty} (1-q)^{\theta (n+1)} \,\mathrm{e}^{-\theta x} \sum_{j=0}^{n-1} \frac{(\theta x)^j}{j!} \right\} \mathrm{d}F_\Theta(\theta) \\
&= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{x^j}{j!} \int \theta^j \mathrm{e}^{-\theta(x - n\ln(1-q))} \mathrm{d}F_\Theta(\theta) - \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{x^j}{j!} \int \theta^j \mathrm{e}^{-\theta(x-(n+1)\ln(1-q))} \mathrm{d}F_\Theta(\theta) \\
&= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{x^j}{j!} (-1)^j \frac{\mathrm{d}^{\mathrm{j}}}{\mathrm{d}x^j} \mathcal{L}_\Theta \left(x - n\ln(1-q)\right) - \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{x^j}{j!} (-1)^j \frac{\mathrm{d}^{\mathrm{j}}}{\mathrm{d}x^j} \mathcal{L}_\Theta \left(x - (n+1)\ln(1-q)\right) \\
&= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \frac{x^j}{j!} (-1)^j \frac{\mathrm{d}^{\mathrm{j}}}{\mathrm{d}x^j} \left\{ \mathcal{L}_\Theta \left(x - n\ln(1-q)\right) - \mathcal{L}_\Theta \left(x - (n+1)\ln(1-q)\right) \right\}, \ x \geq 0. \qquad (41)
\end{aligned}$$

Note that the generator derivatives appearing in (41) are known for different Archimedean copulas with discrete or continuous mixing rvs (see e.g., [Hofert et al., 2012]).

4.2 Hierarchical Archimedean copulas

We can also generalize the class of collective risk model with dependence just presented, to allow for asymmetric dependence relationship between N and X_i, for $i = 1, 2, \ldots$ Let the multivariate survival function of $(N, \underline{X}) = (N, X_1, X_2, \ldots)$ to be defined with a one level hierarchical Archimedean copula C with generators $\mathcal{L}_{\Theta_0}$ and $\mathcal{L}_{\Theta_{0,1}}$ as defined in Section 3.1 and depicted in Figure 1. Then, given $\Theta_0 = \theta_0$ and $\Theta_{0,1} = \theta_{0,1}$, the rvs $(X_i|\Theta_0 = \theta_0, \Theta_{01} = \theta_{01})$ are conditionally independent and exponentially distributed with mean $\frac{1}{\theta_{0,1}}$, for $i = 1, 2, \ldots$. Also, given $\Theta_0 = \theta_0$, the rv $(N|\Theta_0 = \theta_0)$ is conditionally independent of $(X_i|\Theta_0 = \theta_0, \Theta_{01} = \theta_{01})$, for $i = 1, 2, \ldots$, and follows a geometric distribution with parameter $q_{\theta_0} = 1 - (1-q)^{\theta_0}$, where $q \in (0,1)$. Then, $(S|\Theta_0 = \theta_0, \Theta_{01} = \theta_{01})$ has a compound geometric distribution where the counting rv follows a geometric distribution with parameter $q_{\theta_0} = 1 - (1-q)^{\theta_0}$ and the claim amount follows an exponential distribution with parameter θ_{01}. The multivariate survival function of $(N, \underline{X})$ and the conditional survival function of S can hence be respectively written as

$$\overline{F}_{N,\underline{X}}(n, x_1, \ldots, x_n) \quad = \quad \mathcal{L}_{\Theta_0}\left(-(n+1)\ln(1-q) + \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_1}\left(x_1 + \ldots + x_n\right)\right),$$

for $x_1 \geq 0, \ldots, x_n \geq 0$ and $n \in \mathbb{N}$, and,

$$\begin{aligned}
\overline{F}_{S|\Theta_0=\theta_0,\Theta_{01}=\theta_{01}}(x) &= \sum_{n=1}^{\infty} q_{\theta_0}\left(1-q_{\theta_0}\right)^n \overline{F}_{X_1+\ldots+X_n|\Theta_0=\theta_0,\Theta_{01}=\theta_{01}}(x) \\
&= \sum_{n=1}^{\infty} q_{\theta_0}\left(1-q_{\theta_0}\right)^n \mathrm{e}^{-\theta_{01}x} \sum_{j=0}^{n-1} \frac{(\theta_{01}x)^j}{j!} \\
&= \sum_{n=1}^{\infty} \left(1-(1-q)^{\theta_0}\right)(1-q)^{n\theta_0} \mathrm{e}^{-\theta_{01}x} \sum_{j=0}^{n-1} \frac{(\theta_{01}x)^j}{j!},\ x \geq 0. \qquad (42)
\end{aligned}$$

The unconditional survival function of S can be deduced from (42) as follows

$$\begin{aligned}
\overline{F}_S(x) &= \int\int \left\{\sum_{n=1}^{\infty}\left(1-(1-q)^{\theta_0}\right)(1-q)^{n\theta_0} \mathrm{e}^{-\theta_{01}x} \sum_{j=0}^{n-1} \frac{(\theta_{01}x)^j}{j!}\right\} \mathrm{d}F_{\Theta_{01}(\theta_{01})} \mathrm{d}F_{\Theta_0(\theta_0)} \\
&= \int \left\{\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} \frac{x^j}{j!}\left((1-q)^{n\theta_0} - (1-q)^{(n+1)\theta_0}\right)\int \mathrm{e}^{-\theta_{01}x}\theta_{01}^j \mathrm{d}F_{\Theta_{01}(\theta_{01})}\right\} \mathrm{d}F_{\Theta_0(\theta_0)} \\
&= \int\int \left\{\sum_{n=1}^{\infty}\left((1-q)^{n\theta_0} - (1-q)^{(n+1)\theta_0}\right) \mathrm{e}^{-\theta_{01}x} \sum_{j=0}^{n-1} \frac{(\theta_{01}x)^j}{j!}\right\} \mathrm{d}F_{\Theta_{01}(\theta_{01})} \mathrm{d}F_{\Theta_0(\theta_0)} \\
&= \int \left\{\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} \frac{x^j}{j!}\left((1-q)^{n\theta_0} - (1-q)^{(n+1)\theta_0}\right)(-1)^j \frac{\mathrm{d}^j}{\mathrm{d}x^j}\mathcal{L}_{\Theta_{01}}(x)\right\} \mathrm{d}F_{\Theta_0(\theta_0)}, \qquad (43)
\end{aligned}$$

for $x \geq 0$.

In order to investigate the general expression for the derivatives of the LST of $\Theta_{0,1}$, we rewrite $\mathcal{L}_{\Theta_{0,1}}$ as a composition of two functions f and g, where $f(x) = \mathrm{e}^{-\theta_0 x}$ and $g(x) = \mathcal{L}_{\Theta_0}^{-1} \circ \mathcal{L}_{\Theta_1}(x)$, $\forall x > 0$. Using [McKiernan, 1956], the j^{th} derivative of $\mathcal{L}_{\Theta_{01}}$ is given by

$$\begin{aligned}
\frac{\mathrm{d}^j}{\mathrm{d}x^j}\mathcal{L}_{\Theta_{01}}(x) &= \sum_{r=1}^{j} f^{(r)}(g(x)) \sum_{s=0}^{r} \frac{(-1)^{r-s}}{s!(r-s)!} g^{r-s}(x) \times (g^s)^{(j)}(x) \\
&= \sum_{r=1}^{j} (-1)^r \theta_0^r \mathrm{e}^{-\theta_0 \times g(x)} \sum_{s=0}^{r} \frac{(-1)^{r-s}}{s!(r-s)!} g^{r-s}(x) \times (g^s)^{(j)}(x). \qquad (44)
\end{aligned}$$

Combining (43) and (44), we obtain

$$\overline{F}_S(x) \quad = \quad \int \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} \frac{x^j}{j!}\left((1-q)^{n\theta_0} - (1-q)^{(n+1)\theta_0}\right)(-1)^j$$

$$
\begin{aligned}
&\times \quad \sum_{r=1}^{j}(-1)^r\theta_0^r\,\mathrm{e}^{-\theta_0\times g(x)}\sum_{s=0}^{r}\frac{(-1)^{r-s}}{s!(r-s)!}g^{r-s}(x)\times(g^s)^{(j)}(x)\mathrm{d}F_{\Theta_0(\theta_0)}\\
&= \quad \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\sum_{r=1}^{j}(-1)^{j+r}\frac{x^j}{j!}\int\left(\mathrm{e}^{-\theta_0(g(x)-n\ln(1-q))}-\mathrm{e}^{-\theta_0(g(x)-(n+1)\ln(1-q))}\right)\theta_0^r\mathrm{d}F_{\Theta_0(\theta_0)}\\
&\times \quad \left\{\sum_{s=0}^{r}\frac{(-1)^{r-s}}{s!(r-s)!}g^{r-s}(x)\times(g^s)^{(j)}(x)\right\}\\
&= \quad \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\sum_{r=1}^{j}(-1)^{j}\frac{x^j}{j!}\left\{\frac{\mathrm{d}^\mathrm{r}}{\mathrm{d}t^r}\mathcal{L}_\Theta(t)\Big|_{g(x)-n\ln(1-q)}-\frac{\mathrm{d}^\mathrm{r}}{\mathrm{d}t^r}\mathcal{L}_\Theta(t)\Big|_{g(x)-(n+1)\ln(1-q)}\right\}\\
&\times \quad \left\{\sum_{s=0}^{r}\frac{(-1)^{r-s}}{s!(r-s)!}g^{r-s}(x)\times(g^s)^{(j)}(x)\right\},
\end{aligned}
$$

where $g(x)=\mathcal{L}_{\Theta_0}^{-1}\circ\mathcal{L}_{\Theta_1}(x)$, $\forall x>0$. Note that the derivatives of $g^s(x)=\left(\mathcal{L}_{\Theta_0}^{-1}\circ\mathcal{L}_{\Theta_1}\right)^s(x)$, for $s\in\mathbb{N}$, can be obtained numerically.

5 Conclusion

Collective risk models under hierarchical Archimedean dependence settings were presented. A sampling algorithm for the random sum S, and stochastic ordering inequalities on S for different setups were derived. Moreover, a computational methodology for the pmf of S was presented. To further complement our results, explicit formulas have been derived for the cdf of S for specific classes of collective risk models based on Archimedean and hierarchical Archimedean copulas.

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