Asset Price Bubbles, Market Liquidity and Systemic Risk

Robert Jarrow*and Sujan Lamichhane†

February 15, 2018

Abstract

This paper studies an equilibrium model with heterogeneous agents, asset price bubbles, and trading constraints. Market liquidity is modeled as a stochastic quantity impact from trading on the price. We introduce a different framework for analyzing rational asset price bubbles, which are shown to exist in equilibrium due to heterogeneous beliefs, heterogeneous preferences, and binding trading constraints. Positive price bubbles are larger in illiquid markets and when trading constraints are more binding. A realization of systemic risk, defined as the risk of market failure due to an exogenous shock to the economy, results in a significant loss of wealth in the economy as agents are unable to meet their trading constraints and default. Systemic risk is shown to increase as: (i) the fraction of agents seeing an asset price bubble increases, (ii) as the market becomes more illiquid, and (iii) as trading constraints are relaxed.

Key Words: asset price equilibrium, bubbles, market liquidity, systemic risk, heterogeneous agents/beliefs, borrowing/trading constraints

JEL Codes: G12, E44, E58

*Samuel Curtis Johnson Graduate School of Management, Cornell University, Ithaca, N.Y. 14853 and Kamakura Corporation, Honolulu, Hawaii 96815. email: raj15@cornell.edu.

†Department of Economics, Cornell University, Ithaca, N.Y. 14853. email: sl2563@cornell.edu.

1 Introduction

The purpose of this paper is to study the effects of financial markets on the real macroeconomy. In the process, we introduce a different framework for analyzing rational asset price bubbles in equilibrium, which should prove useful for formulating policy prescriptions. In particular, we study how asset price bubbles, market liquidity, and trading constraints affect systemic risk, a term often associated with the breakdown of the economic and financial system (Billio et al. (2010) [10]). Indeed, many economic crises have been associated with the failure of market liquidity and the bursting of asset price bubbles. The two most recent examples are the liquidity crisis of 1998 due to the failure of LTCM and the 2007 credit crisis due to bursting of the housing price bubble. Government intervention was necessary in both cases to ensure continued market liquidity and (allegedly) to avoid market failure (see Brunnermeier and Pedersen (2008) [14] for more details on these crises).

The sheer size of financial markets evidences its potential for impacting the real economy. In this regard, consider some US household asset allocation data from the Federal Reserve's quarterly release of the Financial Accounts of the United States (Z.1) data [29], which includes flow of funds, balance sheet, and integrated macroeconomic accounts. The June 2017 release shows that the net-worth of US households and nonprofits rose to \$94.8 trillion during the first quarter of 2017. For this time period, total asset values were about \$109.98 trillion and liabilities were about \$15.15 trillion. Of this, financial assets were about \$77.11 trillion while the market value of equity shares were \$26.88 trillion. This means that financial assets constituted about 70% of total asset value and equities about 24.44%. These numbers document the significance of financial assets as a percent of total assets, and the importance of equity. In contrast, for the same time period, non-financial assets constituted about \$32.87 trillion and among such assets, real estate accounted for about \$26.86 trillion, which is 81.71% of non-financial assets or 24.44% of total assets. Thus, equity and real estate represent roughly same percent of US households' total assets.

Given these magnitudes, it should come as no surprise that shocks to prices in financial and related asset markets can easily affect the real macroeconomy. The bursting of price bubbles are an undeniable phenomena in asset markets, be it equities, real estate/housing, or commodities[1]. One of the first recorded price bubbles was the Dutch

[1] All these markets are connected to the financial sector that intermediates the flow of funds within the economy. We will use the terms financial sector or markets interchangeably. This should cause no

tulip mania of the mid 1600s. Other notable bubbles were related to the South Sea company in 1720 (Garber 1989,1990 [31][32]), the equity price bubble preceding the Great Depression in the U.S. (White 1990 [62]), the dot com bubble (Brunnermeier and Nagel, 2004 [13]), and the US housing price bubble before 2007 (Clark and Coggin, 2011 [24]). It has been argued that the Great Recession of 2007-9 was precipitated by massive declines in both equity and real estate prices - the bursting of price bubbles in these markets.

Market liquidity is another dimension of asset markets that interacts with asset price bubbles to affect the macroeconomy. The liquidity of asset markets can affect both the magnitude and severity of asset price bubbles. To understand this interaction, consider housing versus equity markets, in particular the equity futures market. According to the recent U.S. Census Bureau report [18], the median price of new homes sold in the U.S. in June 2017 was $311,600. Compare that to the E-mini S&P 500 Index futures, one of the most liquid index futures contracts, which traded around 2500 points (rounding up) during the same period. The notional value of one contract is 2500 times 50 (the multiplier) or $125,000. So three futures contracts total $375,000 in notional; about the same value as the median home.

Now assume that an agent wants to sell their home. One can not ignore various frictions associated with the sale, such as commissions and search cost. These costs reflect the illiquid nature of the housing market. In contrast, assume that an agent wants to sell three S&P 500 futures position. The process is literally just one click of a button (or two depending on the broker), and the transaction costs, in every aspect are negligible. Most brokers charge about $2.50 to $3 per-contract to open or close the trade, which includes commissions and various fees. Consequently, one suspects that price bubbles should be more prevalent in the S&P futures markets.

This paper constructs an equilibrium model to study the impact of asset price bubbles and market liquidity on systemic risk. The model is an extension of Jarrow (2017) [49]. The setting is discrete time with a finite horizon. The economy is populated by heterogeneous agents/households facing trading constraints. Market liquidity costs are modeled as a stochastic quantity impact from trading on price, where the size of the impact depends on the trade size. Traded are two assets: a bond/money market account and a risky asset/stock. Borrowing and lending occurs where agents can only borrow up to a certain fraction of their equity value. Short sales are allowed but margin

confusion as asset markets essentially operate through the financial sector, be it lending, borrowing, or insurance.

must be posted to insure coverage of the short position at a future date[2]. All agents are risk averse, maximizing the expected utility of terminal consumption.

In this economy, we show that asset price bubbles can arise endogenously in a rational equilibrium due to heterogeneous beliefs, heterogeneous preferences, and binding trading constraints. A bubble is defined as the difference between the actual/observed market price of the asset and the fundamental value that agents assign given their own beliefs, preferences, and optimal (constrained) trading strategy. Due to this economic structure, some agents may see bubbles while others may not. We define systemic risk as the risk of market failure due to an exogenous shock to the economy that results in funding illiquidity, which is the conjunction of market illiquidity (i.e. liquidity risk) and binding trading constraints. In our setting this is equivalent to the shock resulting in the inability of agents to meet their trading constraints, leading to default and the nonexistence of an economic equilibrium. Such a market failure implies a large loss of wealth in the economy.

Consistent with the previous intuition, we show the following.

- Due to borrowing constraints, negative price bubbles exist (assets are undervalued), and they are larger (smaller) in more liquid markets (illiquid markets).
- Due to short sale constraints, positive price bubbles exist (assets are overvalued), and they are smaller (larger) in more liquid markets (illiquid markets).
- The percentage of agents in the economy who view negative asset price bubbles increase as a market becomes more liquid.
- The percentage of agents in the economy who view positive asset price bubbles decreases as a market becomes more liquid.
- The magnitude of a bubble increases when trading constraints are more restrictive.
- Systemic risk increases as the percentage of agents who see bubbles increases.
- Systemic risk increases as the market becomes more illiquid.

[2]Note that our trading constraint has a very general structure. First, it limits direct borrowing by requiring collateral, a widely used approach in the macro literature. Second, it limits indirect borrowing by restricting the magnitude of short positions. Although short sale restrictions are widely used in the finance literature, this constraint is largely ignored in macro. Our formulation jointly considers both of these constraints, and hereafter the term *trading constraint* will be used to describe both of these restrictions.

- Systemic risk decreases (increases) when the trading constraints are more restrictive (relaxed).

After the financial crisis of 2007-09, many scholarly papers in macro-finance have qualitatively discussed the interaction among systemic risk, market liquidity and asset price bubbles, generally understood to be the difference between the market price and a unique fundamental price (for example, see Brunnermeier and Oehmke (2012)[15], Hall (2011)[34] and references therein.). The implication, of course, is that everyone sees (or should see) the same price bubble. In contrast, in the context of our heterogeneous agents economy, agents can differ both in whether a price bubble exists and if it does, its magnitude[3]. The systemic risk implications of agents seeing bubbles is important because the existence of bubbles increases systemic risk, i.e. massive agent defaults and market failure, which leads to a significant loss of wealth in the economy.

This insight implies that what is relevant for macro/monetary policy is gauging the total *fraction of agents that see price bubbles* and *not* whether the level of market prices is too large[4]. In fact, it may be counterproductive for policy makers to seek a universal price bubble that all agents agree on. To take action based on the level of market prices, policy makers need to verify its existence and magnitude, which is a difficult task[5]. Yet, with heterogeneous beliefs, policy makers need to only focus on the fraction of agents that see bubbles, monitoring market sentiment using the financial press, surveys, and other relevant borrowing and short sale interest data, and act when that fraction becomes too large. Because bubbles are affected by market liquidity and trading constraints, as detailed above, policy makers can affect the fraction of agents seeing bubbles by making markets either more liquid or by making trading constraints more restrictive. This indirect channel can be very effective in changing agents' beliefs regarding price bubbles, perhaps even more effective than interest rate monetary policy.

Our paper relates to the macroeconomics literature studying the impact of financial frictions. The classic papers in this area include Bernanke and Gertler (1989)

[3]This link between bubbles and heterogeneous beliefs was started by Harrison and Kreps (1978) [35]. In this literature, bubbles arise because agents disagree about an asset's fundamental value and they are trading/short sale constrained. See the review papers by Brunnermeier and Oehmke (2012)[15] and Xiong (2013)[63] for more details.

[4]This statement formally applies only if we restrict ourselves to equilibrium prices. In disequilibrium it is possible that all agents see the same uniform price bubble. This would occur, for example, in a complete and arbitrage-free market, see Jarrow (2015)[44]. Characterizing systemic risk in terms of the fraction of economic agents was first introduced in Lamichhane (2017)[52].

[5]For example, see the discussion by the president of the Federal Reserve Bank of Minneapolis Neel Kashkari [50] on monetary policy and bubbles.

[8], Kiyotaki and Moore (1997) [51], and Bernanke, Gertler, and Gilchrist (1999) [9]. These papers analyze how temporary financial shocks can have persistent effects on the economy given the existence of financial frictions, such as borrowing or collateral constraints. Some recent papers include He and Krishnamurthy (2012 [37], 2013 [38], 2014 [39]) and Brunnermeier and Sannikov (2014)[17]. For a review of this literature see Brunnermeier, Eisenbach and Sannikov (2013) [16]. Our paper is closest to Brunnermeier and Sannikov (2014) [17], He and Krishnamurthy (2014) [39], and Lamichhane (2017) [52] because these papers study systemic risk with a financial sector. Our paper is also related to the papers by Huggett (1993) [41], Aiyagari (1994) [5], Benhabib, Bisin and Zhu (2011 [6], 2014 [7]), and Achdou et al. (2015) [3]. Our model has heterogeneous beliefs across agents, as opposed to heterogeneous income or productivity shocks considered in these papers.

With respect to finance, our paper relates to the literature studying an investor's optimal trading strategy with liquidity risk, see Cetin and Rogers [20], Vath, Mnif, and Pham [61], Chebbi and Soner[21], and Pennanen [54]. We use a similar formulation as contained in Pennanen [54]. In this literature, there are two papers studying dynamic Radner equilibrium. These are the overlapping generations model of Acharya and Pedersen [1] and the discrete time model of Jarrow[47]. Acharya and Pedersen [1] include no short sales and liquidity risk is characterized by a fixed but stochastic transaction cost that is independent of trade size. Jarrow [47] includes a stochastic liquidity cost that depends on trade size, but there are no trading constraints. Standard transaction costs are a special case of our formulation (see Jarrow and Protter [43] for a detailed explanation). Our notion of market liquidity is related to but more general than the market and funding liquidity as considered in Brunnermeier and Pedersen (2008)[14].

An outline for the paper is as follows. The next section presents the model structure. Section 3 characterizes price bubbles and their relation to market liquidity. Section 4 analyzes systemic risk and section 5 concludes.

2 The Model

This section presents the details of the model, which is an extension of Jarrow (2017b) [49].

2.1 The Market

We consider a discrete time economy with a finite horizon $t \in \{0, 1, ..., T\}$. The randomness in the economy is characterized by a given complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where Ω is state space, $\mathbb{P}$ is the (statistical) probability measure, and $\mathbb{F} = \{\mathcal{F}\}_{t\in\{0,1,...,T\}}$ is the filtration with $\mathcal{F} = \mathcal{F}_T$. The economy is populated with heterogeneous agents of total mass one. These agents are partitioned into a finite number of types, each type with a strictly positive mass, indexed by $i = 1, \ldots, I$. The discrete mass of a type i agent is denoted $\mathbb{I}(i)$ for $i = 1, \ldots, I$ with $\sum_{i=1}^{I} \mathbb{I}(i) = 1$. Agent types are heterogeneous in their beliefs $\mathbb{P}^i$, wealth W^i (to be defined later), and their preferences $U_i : \mathbb{R} \times \Omega \to \mathbb{R}$ which are defined over terminal consumption. We also assume that $\mathbb{P}_i$ are equivalent to $\mathbb{P}$ for all i. This means that agents' beliefs and the statistical probability measure agree on zero probability events. The filtration $\mathbb{F}$ corresponds to the information set of each agent implying that there is symmetric information.

Agents can trade two assets: a riskless bond, representing by a money market account (mma) and a risky asset. The risky asset is arbitrary, so it can represent equity (stock), physical capital, or even real estate depending upon the market analyzed. Here, for convenience, we will use the term risky asset or stocks interchangeably. Borrowing and lending takes place through a bond market that is in zero net supply. Agents also face a borrowing constraint (discussed below). The risky asset is in positive supply, which is constant across time. Short sales are allowed, but margin requirements are imposed to insure that the short position can be covered at a future date. Without loss of generality, prices are normalized by the value of the mma, i.e. the mma is the numeraire. Alternatively, we can think of this numeraire as the units of consumption good redeemable at the last date in the model.

We assume that the stock pays no dividends over times $\{0, \ldots, T-1\}$, but it pays a liquidating dividend at time T.

Assumption. *(Liquidating Dividends)*

The risky asset S_t is assumed to have no cash flows (dividends) over times $t \in \{0, 1, ..., T\}$ and has an exogenous liquidating dividend at time T, i.e. there exists a $\xi : \Omega \to \mathbb{R}$ at time T such that $S_T = \xi > 0$.

We assume that the stock is in positive supply with $N > 0$ shares outstanding and that the mma is in zero net supply, for all times. At the terminal time T, all debts, if any, must be paid, all the remaining positions liquidated, and the proceeds used for consumption. Since all positions must be liquidated at time T, liquidity costs are necessarily incurred by agents prior to consumption. Time $T+1$ is a non-trading date when consumption occurs.

2.2 The Liquidity Cost Process

Market liquidity/illiquidity is captured by a stochastic quantity impact on the price from trading, where the trade size affects the magnitude of the impact. In this regard, we use the liquidity cost process of Cetin, Jarrow, Protter [19] as modified by Cetin and Rogers [20]. Our approach is closely related to market liquidity as defined in Brunnermeier and Pedersen (2008)[14][6].

Define $s_t(x,\omega)$ to be the *per share* market price of the stock for a trade of size $x \in \mathbb{R}$. Both purchases and short sales are allowed. This is sometimes called the supply curve. We assume that: (i) $s_t(x,\omega)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$ measurable for each t where $\mathcal{B}(\mathbb{R})$ is the Borel sigma-algebra on $\mathbb{R}$, (ii) $s_t(x,\omega) > 0$ in x for for all t a.e. $\mathbb{P}$, and (iii) $s_t(x,\omega)$ is strictly increasing in x for for all t a.e. $\mathbb{P}$.

Define $s_t(0,\omega) \equiv S_t$ to be the market price for zero trades. This corresponds to the market price in a world with no quantity impact from trading. S_t is the *marked-to-market* price of the stock. We add the following assumption to characterize the liquidity costs of trading.

Assumption. *(Liquidity Cost Process)*

Define $\varphi_t(x,\omega)$ to be the liquidity cost for selling/buying x shares at time t given ω, i.e.

$$xs_t(x,\omega) \equiv \varphi_t(x,\omega)s_t(0,\omega) = \varphi_t(x,\omega)S_t.$$

We assume that[7]

1. *$\varphi_t(x,\omega)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$ measurable for each t, and*

2. *for a fixed $t \in \{0,1,\ldots,T\}$ and $\omega \in \Omega$, $\varphi_t(x,\omega) : \mathbb{R} \to (-\infty,\infty]$ is strictly convex, strictly increasing where $\varphi_t(0) = 0$, and φ_t is differentiable for all x.*[8]

[6]See Nikolau (2009)[53] for a discussion of various liquidity notions. Our liquidity cost can also be interpreted as an endogenous transaction cost. In equity markets this cost is incurred by having to sell below the fair/mid-price or buying above the fair/mid-price. In housing or real estate markets this reflects the costs associated with either buying or selling the real estate property. This implies that standard transaction costs are a special case of our market liquidity formulation.

[7]For simplicity of notation, we will often drop the dependence of φ_t on ω.

[8]This implies that $\tilde{S}_t(x,\omega) = xs_t(x,\omega)$ is convex, lower semicontinuous with $\tilde{S}_t(0,\omega) = 0$ for every ω. Hence, by Pennannen [54], p. 747, $\tilde{S}_t$ is a $\mathcal{F}_t$ - measurable normal integrand.

This is a very general liquidity cost process. The functional form of $\varphi_t(x)$ is given exogenously, but validated endogenously in equilibrium. By condition (1), the larger the quantity purchased, the larger the price paid per share. Conversely, the larger the quantity sold, the less the price received per share. This captures the inelastic nature of the supply curve for shares. The increasing condition, in conjunction with $\varphi_t(0) = 0$, implies that $\varphi_t(x) > 0$ for $x > 0$ and $\varphi_t(x) < 0$ for $x < 0$.

The convexity condition is needed to incorporate nonlinearities in liquidity costs. Indeed, when there is no quantity impact on the price from trading (the traditional model), the quantity impact function is linear in the trade size, i.e.

$$\varphi_t(x) = x.$$

The strict convexity assumption insures that the liquidity cost is larger than proportional as the trade size increases.

The marginal cost from trading dx additional shares in a trade of size x is

$$\frac{d\left(\varphi_t(x)S_t\right)}{dx} = \varphi_t'(x)S_t > 0.$$

This represents the *quoted* or *transaction price* when trading x shares. It is the price paid/received for the last share traded. In this representation there is no distinction between a quoted and transaction price, both are the same. Since $\varphi_t'(0) = 1$ and $\varphi_t(x)$ is convex, $\varphi_t'(x) > 1$ for $x > 0$ and $\varphi_t'(x) < 1$ for $x < 0$.

Given this interpretation, we see that $\varphi_t'(0)S_t = S_t$ corresponds to the transaction price when trading zero shares, or equivalently, the marked-to-market price. This implies that $\varphi_t'(0+)S_t$ corresponds to the ask price paid for buying $0+$ (an infinitesimal quantity of) shares and $\varphi_t'(0-)S_t$ corresponds to the bid price received for selling $0-$ (an infinitesimal quantity of) shares. The condition $\varphi_t'(0) = 1$ along with the differentiability of $\varphi_t(x)$ for all x implies that $\varphi_t'(x)$ is continuous[9], hence $\varphi_t'(0+)S_t = S_t = \varphi_t'(0-)S_t$, i.e. the ask price equals the marked-to-market price which equals the bid price. Finally, note that if there is no quantity impact on the price from trading, then $\varphi'(x) = 1$ for all x, and the transaction and marked-to-market prices are always equal. An example helps to clarify this assumption.

Example. *(Liquidity Cost Process)* Consider the following stochastic liquidity cost

[9]See Rockafellar [56], p. 246.

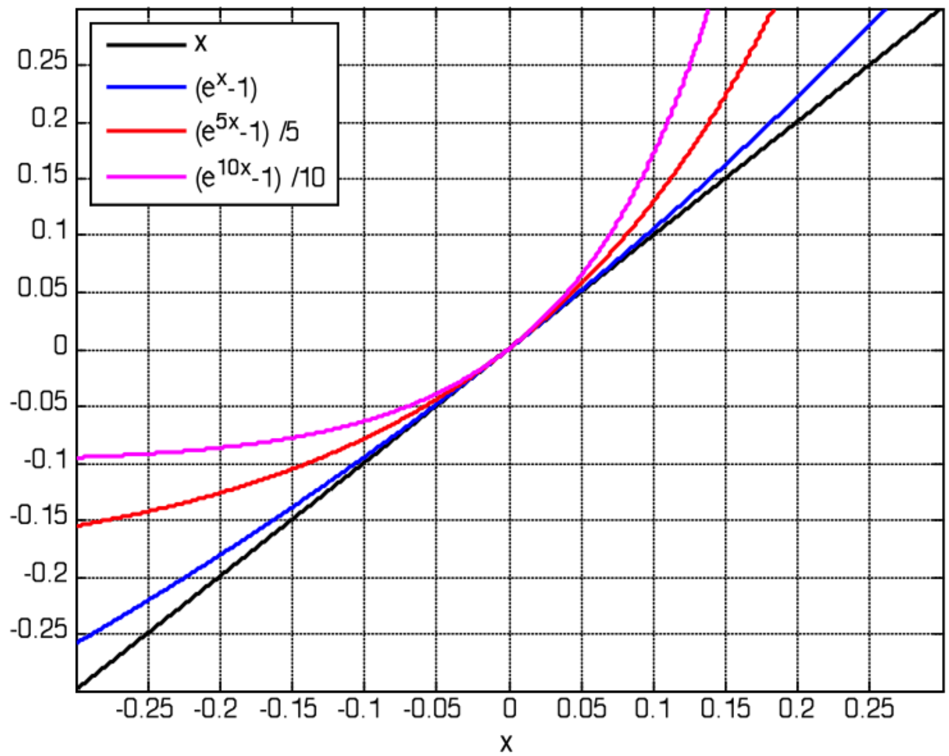

Figure 1: Liquidity Cost Functions

process:

$$\varphi_t(x, \omega) = \frac{e^{\alpha_t(\omega)x} - 1}{\alpha_t(\omega)}$$

where $\alpha_t(\omega) > 0$ is $\mathcal{F}_t$ - measurable. It can be easily checked that this process satisfies all the previous assumptions. We see that

$$\varphi_t'(x, \omega) = e^{\alpha_t(\omega)x}$$

with $\varphi_t'(0) = 1$. This liquidity cost process is graphed in Figure (1). The 45 degree line corresponds to the no liquidity cost case where $\varphi_t(x, \omega) = x$ for all (t, ω). As we will discuss in subsequent sections, the liquidity cost curves associated with higher $\alpha_t(\omega)$ are relatively more illiquid compared to the curves with lower $\alpha_t(\omega)$.

This liquidity cost function is analogous to the non-linear adjustment cost of investment or capital in the macroeconomics literature. Firms or households that own the capital face such costs when replacing old capital with new (physical) capital. Adjustment cost functions are often represented by $\varphi(I_t/k_t)$ where $\varphi(\delta) = 0$, $\varphi'(.) > 0$, $\varphi''(.) > 0$, and δ is the rate of capital depreciation. Here each unit of investment is transformed into less than one unit of capital, reflecting adjustment costs. $\varphi_t(.)$ usually enters the capital evolution process as $k_{t+1} = (1-\delta)k_t + I_t - k_t\varphi(I_t/k_t)$. In macro models, when this approach is used to analyze a firm's investment decision, we get Tobin's Q-theory, as in Tobin (1969)[59] and Hayashi (1982)[36]. Further, such adjustment costs can also arise due to investment changes between periods as in Christiano, Eichenbaum and Evans (2005) where the investment adjustment cost function is $\varphi(I_t/I_{t-1})$ and in steady state $\varphi(1) = 0$, $\varphi'(1) = 0$, and $\varphi''(1) > 0$. Here, the capital accumulation is given as $k_{t+1} = (1 - \varphi(I_t/I_{t-1}))I_t + (1-\delta)k_t$. The adjustment cost function is often assumed to be quadratic[10].

An advantage of using this liquidity cost process to characterize market liquidity is that it allows us to consider various asset market types with different liquidities. For example, equities markets are more liquid than housing markets. This difference is represented by different functional forms for the liquidity cost function $\varphi_t(x)$. We will exploit this benefit below. Additionally, this liquidity cost function can be used to generate liquidity policies in an analogous function to how the Taylor rule is used in monetary policy to set nominal interest rates depending on output/inflation gaps and the unemployment rate (see Eggertsson and Krugman (2012)[28], Guerrieri and Iacoviello (2017)[33]; also Christiano et al. (2010)[23] for a review of DSGE models with monetary policy).

2.3 The Budget Set

In an asset pricing model, the budget set is characterized by the notion of a *self-financing trading strategy*. With a liquidity cost function, defining an agent's budget is subtle. This is because a trading strategy's wealth is not well defined. A trading strategy's wealth depends on the stock price, which depends on a trade quantity x. To avoid this ambiguity, we focus on the shares in the budget set.

A *trading strategy* is defined to be a $\mathcal{F}_{t-1}$ - measurable stochastic process (X_t, Y_t) representing the aggregate shares in stock and mma, given an initial position (x, y).

[10]Bernanke, Gertler and Gilchrist (1999)[9] use a similar non-linear adjustment cost to capital, allowing for dynamic amplification of a negative shock with $k_{t+1} = K_t\varphi(I_t/k_t) + (1-\delta)k_t$. Here, $\varphi(.)$ there is increasing and concave. Brunnermeier and Sannikov (2014)[17] also use a similar approach interpreting the concavity of such an adjustment function as capturing a technological illiquidity when converting output to new capital and vice versa.

The budget set, a *self-financing* trading strategy, for a generic (X_t, Y_t) can be written as

$$Y_{t+1} = Y_t - \Delta X_{t+1} s_t(\Delta X_{t+1}) \tag{1}$$

where $\Delta X_{t+1} = X_{t+1} - X_t$. Using the liquidity cost function we get

$$Y_{t+1} = Y_t - \varphi_t(\Delta X_{t+1}) S_t \tag{2}$$

Starting from (x, y), this is a difference equation whose solution is

$$Y_{t+1} = y - \sum_{j=0}^{t} \varphi_j(\Delta X_{j+1}) S_j. \tag{3}$$

By construction, the last trade occurs at time T, when the stock holdings are liquidated, i.e.

$$\Delta X_{T+1} = -x - \sum_{t=0}^{T-1} \Delta X_{t+1} = -X_T.$$

This implies that

$$Y_{T+1} = y - \varphi_T \left(-x - \sum_{t=0}^{T-1} \Delta X_{t+1} \right) S_T - \sum_{t=0}^{T-1} \varphi_t(\Delta X_{t+1}) S_t. \tag{4}$$

Given the self financing condition, we have that

$$\{(x, y), X_{t+1} \text{ for } t \in \{0, 1, \ldots, T-1\}\}$$

uniquely determines Y_{T+1}[11].

As seen above, the change in the budget set is determined by the change in the mma and risky asset as agents borrow and trade amongst each other. Later on we will see that this is related to the changes in the wealth of agents.

2.4 The Trading Constraints

We now characterize the trading constraints in the economy. This closely follows the trading constraints in Jarrow (2017)[49]. We restrict the *self-financing* trading strategy

[11]Note that we do not need to include X_{T+1} in this expression because by construction it equals zero.

(X_t, Y_t) to satisfy the following condition:

$$\{X_{t+1} \in K_t(\omega) \text{ for } t \in \{0, 1, \dots, T-1\} \text{ a.e. } \mathbb{P}\}$$

where $K_t(\omega) \subset \mathbb{R}$ is a $\mathcal{F}_t$ - measurable[12], nonempty, closed, convex set with $0 \in K_t$ and $\{X_{t+1} = X_t\} \in int\,(K_t)$ for $t \in \{0, 1, \dots, T-1\}$ a.e. $\mathbb{P}$ where $int(\cdot)$ denotes the interior in the usual topology on $\mathbb{R}$.

In our paper, we represent the trading constraints in the economy by defining the set $K_t(\omega)$ as follows.[13].

Assumption. *(Trading Constraints)*

$$K_t(\omega) = \left\{X_{t+1} \in \mathbb{R} :\ Y_{t+1} \geq -S_t X_{t+1} \left(\gamma 1_{X_{t+1} \geq 0} + (1+\gamma) 1_{X_{t+1} \leq 0}\right)\right\}$$

where $0 \leq \gamma \leq 1$, and $K_t(\omega) \subset \mathbb{R}$ is a $\mathcal{F}_t$ measurable, nonempty, closed, convex set.

Proof. It is nonempty because $0 \in K_t(\omega)$. We can rewrite the constraint set as $K_t(\omega) = \{X_{t+1} \geq 0 :\ Y_{t+1} \geq -\gamma S_t X_{t+1}\} + \{X_{t+1} \leq 0 :\ Y_{t+1} \geq -(1+\gamma) S_t X_{t+1}\}$.

This set is closed and since each subset in the sum is convex, the sum is convex, see Ruszczynski [58], p. 18. □

Marked-to-market prices S_t are used to define the trading constraint because they reflect prices readily observable in the market. This is contrasted with transaction prices $\varphi_t'(x) S_t$ which depend on knowledge of the trade size and the liquidity cost paid via the liquidity cost function $\varphi_t(x)$. This constraint limits both borrowing and short selling. To see this, note that if the trading strategy shorts the risky asset $(X_{t+1} < 0)$, then it simplifies to $Y_{t+1} \geq -(1+\gamma) S_t X_{t+1} > 0$. This implies that to cover the short position, a margin account must be held in the riskless asset consisting of the marked-to-market value of the short position plus a fraction $\gamma > 0$ more. Next, if the trading strategy is to buy the risky asset $(X_{t+1} > 0)$, then it simplifies to $Y_{t+1} \geq -\gamma S_t X_{t+1}$ where $Y_{t+1} < 0$ is the borrowing. This implies that to finance the purchase, a borrowing of no more than a fraction $\gamma > 0$ of the market-to-market value of the long position is allowed.

Note that the trading constraint only applies for times $t \in \{0, 1, ..., T-1\}$ because at time T all the positions are liquidated. The assumption $\{0\} \in K_t$ implies that

[12] K_t being $\mathcal{F}_t$ - measurable means that $\{\omega \in \Omega : K_t(\omega) \cap A \neq \emptyset\} \in \mathcal{F}_t$ for every open set $A \subset \mathbb{R}^2$.

[13] See Jarrow (2017b)[49] for some examples of different types of trading constraint sets that frequently arise in financial applications.

zero holdings in the stock always satisfies the borrowing constraint. $X_{t+1} = X_t \in int(K_t)$ means that no trading at any time t is always feasible. Here, the restrictions on the mma position are not explicit. However, the holdings in the mma are implicitly restricted because Y_{t+1} is uniquely determined by the position in the stock and the budget constraint.

Let us now define the normal cone to the set K_t for a given $\omega \in \Omega$ as

$$N_{K_t}(X) = \Big\{ \kappa \in \mathbb{R} : \kappa(Z - X) \geq 0 \quad \text{for all} \quad Z \in K_t \Big\}$$

where $X \in K_t$ and $t \in \{0, 1, ..., T-1\}$. This set is $\mathcal{F}_t$ - measurable and by construction $N_{K_T}(X) = N_{\mathbb{R}}(X) = \{0\}$.

The following lemma will be needed to understand the shadow costs of the trading constraints in subsequent sections.

Lemma 1. *(Signs of the Elements $\kappa \in N_{K_t}(X_{t+1})$)*

Let X_{t+1} be a trading strategy and $\kappa \in N_{K_t}(X_{t+1})$.

(1) (Non-binding Constraint) If $X_{t+1} \in int(K_t)$, then $\kappa = 0$.

(2) (Binding Constraint) If $X_{t+1} \in bd(K_t)$ where $bd(\cdot)$ denotes the boundary in the usual topology on $\mathbb{R}$.

Given $\kappa \in N_{K_t}(X)$ with $\kappa \neq 0$. Then,

- *(a) if $X_{t+1} > 0$, then $\kappa > 0$.*
- *(b) if $X_{t+1} = 0$ and $[-\epsilon, 0] \subset K_t$ for $\varepsilon > 0$, then $\kappa > 0$.*
- *(c) if $X_{t+1} < 0$, then $\kappa < 0$.*
- *(d) if $X_{t+1} = 0$ and $[0, \varepsilon] \subset K_t$ for $\varepsilon > 0$, then $\kappa < 0$.*

Proof. If $X_{t+1} \in int(K_t)$, then $N_{K_t}(X_{t+1}) = \{0\}$, see Tuy [60], p. 22. Hence, if $\kappa \in N_{K_t}(X_{t+1})$, then $\kappa = 0$.

(Case a) If $X_{t+1} \in bd(K_t)$, and $X_{t+1} > 0$, then by the convexity of K_t, $[0, X_{t+1}] \subset K_t$. This implies $N_{K_t}(X_{t+1}) \neq \{0\}$. Hence, if $\kappa \neq 0$, then $\kappa(Z - X_{t+1}) \leq 0$ for $0 \leq Z \leq X_{t+1}$, implying $\kappa > 0$.

(Case b) follows similarly.

(Case c) If $X_{t+1} \in bd(K_t)$, and $X_{t+1} < 0$, then by the convexity of K_t, $[X_{t+1}, 0] \subset K_t$. This implies $N_{K_t}(X_{t+1}) \neq \{0\}$. Hence, if $\kappa \neq 0$, then $\kappa(Z - X_{t+1}) \leq 0$ for $0 \leq X_{t+1} \leq Z$, implying $\kappa < 0$.

(Case d) follows similarly. This completes the proof. □

This lemma shows that if X_{t+1} is on the boundary and cannot increase (cases (a) and (b)), then $\kappa \neq 0 \in N_{K_t}(X_{t+1})$ is strictly positive. Conversely, if X_{t+1} is on the boundary and cannot decrease (cases (c) and (d)), then $\kappa \neq 0 \in N_{K_t}(X_{t+1})$ is strictly negative. This abstraction will be explicitly used below to characterize bubble. A concrete version of this abstraction is the usual Lagrange multipliers in the optimization problems, where the multiplier $\kappa = 0$ means the constraint is not binding, and $\kappa \neq 0$ means the constraint is binding.

2.5 The Optimization Problem

As noted earlier, the economy is populated by a finite number of heterogeneous agents types indexed by i, with beliefs $\mathbb{P}_i$ and preferences $U_i : \mathbb{R} \times \Omega \to \mathbb{R}$ defined over terminal consumption. The agents of type i are initially endowed with shares in the stock and mma (x_i, y_i). We assume that for all $\omega \in \Omega$, $U_i(z, \omega)$ is strictly increasing, strictly concave, and satisfies the Inada conditions

$$\underset{x \to -\infty}{lim} U_i'(z, \omega) = \infty, \quad \underset{x \to 0}{lim} U_i'(z, \omega) = 0.$$

In addition, we assume that $U_i(0, \omega) = 0$ and that $U_i(z, \omega), U_i'(z, \omega)$ are bounded above by $\mathbb{P}_i$ integrable random variables (independent of z) for all i.[14]

Agents choose shares in the stock and mma, subject to borrowing constraints, to maximize their expected utility of terminal consumption:

$$u_i(x, y) = \sup_{\{X_{t+1} \in K_t :\, t \in \{0,1,\ldots,T-1\}\}} E^{\mathbb{P}_i}[U_i(Y_{T+1})] \tag{5}$$

where expectation is taken under $\mathbb{P}_i$.

2.6 Equilibrium

In an equilibrium, agents maximize their expected utility subject to their constraints, such that the price of risky asset is endogenously determined in the equilibrium. The aggregate holdings in stocks and mma are represented by the expectation with respect to the distribution $\mathbb{I}(i)$ across agents. To formalize this description, we need some definitions.

[14] These assumption imply that $U_i(z, \omega)$ is a normal integrand on $\mathbb{R} \times \Omega$ and when taking the derivative of $E^i\left[U_i(z)\right]$ with respect to z, one can exchange the expectation and derivative operators.

Definition 2. *(An Asset Market)*

An *asset market* is a collection $\left((\varphi, K, N), (\mathbb{P}_i, U_i, (x_i, y_i))_{i=1}^{I}\right)$ where $\mathbb{E}^{\mathbb{I}}[y^i] = 0$ and $\mathbb{E}^{\mathbb{I}}[x^i] = N$.

As defined, an asset market is a liquidity cost process, a trading constraint, a supply of shares outstanding i.e. (φ, K, N) and a set of economic agents $(\mathbb{P}_i, U_i, (x_i, y_i))$ of total mass one. The mma is in zero net supply and the supply of the risky asset is strictly positive and equal to N. By construction, the total supply of shares and mma endowed at time 0 must equal the total supply of shares and mma outstanding. Finally, the underlying filtration and probability measure $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\{0,\ldots,T\}}, \mathbb{P})$ are implicit in the specification of a market.

Definition 3. *(Equilibrium)*

Given an asset market $\left((\varphi, K, N), (\mathbb{P}_i, U_i, (x_i, y_i))_{i=1}^{I}\right)$ with $\mathbb{E}^{\mathbb{I}}[y^i] = 0$ and $\mathbb{E}^{\mathbb{I}}[x^i] = N$.

An *equilibrium* is a price process S and risky asset demands $(X_{t+1}^i : t \in \{0, \ldots, T-1\})_{i=1}^I$ a.e. $\mathbb{P}$ such that

(i). $(X_{t+1}^i \in K_t : t \in \{0, \ldots, T-1\})$ are optimal for all i, and

(ii) $\mathbb{E}^{\mathbb{I}}[\Delta X_{t+1}^i] = 0$ for $t \in \{0, \ldots, T-1\}$ a.e. $\mathbb{P}$ such that the market clears i.e. supply equals demand.

There is no trading at time $T + 1$.

This is standard definition of a dynamic competitive Radner equilibrium (Radner (1982)[57]). Because the supply of shares is constant N for all times, in conjunction with $\mathbb{E}^{\mathbb{I}}[x^i] = \mathbb{E}^{\mathbb{I}}[X_0^i] = N$ this implies that $\mathbb{E}^{\mathbb{I}}[X_{t+1}^i] = N$ for $t \in \{0, \ldots, T-1\}$. Note that by the budget/self-financing condition (1), this implies that mma market decreases in aggregate value if $\Delta X_{t+1}^i \neq 0$ for some i, i.e.

$$\mathbb{E}^{\mathbb{I}}[\Delta Y_{t+1}^i] = -\mathbb{E}^{\mathbb{I}}[\varphi_t(\Delta X_{t+1}^i) S_t] < 0 \quad \text{for } t \in \{0, \ldots, T-1\} \text{ a.e. } \mathbb{P}.$$

This strict decrease in value follows because if $\Delta X_{t+1}^i \neq 0$ for some i, then by the strict convexity of $\varphi_t(\Delta X_{t+1}^i)$, $\varphi_t(\Delta X_{t+1}^i) > \varphi_t(0) + \varphi_t'(0)\Delta X_{t+1}^i$. The fact that $\varphi_t(0) = 0$ in conjunction with $\mathbb{E}^{\mathbb{I}}[\Delta X_{t+1}^i] = 0$ yields

$$-\mathbb{E}^{\mathbb{I}}[\varphi_t(\Delta X_{t+1}^i) S_t] < -\varphi_t'(0) S_t \mathbb{E}^{\mathbb{I}}[\Delta X_{t+1}^i] = 0 \quad \text{for } t \in \{0, \ldots, T-1\} \text{ a.e. } \mathbb{P}.$$

Thus, the decrease in the value of the aggregate mma market captures the the liquidity costs incurred by the aggregate trading in the economy. The parties to whom the liquidity costs are paid are the market makers in the exchange market. The equilibrium in the exchange market endogenously determines the equilibrium liquidity cost function φ. For the details of the exchange market equilibrium see Jarrow (2017b)[49]. In this paper we take the liquidity cost function as given, but the functional form is itself determined in equilibrium.

3 Bubbles and Market Liquidity

This section characterizes equilibrium asset price bubbles and their relation to market liquidity. First, it can be shown, under a set of sufficient conditions, that there exists an equilibrium for the economy previously described (see Jarrow (2017b)[49]). For the subsequent analysis, therefore, we assume the existence of such an equilibrium and we characterize this equilibrium.

3.1 Characterizing Asset Price Bubbles

We start with a theorem characterizing an agent's first order conditions. This characterization is given in the following theorem.

Theorem 4. *(Individual Agent's Optimal allocation)*

Let S be an equilibrium price process.

Then, there exists a unique optimal trading strategy $\{X_{t+1} \in K_t : t \in \{0, \ldots, T-1\}\}$ characterized by the following equations

$$0 \in \partial E_t^{\mathbb{P}_i}\left[U_i\left(Y_{T+1}^i\right)\right] + N_{K_t}(X_{t+1}^i) \tag{6}$$

for $t \in \{0, \ldots, T-1\}$ a.e. $\mathbb{P}_i$

where[15] $Y_{T+1}^i = y_i - \varphi_T\left(-x_i - \sum_{t=0}^{T-1} \Delta X_{t+1}^i\right) S_T - \sum_{t=0}^{T-1} \varphi_t(\Delta X_{t+1}^i) S_t$, $\Delta X_{T+1}^i = -X_T^i$, $\partial(\cdot)$ *is the subdifferential, and* $E_t^{\mathbb{P}_i}[\cdot] = E^{\mathbb{P}_i}[\cdot\,|\mathcal{F}_t]$.

Or equivalently,

[15] Note that when $t = T$, expression (6) is identically zero.

$\varphi_t'(\Delta X^i_{t+1})(S_t - \nu^i_t)$ *is a martingale under*

$$\frac{d\mathbb{Q}_i}{d\mathbb{P}_i} = \frac{U_i'(Y^i_{T+1})}{E^{\mathbb{P}_i}\left[U_i'(Y^i_{T+1})\right]} > 0$$

for $t \in \{0, \dots, T\}$ *where* $v^i_t = \frac{\kappa^i_t}{E^{\mathbb{P}_i}_t\left[U_i'\left(Y^i_{T+1}\right)\right]\varphi_t'(\Delta X^i_{t+1})}$ *and* $\kappa^i_t \in N_{K_t}(X^i_{t+1})$.

Proof. (Step 1) *Existence and Uniqueness*

The above assumptions in conjunction with Examples 2.1 and 5.2 of Pennanen [54] imply that the hypothesis of Theorem 5.1 in Pennanen [54] hold. This theorem states that an optimal trading strategy exists. It is unique by the concavity of the utility function.

(Step 2) *Characterization of the Solution*

We use backward induction.

At time T, with share holdings (X^i_T, Y^i_T), the optimal trading strategy is $\Delta X_{T+1} = -X_T$ since the portfolio must be liquidated.

At time $t < T$ with share holdings (X^i_t, Y^i_t), having determined the optimal $\{X^i_{j+1} : j \in \{t+1, \dots, T-1\}\}$, the optimal $Z \in K_t$ must maximize

$$\begin{aligned} E^{\mathbb{P}_i}_t\left[U_i\left(Y^i_{T+1}(Z)\right)\right] = E^{\mathbb{P}_i}_t\Big[U_i\Big(&y_i - \varphi_T(-\textstyle\sum_{j=t+1}^{T-1}\Delta X^i_{j+1} - Z)S_T \\ &- \textstyle\sum_{j=t+1}^{T-1}\varphi_j(\Delta X^i_{j+1})S_j - \varphi_t\left(Z - X_t\right)S_t + (Y^i_t - y_i)\Big)\Big]. \end{aligned}$$

The first order condition, which is necessary and sufficient (see Tuy [60], p. 75), is that

$$0 \in \partial E^{\mathbb{P}_i}_t\left[U_i\left(Y^i_{T+1}(X_{t+1})\right)\right] + N_{K_t}(X^i_{t+1}).$$

But,

$\partial E^{\mathbb{P}_i}_t\left[U_i\left(Y^i_{T+1}(X_{t+1})\right)\right] = \frac{dE^{\mathbb{P}_i}_t\left[U_i\left(Y^i_{T+1}(X_{t+1})\right)\right]}{dX^i_{t+1}}$

$= E^{\mathbb{P}_i}_t\left[U_i'(Y^i_{T+1})\left(\varphi_T'(\Delta X^i_{T+1})S_T - \varphi_t'(\Delta X^i_{t+1})S_t\right)\right]$.

Hence, there exists a $\kappa^i_t \in N_{K_t}(X^i_{t+1})$ such that

$$E^{\mathbb{P}_i}_t\left[U_i'\left(Y^i_{T+1}\right)\left(\varphi_T'(\Delta X^i_{T+1})S_T - \varphi_t'(\Delta X^i_{t+1})S_t\right)\right] + \kappa_t = 0. \tag{7}$$

$\frac{E^{\mathbb{P}_i}_t\left[U_i'\left(Y^i_{T+1}\right)\left(\varphi_T'(\Delta X^i_{T+1})S_T - \varphi_t'(\Delta X^i_{t+1})S_t\right)\right]}{E^{\mathbb{P}_i}_t\left[U_i'(Y^i_{T+1})\right]} + \frac{\kappa_t}{E^{\mathbb{P}_i}\left[U_i'(Y^i_{T+1})\right]} = 0.$

$E^{\mathbb{Q}_i}_t\left[\varphi_T'(\Delta X^i_{T+1})S_T - \varphi_t'(\Delta X^i_{t+1})S_t\right] + \frac{\kappa_t}{E^{\mathbb{P}_i}_t\left[U_i'(Y^i_{T+1})\right]}\frac{\varphi_t'(\Delta X^i_{t+1})}{\varphi_t'(\Delta X^i_{t+1})} = 0$

where $E_t^{\mathbb{Q}_i}[\cdot] = E^{\mathbb{Q}_i}[\cdot\,|\mathcal{F}_t]$ is conditional expectation under $\mathbb{Q}_i$.

$E_t^{\mathbb{Q}_i}\left[\varphi_T'(\Delta X_{T+1}^i)S_T - \varphi_t'(\Delta X_{t+1}^i)S_t + \varphi_t'(\Delta X_{t+1}^i)\frac{\kappa_t}{E_t^{\mathbb{P}_i}\left[U_i'(Y_{T+1}^i)\right]\varphi_t'(\Delta X_{t+1}^i)}\right] = 0.$

Define $v_t^i = \frac{\kappa_t^i}{E_t^{\mathbb{P}_i}\left[U_i'(Y_{T+1}^i)\right]\varphi_t'(\Delta X_{t+1}^i)}$. Note that $\kappa_T^i = 0$ since $N_{K_T}(X) = \{0\}$ for all $X \in K_T$. Then,

$E_t^{\mathbb{Q}_i}\left[\varphi_T'(\Delta X_{T+1}^i)(S_T - v_T^i) - \varphi_t'(\Delta X_{t+1}^i)(S_t - v_t^i)\right] = 0.$

This implies that $\varphi_t'(\Delta X_{t+1}^i)(S_t - \nu_t^i)$ is a martingale for $t \in \{0, \ldots, T\}$ under $\frac{d\mathbb{Q}_i}{d\mathbb{P}_i} = \frac{U_i'(Y_{T+1}^i)}{E^{\mathbb{P}_i}\left[U_i'(Y_{T+1}^i)\right]} > 0$. This completes the proof. □

Given a characterization of the agent's first order conditions in equilibrium, we now turn to a characterization of asset price bubbles. We first need to define an asset's fundamental value.

Definition 5. *(Fundamental Value)*

The *fundamental value* of an asset to a type i agent is

$$E_t^{\mathbb{Q}_i}[\varphi_T'(\Delta X_{T+1}^i)S_T]$$

where $E_t^{\mathbb{Q}_i}[.] = E^{\mathbb{Q}_i}[.|\mathcal{F}_t]$ is conditional expectation taken under $\mathbb{Q}_i$.

The fundamental value is the time t expected liquidation value of the agent's portfolio at time T, given their optimal trading strategy. This liquidation value reflects *the transaction or market price for the last shares traded* by the agent, i.e. $\varphi_T'(\Delta X_{T+1}^i)S_T$. The personalized state price density $\frac{d\mathbb{Q}_i}{d\mathbb{P}_i}$ is used to value the future transaction because it includes an adjustment for risk.[16] The standard definition of a price bubble now follows.

Definition 6. *(Asset Price Bubble)*

The asset price bubble for a type i agent is

$$\beta_t^i = \varphi_t'(\Delta X_{t+1}^i)S_t - E_t^{\mathbb{Q}_i}[\varphi_T'(\Delta X_{T+1}^i)S_T].$$

From this definition it is clear that a price bubble is related to the beliefs of an individual agent which translates into their personalized fundamental value for the

[16]The state price density is also called the stochastic discount factor.

risky asset. Consequently, some agents may see a price bubble based on their beliefs, while others may not. Note that if there were no liquidity costs to trading shares, i.e. $\varphi_t'(\Delta X_{t+1}^i) = 1$, then the market and transaction prices would equal S_t, and this definition collapses to the standard definition of an asset price bubble appearing in the literature (see Jarrow [44]). We can now characterize an agent's price bubble.

Corollary 7. *(Bubbles in Equilibrium)*

For all $i = 1, \ldots, n$ and $t = 1, \ldots, T$,

$$\beta_t^i = \frac{-\kappa_t^i}{E_t^{\mathbb{P}_i}\left[U_i'\left(Y_{T+1}^i\right)\right]\varphi_t'(\Delta X_{t+1}^i)} \tag{8}$$

for $\kappa_t^i \in N_{K_t}(X_{t+1}^i)$.

If the constraint is non-binding, then $\beta_t^i = 0$. No asset price bubble.

If the constraint is binding, then

(a) if $X_{t+1} > 0$, then $\beta_t^i < 0$.

(b) if $X_{t+1} = 0$ and $[-\epsilon, 0] \subset K_t$ for $\varepsilon > 0$, then $\beta_t^i < 0$.

In cases (a) and (b), the transaction price for the last share traded is a $\mathbb{Q}_i$ submartingale.

(c) if $X_{t+1} < 0$, then $\beta_t^i > 0$.

(d) if $X_{t+1} = 0$ and $[0, \varepsilon] \subset K_t$ for $\varepsilon > 0$, then $\beta_t^i > 0$.

In cases (c) and (d) the transaction price for the last share traded is a $\mathbb{Q}_i$ supermartingale.

Proof. Given $E_t^{\mathbb{P}_i}\left[\varphi_T'(\Delta X_{T+1}^i)(S_T - v_T^i) - \varphi_t'(\Delta X_{t+1}^i)(S_t - v_t^i)\right] = 0$.

$E_t^{\mathbb{P}_i}\left[\varphi_T'(\Delta X_{T+1}^i)S_T\right] - \varphi_t'(\Delta X_{t+1}^i)S_t + \varphi_t'(\Delta X_{t+1}^i)v_t^i = 0$.

Algebra yields

$\varphi_t'(\Delta X_{t+1}^i)S_t - E_t^{\mathbb{P}_i}\left[\varphi_T'(\Delta X_{T+1}^i)S_T\right] = -\varphi_t'(\Delta X_{t+1}^i)v_t^i$, i.e.

$\beta_t^i = -\varphi_t'(\Delta X_{t+1}^i)v_t^i$. But, $v_t^i = \frac{\kappa_t^i}{E_t^{\mathbb{P}_i}\left[U_i'(Y_{T+1}^i)\right]\varphi_t'(\Delta X_{t+1}^i)}$.

Hence $\beta_t^i = \frac{-\kappa_t^i}{E_t^{\mathbb{P}_i}\left[U_i'\left(Y_{T+1}^i\right)\right]\varphi_t'(\Delta X_{t+1}^i)}$.

Using Lemma 1 completes the proof. □

This corollary characterizes an individual agent's asset price bubble. If trading constraints are non-binding and $\kappa_t^i = 0$, then there are no asset price bubbles. However, if trading constraints are binding with $\kappa_t^i \gtrless 0$, then there are four cases. With respect to cases (a) and (b), the trader desires to buy more of the stock, but is constrained. Hence, the stock price is too low relative to the trader's valuation given their state price density. Here the asset price bubble is negative (the stock is undervalued) and the stock price process is a $\mathbb{Q}_i$ submartingale. With respect to cases (c) and (d), the trader desires to short more of the stock, but is constrained. Hence, the stock price is too high relative to the trader's valuation given their state price density. Here the asset price bubble is positive (the stock is overvalued) and the stock price process is a $\mathbb{Q}_i$ supermartingale. This is analogous to the characterization of asset price bubbles in continuous time, competitive, and frictionless markets (see Jarrow [44]).

This corollary can also be used to understand the birth and death of asset price bubbles in this market. In an asset market with no price bubbles, a bubble starts if trading constraints suddenly become binding. This might occur because of exogenous shocks to the constraints imposed by regulators, or because beliefs and or preferences change so that a previously non-binding constraint becomes binding. Conversely, analogous random shocks to the market can also cause bubbles to burst.

3.2 Characterizing Market Liquidity

Now we move on to characterize the relation between bubbles and market liquidity. This characterization requires another definition.

Definition 8. *(Liquid versus Illiquid Markets)*

For a given t, let φ_t^l and φ_t^{nl} be two liquidity cost processes associated with different market types that satisfy the assumptions given above, and such that

$$\begin{array}{lll} \frac{d\varphi_t^l(x)}{dx} < \frac{d\varphi_t^{nl}(x)}{dx} & if & x > 0 \\ \frac{d\varphi_t^{nl}(x)}{dx} < \frac{d\varphi_t^l(x)}{dx} & if & x < 0 \end{array} \tag{9}$$

for all $x \neq 0$ a.e. $\mathbb{P}$. Then, we say that the time t asset market with liquidity cost $\varphi_t^{nl}(x)$ is illiquid (not liquid) relative to the market with liquidity cost $\varphi_t^l(x)$.

As defined above, a market is more liquid the smaller is the transaction price paid $\frac{d(\varphi_t(x)S_t)}{dx} = \varphi_t'(x)S_t$ for a purchase $x > 0$, and the larger the transaction price received for a sale $x < 0$. Given the marked-to-market value of the stock, this is measured by $\varphi_t'(x) \neq 1$, hence the definition. It is easy to show that the more liquid the market, the smaller is the liquidity cost function for a trade of size $x \neq 0$.

Lemma 9. *(Price Impact Costs)*

Let the liquidity cost function φ_t^l be more liquid than φ_t^{nl}. Then

$$\varphi_t^l(x) < \varphi_t^{nl}(x) \tag{10}$$

for all $x \neq 0$ a.e. $\mathbb{P}$.

Proof. Using the fundamental theorem of calculus and the assumption that $\varphi_t^k(0) = 0$, we have $\varphi_t^k(x) = \int_0^x \frac{d\varphi_t^k(u)}{du} du$ for $k \in \{l, nl\}$ for $x > 0$ and $\varphi_t^k(x) = -\int_0^x \frac{d\varphi_t^k(u)}{du} du$ for $k \in \{l, nl\}$ for $x < 0$. The result follows because the integral is a positive linear operator. □

We have already discussed asset markets that can have different liquidities (say housing versus equity markets). The implication of the above lemma is that, if an agent is buying or selling (shorting) a risky asset in a relatively illiquid market, the liquidity cost is higher, i.e. an illiquid market is unfavorable to buyers or sellers in terms of overall cost. This observation is central to many of the results below. This lemma is also analogous to the liquidity cost function example in Figure (1). Larger values of $\alpha_t(\omega)$ represent more illiquid markets. With above insights, we now analyze the nature of bubbles in liquid versus illiquid markets.

Theorem 10. *(Bubbles in Liquid vs. Illiquid Markets)*
(Borrowing Constrained Bubbles, $\beta_t^i < 0$)

If $X_{t+1}^i > 0$, then $\beta_t^i(\varphi_t^l) < \beta_t^i(\varphi_t^{nl}) < 0$, i.e. bubbles in liquid markets are larger in absolute value, all else constant.
(Short Selling Constrained Bubbles, $\beta_t^i > 0$)

If $X_{t+1}^i < 0$, then $0 < \beta_t^i(\varphi_t^l) < \beta_t^i(\varphi_t^{nl})$, i.e. bubbles in illiquid markets are larger, all else constant.

Proof. From expression (8),

$$\beta_t^i = \frac{-\kappa_t^i}{E_t^{\mathbb{P}_i}\left[U_i'\left(Y_{T+1}^i\right)\right]\varphi_t'(\Delta X_{t+1}^i)}.$$

Assume that Y_{T+1}^i, κ_t^i, X_{t+1}, and ΔX_{t+1}^i are fixed. There are two cases to consider.

(Case 1) Assume that $X_{t+1}^i > 0$, then we know from above $\kappa_t^i > 0$, and $\Delta X_{t+1}^i > 0$. Thus, $\beta_t^i < 0$ since $\varphi_t'(\Delta X_{t+1}^i) > 0$ for all ΔX_{t+1}^i.

By the definition (8), $1 > \frac{d\varphi_t^{nl}(x)}{dx} > \frac{d\varphi_t^{l}(x)}{dx} > 0$ for $x > 0$. This means the denominator is larger for φ_t^{nl} than for φ_t^{l}.

Thus, the absolute value of the bubble $|\beta_t^i|$ is larger for φ_t^l, implying $\beta_t^i(\varphi_t^l) < 0$ is a larger negative number.

(Case 2) Assume that $X_{t+1}^i < 0$, then we know from above $\kappa_t^i < 0$, and $\Delta X_{t+1}^i < 0$. Thus, $\beta_t^i > 0$ since $\varphi_t'(\Delta X_{t+1}^i) > 0$ for all ΔX_{t+1}^i.

By the definition (8), $1 < \frac{d\varphi_t^{nl}(x)}{dx} < \frac{d\varphi_t^{l}(x)}{dx}$ for $x < 0$. This means that the denominator is smaller for φ_t^{nl} than for φ_t^{l}.

Thus, the bubble $\beta_t^i > 0$ is larger for φ_t^{nl}. □

This theorem shows that market liquidity and the magnitude of asset price bubbles have an asymmetric relation depending on the sign of the price bubble. When the risky asset is undervalued due to borrowing constraints and the price bubble is negative, bubbles are smaller in illiquid markets. In contrast, when the risky asset is overvalued due to short selling constraints and the price bubble is positive, bubbles are larger in illiquid markets. This asymmetry is due to the difference in the agents' actions to exploit the perceived mispricing and the asymmetry in the liquidity cost function's transaction price $\varphi_t'(x)S_t$ across the two actions.

When agents view the risky asset as undervalued, they borrow to *purchase* the asset. The more risky assets they purchase, the *larger* the transaction price paid per share in an illiquid market. Due to the convexity of $\varphi_t'(x)$, this increase in the transaction price occurs at an increasing rate in an illiquid market relative to a liquid market. This increasing marginal cost of a purchase at an increasing rate, relative to the marginal benefit attained, reduces the optimal constrained purchase quantity in an illiquid market relative to a liquid market, consequently reducing the size of the price bubble in a more illiquid market.

In contrast, when agents' view the risky asset as overvalued, they want to *short sell* the risky asset and provide margin to cover the short position. The more assets they short sell, the *smaller* the transaction price proceeds received per share in an illiquid market. Due to the convexity of $\varphi_t'(x)$, this decrease in the transaction price is at an decreasing rate in an illiquid market relative to a liquid market. This decreasing marginal cost of a sale at a decreasing rate, relative to the marginal benefit attained, increases the optimal constrained sale quantity in an illiquid market relative to a liquid market, consequently increasing the size of the price bubble in a more illiquid market.

We now move on to characterize the percentage of agents in an economy that see bubbles at any given time. A direct application of Theorem 10 yields the following result.

Theorem 11. *(Liquidity and Size of the Price Bubble Distribution)*

Let $\mathbb{I}^-_{\beta_t} \equiv \mathbb{I}(i : \beta^i_t < 0)$ *denote the percentage of agents in the economy that see negative bubbles at time* t*, and* $\mathbb{I}^+_{\beta_t} \equiv \mathbb{I}(i : \beta^i_t > 0)$ *denote the percentage of agents in the economy that see positive bubbles at time* t*. Then,*

(i) $\mathbb{I}^-_{\beta_t}(\varphi^l_t) > \mathbb{I}^-_{\beta_t}(\varphi^{nl}_t)$*, i.e. the percentage of agents viewing negative bubbles increase in more liquid markets, and*

(ii) $\mathbb{I}^+_{\beta_t}(\varphi^l_t) < \mathbb{I}^+_{\beta_t}(\varphi^{nl}_t)$*, i.e. the percentage of agents viewing positive bubbles increase in more illiquid markets.*

Proof. This follows from the following facts. For case (i), if $X^i_{t+1} > 0$, then $\beta^i_t(\varphi^l_t) < \beta^i_t(\varphi^{nl}_t) < 0$. Thus, more agents will see bubbles in liquid markets, everything else constant, because there will be some agents with $\beta^i_t(\varphi^{nl}_t) = 0$ but $\beta^i_t(\varphi^l_t) < 0$. The same argument holds for case (ii). □

This theorem states that when market liquidity declines in a given asset market, the percentage of agents seeing price bubbles declines as well. Consequently, Theorems (10) and (11) have policy implications with respect to the existence of asset price bubbles. This is especially true for monetary policy, which can immediately affect market liquidity. In our model, in equilibrium, bubbles depend on agents' beliefs and preferences. This means that what is relevant for macro-policy is the *percentage of agents in the economy that see bubbles*, and not whether a uniform price bubble exists. This frees policy makers from determining the existence and magnitude of a uniform price bubble. Instead, policy makers can simply monitor the financial markets and statistically infer traders' beliefs concerning bubbles and react accordingly.

While a separate model is needed to fully explore the policy implications we have just discussed, our results provide a framework for analyzes these issues. In the next section of the paper we show a direct relation between the percentage of agents in the economy viewing price bubbles and systemic risk. We now explore the relation between the magnitude of the trading constraints and the size of asset price bubbles.

3.3 Characterizing the Shadow Price of Trading Constraints

This subsection characterizes the shadow price of the trading constraints in equilibrium.

Theorem 12. *(Trading Constraint and Shadow Price)*

Let the trading constraint be binding at time t.

Then, as γ increases,

(i) $\kappa_t^i > 0$ decreases when $X_{t+1}^i > 0$, and

(ii) $\kappa_t^i < 0$ increases in absolute value when $X_{t+1}^i < 0$.

Proof. At time t, the optimal share holdings are either $X_{t+1}^i > 0$ or $X_{t+1}^i < 0$. Since the constraint is binding at the optimum, when $X_{t+1}^i > 0$, $Y_{t+1}^i < 0$ and when $X_{t+1}^i < 0$, $Y_{t+1}^i > 0$. This follows from our trading constraint set $K_t(\omega)$ as discussed before. Consequently, at time t, only one side of the constraint

$$K_t(\omega) = \left\{X_{t+1} \in \mathbb{R} : \; Y_{t+1} \geq -S_t X_{t+1} \left(\gamma 1_{X_{t+1} \geq 0} + (1+\gamma) 1_{X_{t+1} \leq 0}\right)\right\}$$

is binding for each agent i. Thus, in the time t optimization problem, the original constraint can be replaced by the constraint

$$K_t(\omega) = \{X_{t+1} \in \mathbb{R}_{++} : -\gamma S_t X_{t+1} \leq Y_{t+1}\} = \left\{X_{t+1} \in \mathbb{R}_{++} : -\frac{Y_{t+1}}{S_t X_{t+1}} \leq \gamma\right\}$$

for $X_{t+1} > 0$, and

$$K_t(\omega) = \{X_{t+1} \in \mathbb{R}_{--} : -(1+\gamma) S_t X_{t+1} \leq Y_{t+1}\} = \left\{X_{t+1} \in \mathbb{R}_{--} : \frac{Y_{t+1}}{S_t X_{t+1}} \leq -(1+\gamma)\right\}$$

for $X_{t+1} < 0$ without changing the optimal solution.

Note that $Y_{t+1} = Y_t - \varphi_t(X_{t+1} - X_t) S_t$ from expression (2).

Define the function $f(X_{t+1}) = -\left(\frac{Y_t - \varphi_t(X_{t+1} - X_t) S_t}{S_t X_{t+1}}\right)$ on $X_{t+1} \in \mathbb{R}$.

We note that

$f'(X_{t+1}) = \frac{Y_t - \varphi_t(X_{t+1} - X_t) S_t}{S_t X_{t+1}^2} + \frac{\varphi_t'(X_{t+1} - X_t) S_t}{S_t X_{t+1}} = \frac{Y_{t+1} + \varphi_t'(X_{t+1} - X_t) S_t X_{t+1}}{S_t X_{t+1}^2}$.

We have, $f'(X_{t+1}) > 0$ when $X_{t+1} > 0$.

Indeed, $\varphi_t'(X_{t+1} - X_t) S_t X_{t+1} > \gamma S_t X_{t+1}$ when $X_{t+1} > 0$ because $\varphi_t'(x) > 1 > \gamma$ when $x > 0$.

Thus, $f'(X_{t+1}) = \frac{Y_{t+1} + \varphi_t'(X_{t+1} - X_t) S_t X_{t+1}}{S_t X_{t+1}^2} > \frac{Y_{t+1} + \gamma S_t X_{t+1}}{S_t X_{t+1}^2} \geq 0$ because $Y_{t+1} + \gamma S_t X_{t+1} \geq 0$ when $X_{t+1} > 0$.

We have, $f'(X_{t+1}) > 0$ when $X_{t+1} < 0$.

Indeed, $(1+\gamma)\, S_t X_{t+1} < \varphi_t'(X_{t+1} - X_t) S_t X_{t+1}$ when $X_{t+1} < 0$ because $\varphi_t'(x) < 1 < 1+\gamma$ when $x < 0$.

Thus, $f'(X_{t+1}) = \frac{Y_{t+1} + \varphi_t'(X_{t+1} - X_t) S_t X_{t+1}}{S_t X_{t+1}^2} > \frac{Y_{t+1} + (1+\gamma) S_t X_{t+1}}{S_t X_{t+1}^2} \geq 0$ because $Y_{t+1} + (1+\gamma)\, S_t X_{t+1} \geq 0$ when $X_{t+1} < 0$.

We can rewrite the time t constraint as:

(i) $f(X_{t+1}) \leq \gamma$ when the optimum is $X_{t+1} > 0$, and

(ii) $g(X_{t+1}) \leq \delta$ when the optimum is $X_{t+1} < 0$ with $g(X_{t+1}) = -f(X_{t+1})$ and $\delta = -(1+\gamma)$.

(Case 1) At time t the optimum is $X_{t+1}^i > 0$ with a binding constraint and $f'(X_{t+1}) > 0$.

Looking at the proof of Theorem 4, we see that at time t the optimization problem is

$$\upsilon(\gamma) = \sup_{\{X_{t+1} \in \mathbb{R}\}} E^{\mathbb{P}_i}[U_i(Y_{T+1}(X_{t+1})] \quad \text{subject to} \quad f(X_{t+1}) = \gamma$$

where the constraint is binding. As written, the Lagrangian is

$$\mathscr{L} = E^{\mathbb{P}_i}\left[U_i\left(Y_{T+1}^i(X_{t+1})\right)\right] + \lambda_t\left(f(X_{t+1}) - \gamma\right).$$

The first order (necessary and sufficient) condition is

$$\frac{dE^{\mathbb{P}_i}\left[U_i\left(Y_{T+1}^i(X_{t+1})\right)\right]}{dX_{t+1}} + \lambda_t f'(X_{t+1}) = 0.$$

From the proof of Theorem 4, expression (7), we have that $\kappa_t = \lambda_t f'(X_{t+1})$, i.e. $\lambda_t = \frac{\kappa_t}{f'(X_{t+1})}$.

But, standard results yield $\upsilon'(\gamma) = \lambda_t$, see Holmes [40], p. 39. Hence, $\upsilon'(\gamma) = \lambda_t = \frac{\kappa_t}{f'(X_{t+1})}$. By Lemma 1, $\kappa_t > 0$ for $X_{t+1} > 0$, and $f'(X_{t+1}) > 0$ implies that $\lambda_t > 0$.

Also, it is well known that the value function is concave (in the parameter γ), see Holmes [40], p. 37. Hence, $\upsilon''(\gamma) < 0$, which implies that as γ increases the slope $\upsilon'(\gamma) = \lambda_t = \frac{\kappa_t}{f'(X_{t+1})}$ decreases. Since $f'(X_{t+1}) > 0$, all else equal, this implies that $\kappa_t > 0$ decreases.

(Case 2) At time t the optimum is $X_{t+1}^i < 0$ with a binding constraint and $g'(X_{t+1}) = -f'(X_{t+1}) < 0$.

A similar argument to Case 1 yields that $\lambda_t = \frac{\kappa_t}{g'(X_{t+1})}$. By Lemma 1, $\kappa_t < 0$ for $X_{t+1} < 0$. So, $g'(X_{t+1}) < 0$ implies that $\lambda_t > 0$.

By standard results $v'(\delta) = \lambda_t$, hence $v'(\delta) = \frac{\kappa_t}{g'(X_{t+1})} > 0$. By the concavity of $v(\delta)$, $v''(\delta) < 0$, which implies that as δ increases, the slope $v'(\delta) = \lambda_t = \frac{\kappa_t}{g'(X_{t+1})}$ decreases. Since $g'(X_{t+1}) < 0$, this implies that $\kappa_t < 0$ becomes less negative, i.e. decreases in absolute value. But, $\delta = -(1+\gamma)$ increasing implies that γ decreases. Hence, as γ increases, $\kappa_t < 0$ becomes more negative, i.e. increases in absolute value.

This completes the proof. □

Using expression (8), we get an immediate corollary relating trading constraints to the size of asset price bubbles.

Corollary 13. *(Trading Constraints and Bubble Magnitudes)*

Suppose that the trading constraint is made more restrictive. Then, the absolute value of a bubble's magnitude increases, i.e. $|\beta_t^i|$ *increases.*

Proof. Given $\beta_t^i = \frac{-\kappa_t^i}{E_t^{\mathbb{P}^i}\left[U_i'\left(Y_{T+1}^i\right)\right]\varphi_t'(\Delta X_{t+1}^i)}$, we have that γ increasing (decreasing) implies: (i) $\kappa_t^i > 0$ decreases (increases) when $X_{t+1}^i > 0$, which implies $\beta_t^i < 0$ decreases (increases) in absolute value, and (ii) $\kappa_t^i < 0$ increases in absolute value when $X_{t+1}^i < 0$, which means $\beta_t^i > 0$ increases as there is a negative sign in the above expression.

We note the asymmetry in the increase of γ. In case (i) when $X_{t+1}^i > 0$, γ *decreasing* means the constraint is becoming more binding/restrictive (less borrowing possible). In case (ii) when $X_{t+1}^i < 0$, γ *increasing* means the constraint is becoming more binding/restrictive (less short selling possible). Thus, as γ becomes more binding, in both cases the bubble increases in absolute value. □

This result is subtle and might appear to be counter-intuitive. One might argue that relaxing the trading constraints should lead to more speculative behavior and larger bubbles. However, this argument ignores the reason bubbles exist in our economy. As we have shown, there are no bubbles in an economy when trading constraints are non-binding i.e. $\kappa_t^i = 0$. They only appear when the constraints are binding. Hence, given their beliefs, agents would like to either buy or short sell more stocks to reduce their personalized under- and over-valuation. But, they cannot. The more binding the trading constraints, the larger the personalized under- and over-valuation. This is consistent with the fact that bubbles appear when either the short selling or borrowing constraints are binding.

4 Systemic Risk

This section studies the relation between asset price bubbles, market illiquidity, trading constraints, and systemic risk. We will show that systemic risk is the risk of market failure resulting from funding risk, which is the conjunction of increased market illiquidity (liquidity risk) due to widespread selling pressure (like a fire-sale), leading to binding trading constraints that generate massive defaults across agents. As implied, funding risk is closely related to the notions of default, an accelerator effect, and the percentage of agents in an economy that see bubbles. To see all of these relationships, we first need to define an agent's wealth.

Definition 14. *(Wealth)*

Let W_t^i denote the wealth (or net-worth of a *self-financing* trading strategy) of an individual agent of type i, defined as

$$W_t^i = X_{t+1}^i S_t + Y_{t+1}^i$$

for all times $t \in \{0, 1, ..., T-1\}$.

Analogous to the definition of the trading constraint, wealth is defined using the marked-to-market value of the risky assets. This makes sense because the transaction price requires keeping track of the share purchases at any time t and the liquidity cost paid $\varphi_t'(x)$. In addition, defining wealth using market-to-market price implies that an agent's wealth is always positive, as the following lemma shows.

Lemma 15. *(Wealth is Always Nonnegative)*

If $X_{t+1}^i > 0$ and $\beta_t^i < 0$, then $Y_{t+1}^i = -\gamma S_t X_{t+1}^i$ and $W_t^i = (1-\gamma) X_{t+1}^i S_t > 0$.

If $X_{t+1}^i < 0$ and $\beta_t^i > 0$, then $Y_{t+1}^i = -(1+\gamma) S_t X_{t+1}^i$ and $W_t^i = -\gamma X_{t+1}^i S_t > 0$.

Intuitively, systemic risk is the risk of market failure in an economy due to an exogenous and unanticipated shock that causes massive defaults. In our setting, we define a market failure as the nonexistence of an equilibrium. The shock could be due to catastrophic events or changes in monetary policy, fiscal policy, regulatory policy (trading constraints), or changes to agents' beliefs and preferences. Thus, in our setting market failure results from an exogenous and unanticipated shock that causes the non-existence of an economic equilibrium due to the inability of some agents to satisfy their trading constraints, thereby defaulting with a corresponding loss in wealth. As characterized, systemic risk is the result of a funding illiquidity (funding risk), which is the conjunction of market illiquidity and binding trading constraints that lead to defaults across agents.

It is important to note that the exogenous shock must be unanticipated and not included in the original specification of the economy. Indeed, if the shock is anticipated and included into the economy's structure, then because our equilibrium is a Radner equilibrium, it holds for all times t a.e. $\mathbb{P}$. Hence, the economy is always in equilibrium and there is never any market failure. Anticipated but random shocks only increase the volatility/risk of consumption and equilibrium prices. In equilibrium, agents' trading constraints are never violated, and there is no default (wealth stays positive). This is a direct implication of the existence of an agent's optimal portfolio subject to the trading constraints. We show below that for systemic risk effects, the price shocks do not have to be large. The aggregate effect of small price shocks can also lead to the realization of funding illiquidity and market failure, leading to a significant wealth loss in the economy.

We use a reduced form approach to represent an exogenous shock to the economy. Rather than explore an exogenous shock to the fundamentals of the economy (beliefs, preferences, endowments, trading constraints) and explore its implications on economic equilibrium prices and consumption, we analyze the implications of an exogenous shock on the price process itself. This reduced form approach simplifies the analytics because it enables us to determine, in a competitive economy where all agents are price takers, whether an optimal portfolio still exists after the shock, given the trading constraints. Alternatively stated, it enables us to determine if such an exogenous shock to prices causes any agents to default, i.e. violate the trading constraints, which corresponds to too much debt or an inability to meet short sale margin calls. Default for an agent type, which represents a positive mass in the economy $\mathbb{I}^-_{\beta_t}$ or $\mathbb{I}^+_{\beta_t}$, implies market failure because it results in the non-existence of an economic equilibrium.

Given this discussion, we analyze an unanticipated random shock to the price process equal to ΔS at time t, which could be a large or small, positive or negative change from S_t to $S_t + \Delta S$. The next theorem characterizes the conditions under which such a price shock results in the non-existence of an economic equilibrium and market failure, which is an *instance* of systemic risk materializing into systemic crisis.

Theorem 16. *(Margin Calls, Market Failure, Wealth Loss)*

(Borrowing Constrained)
If $X_{t+1}^i > 0$ *and* $\beta_t^i < 0$.

Let the price shock be $\Delta S < 0$ *at time t.*

Then, the minimum sale $\Delta X^i < 0$ *necessary to satisfy the borrowing constraint satisfies*

$$\left(\varphi_t(\Delta X^i) - \gamma \Delta X^i\right) = \gamma \frac{\Delta S}{S_t + \Delta S} X_{t+1}^i.$$

A feasible $\Delta X^i < 0$ *exists (staying in equilibrium) if and only if* $\varphi_t(\Delta X^i) < \gamma \Delta X^i$.

Market failure occurs otherwise with a loss of wealth.

(Short Sale Constrained)
If $X_{t+1}^i < 0$ *and* $\beta_t^i > 0$.

Let the price shock be $\Delta S > 0$ *at time t.*

Then, the minimum purchase $\Delta X^i > 0$ *necessary to satisfy the margin constraint satisfies*

$$\left(\varphi_t(\Delta X^i) - (1+\gamma)\,\Delta X^i\right) = (1+\gamma)\,\frac{\Delta S}{S_t + \Delta S} X_{t+1}^i.$$

A feasible $\Delta X^i > 0$ *exists (staying in equilibrium) if and only if* $\varphi_t(\Delta X^i) < (1+\gamma)\,\Delta X^i$.

Market failure occurs otherwise with a loss of wealth.

Proof. Two cases to consider.

(Case 1) $X_{t+1}^i > 0$ and $\beta_t^i < 0$ with $\Delta S < 0$ at time t.

Before the shock borrowings are $Y_{t+1}^i = -\gamma S_t X_{t+1}^i < 0$.

After the shock, the maximum borrowings are $-\gamma\,(S_t + \Delta S)\,X_{t+1}^i > -\gamma S_t X_{t+1}^i$. Hence, the constraint is violated and the change in wealth is negative.

Indeed,
$\left(X_{t+1}^i S_t - \gamma S_t X_{t+1}^i\right) - \left(X_{t+1}^i S_t - \gamma\,(S_t + \Delta S)\,X_{t+1}^i\right) = \gamma X_{t+1}^i \Delta S < 0.$

This implies, borrowers must sell shares $\Delta X^i < 0$ to obtain cash to reduce borrowings so that the constraint is not violated. Next, we determine the minimum shares to sell to stay on the constraint.

After selling shares, the constraint is $-\gamma\,(S_t + \Delta S)\left(X_{t+1}^i + \Delta X^i\right) < 0$.

The cash needed is $-\gamma\,(S_t + \Delta S)\left(X_{t+1}^i + \Delta X^i\right) + \gamma S_t X_{t+1}^i > 0$.

From the liquidity cost of trading, the cash obtained from selling shares is
$-\varphi_t(\Delta X^i)\,(S_t + \Delta S) > 0$.

From above cash needed/obtained conditions, the solution is ΔX^i such that

$\gamma\left(S_t+\Delta S\right)\left(X_{t+1}^i+\Delta X^i\right)-\gamma S_t X_{t+1}^i=\varphi_t(\Delta X^i)\left(S_t+\Delta S\right).$
$\gamma\left(S_t+\Delta S\right)X_{t+1}^i+\gamma\left(S_t+\Delta S\right)\Delta X^i-\gamma S_t X_{t+1}^i=\varphi_t(\Delta X^i)\left(S_t+\Delta S\right).$
$\gamma\Delta S X_{t+1}^i+\gamma\left(S_t+\Delta S\right)\Delta X^i=\varphi_t(\Delta X^i)\left(S_t+\Delta S\right).$
$\gamma\frac{\Delta S}{S_t+\Delta S}X_{t+1}^i+\gamma\Delta X^i=\varphi_t(\Delta X^i).$
$\gamma\frac{\Delta S}{S_t+\Delta S}X_{t+1}^i=\left(\varphi_t(\Delta X^i)-\gamma\Delta X^i\right).$
The left side is negative as $\Delta S<0$ and $X_{t+1}^i>0$.
A solution exists with $\Delta X^i<0$ if and only if $\varphi_t(\Delta X^i)-\gamma\Delta X^i<0$.
(Case 2) $X_{t+1}^i<0$ and $\beta_t^i>0$ with $\Delta S>0$ at time t.
Before the shock the margin is $Y_{t+1}^i=-\left(1+\gamma\right)S_t X_{t+1}^i>0$.

After the shock, the margin is $-\left(1+\gamma\right)\left(S_t+\Delta S\right)X_{t+1}^i>-\left(1+\gamma\right)S_t X_{t+1}^i$. Hence, the constraint is violated and the change in wealth is negative as in case 1 above.

This implies, short sellers must buy shares $\Delta X^i>0$ to reduce the short position so that the margin constraint is not violated. Next, we determine the minimum shares to buy to stay on the constraint.

After buying shares, the constraint is $-\left(1+\gamma\right)\left(S_t+\Delta S\right)\left(X_{t+1}^i+\Delta X^i\right)>0$.
The cash reduction in the margin from purchase is
$\left(1+\gamma\right)\left(S_t+\Delta S\right)\left(X_{t+1}^i+\Delta X^i\right)-\left(1+\gamma\right)S_t X_{t+1}^i>0.$
The cash needed to buy shares is $\varphi_t(\Delta X^i)\left(S_t+\Delta S\right)>0$.
From above cash reduction/needed conditions, the solution is ΔX^i such that
$\left(1+\gamma\right)\left(S_t+\Delta S\right)\left(X_{t+1}^i+\Delta X^i\right)-\left(1+\gamma\right)S_t X_{t+1}^i=\varphi_t(\Delta X^i)\left(S_t+\Delta S\right).$
$\left(1+\gamma\right)\left(S_t+\Delta S\right)X_{t+1}^i+\left(1+\gamma\right)\left(S_t+\Delta S\right)\Delta X^i-\left(1+\gamma\right)S_t X_{t+1}^i$
$=\varphi_t(\Delta X^i)\left(S_t+\Delta S\right).$
$\left(1+\gamma\right)\Delta S X_{t+1}^i+\left(1+\gamma\right)\left(S_t+\Delta S\right)\Delta X^i=\varphi_t(\Delta X^i)\left(S_t+\Delta S\right).$
$\left(1+\gamma\right)\frac{\Delta S}{S_t+\Delta S}X_{t+1}^i+\left(1+\gamma\right)\Delta X^i=\varphi_t(\Delta X^i).$
$\left(1+\gamma\right)\frac{\Delta S}{S_t+\Delta S}X_{t+1}^i=\left(\varphi_t(\Delta X^i)-\left(1+\gamma\right)\Delta X^i\right).$
The left side is negative as $\Delta S>0$ and $X_{t+1}^i<0$.
A solution exists with $\Delta X^i>0$ if and only if $\varphi_t(\Delta X^i)-\left(1+\gamma\right)\Delta X^i<0$. □

Before discussing the theorem, we note that the price shock ΔS considered for both the borrowing and short sale constrained agents results in a loss of wealth, independent of whether or not market failure occurs. As shown below, the magnitude of the wealth loss is larger when market failure occurs.

This theorem characterizes the conditions under which a price shock of size ΔS results in the inability of an agent to satisfy her trading constraint, resulting in default and market failure. There are two such possibilities. The first situation occurs when the

i^{th} agent is borrowing to buy the stock, is borrowing constrained, and therefore believes the stock is undervalued ($X_{t+1}^i > 0$ and $\beta_t^i < 0$). Default and market failure occur if the liquidity cost of selling the stocks $\Delta X^i < 0$ to satisfy the constraint does not generate enough cash. This occurs when $0 > \varphi_t(\Delta X^i) > \gamma \Delta X^i$. This is the standard mechanism often discussed intuitively in the literature (see Brunnermeier and Pedersen (2008)[14] and Brunnermeier et al. (2013)[16]). The second situation is largely unexplored in the macroeconomics literature. It occurs when the i^{th} agent is short selling the stock, is margin constrained, and therefore believes the stock is overvalued ($X_{t+1}^i < 0$ and $\beta_t^i > 0$). Default and market failure occur if the liquidity cost of buying back the shorted stocks $\Delta X^i > 0$ to satisfy the constraint is too costly. This occurs when $\varphi_t(\Delta X^i) > (1+\gamma)\,\Delta X^i > 0$. In both cases, when funding illiquidity (market illiquidity and binding trading constraints) causes an agent's default, market failure occurs and there is a significant loss of wealth in the economy.

Market failure occurs in both of these situations because of funding illiquidity. Indeed, the i^{th} agent cannot sell/buy enough stock to satisfy the trading constraints due to the liquidity costs of executing trades. In practice, both the stock and borrowing positions would be in the same brokerage account. If after the price shock an insufficient quantity of stock/mma exists in the portfolio to satisfy the trading constraint, then the entire brokerage account is necessarily liquidated by the agent's broker. Under this assumption, it is easy to show that under some mild conditions the i^{th} agent's wealth becomes negative, which implies bankruptcy.

Corollary 17. *(Market Failure Implies Bankruptcy)*

Let market failure occur.

(Borrowing Constrained)

Let $X_{t+1}^i > 0$, $\beta_t^i < 0$, and $\Delta S < 0$ at time t.

If $\varphi_t(-X_{t+1}^i)\,[S_t + \Delta S] > \gamma(-X_{t+1}^i) S_t$, then the i^{th} agent is bankrupt.

(Short Sale Constrained)

Let $X_{t+1}^i < 0$, $\beta_t^i > 0$, and $\Delta S > 0$ at time t.

If $Z < -X_{t+1}^i$ where $Z > 0$ is the solution to $\varphi_t(Z)\,[S_t + \Delta S] = -\,(1+\gamma)\,S_t X_{t+1}^i$, then the i^{th} agent is bankrupt.

Proof. (Step 1)

Consider the first case where the agent is borrowing to buy the stock. Here, before the shock, the agent's wealth is

$$W_t^i = S_t X_{t+1}^i + Y_{t+1}^i = (1-\gamma)\, S_t X_{t+1}^i > 0$$

because $Y_{t+1}^i = -\gamma S_t X_{t+1}^i < 0$. After the price shock of $\Delta S < 0$, the agent cannot satisfy the trading constraint. Given that the entire stock position is liquidated ($-X_{t+1}^i < 0$), the wealth after liquidation is

$$W_t^i(after) = -\varphi_t(-X_{t+1}^i)\, [S_t + \Delta S] + Y_{t+1}^i$$

$$= -\varphi_t(-X_{t+1}^i)\, [S_t + \Delta S] - \gamma S_t X_{t+1}^i.$$

Note here that the position in the mma is fixed. This new wealth is negative if $\varphi_t(-X_{t+1}^i)\, [S_t + \Delta S] > -\gamma S_t X_{t+1}^i$.

(Step 2)

Consider the second case where the agent has a margin account to short the stock. Here, before the shock, the agent's wealth is

$$W_t^i = S_t X_{t+1}^i + Y_{t+1}^i = -\gamma S_t X_{t+1}^i > 0$$

because $Y_{t+1}^i = -(1+\gamma)\, S_t X_{t+1}^i > 0$. After the price shock of $\Delta S > 0$, the agent cannot satisfy the trading constraint. Given the entire mma position is liquidated to buy stock to cover the short position, the wealth after buying back stock is

$$W_t^i(after) = [S_t + \Delta S]\, (X_{t+1}^i + Z)$$

where $Z > 0$ is the solution to $\varphi_t(Z)\, [S_t + \Delta S] = -(1+\gamma)\, S_t X_{t+1}^i$. Note in this case the position in the shorted shocks is fixed. This new wealth is negative if $Z < -X_{t+1}^i$. This completes the proof. □

The two sufficient conditions for bankruptcy are quite mild and similar to the market failure conditions $\varphi_t(\Delta X^i) > \gamma \Delta X^i$ for $\Delta X^i < 0$ when the agent is borrowing constrained and $\varphi_t(\Delta X^i) > (1+\gamma)\, \Delta X^i$ for $\Delta X^i > 0$ when the agent is short sale constrained where ΔX^i correspond to the solutions to the respective equations in Theorem 16.

Besides characterizing the conditions under which default and market failure occur, the previous theorem also provides some insights into the impact of the shock, if equilibrium still exists and a market failure does not occur. This implication is highlighted in the following corollary.

Corollary 18. *(An Accelerator Effect and Augmented Loss of Wealth)*

Assuming the minimum purchase necessary to satisfy the trading constraints is feasible in Theorem (16) i.e. staying in equilibrium, then the minimum sale or purchase ΔX^i necessary to satisfy the trading i.e. borrowing or short sale constraint exceeds that in an economy with no liquidity costs. Consequently, the loss of wealth is larger in an economy with liquidity costs.

(Borrowing Constrained)

Let $X^i_{t+1} > 0$, $\beta^i_t < 0$, and $\Delta S < 0$ at time t.

The wealth loss is for $\Delta X^i > 0$,

$$(1-\gamma)\, S_t X^i_{t+1} - (1-\gamma)\,[S_t + \Delta S_t]\left[X^i_{t+1} + \Delta X^i\right] > 0.$$

(Short Sale Constrained)

Let $X^i_{t+1} < 0$, $\beta^i_t > 0$, and$\Delta S > 0$ at time t.

The wealth loss is for $\Delta X^i < 0$,

$$-\gamma S_t X^i_{t+1} + \gamma\,[S_t + \Delta S_t]\left[X^i_{t+1} + \Delta X^i\right] > 0.$$

Proof. (Step 1)

If $X^i_{t+1} > 0$ and $\beta^i_t < 0$. Let the price shock be $\Delta S < 0$ at time t. Then, by the previous Theorem (16) the minimum sale $\Delta X^i < 0$, necessary to satisfy the borrowing constraint satisfies $\Delta X^i = \frac{1}{\left(\frac{\varphi_t(\Delta X^i)}{\Delta X^i} - \gamma\right)} \gamma \frac{\Delta S}{S_t + \Delta S} X^i_{t+1} > \frac{1}{(1-\gamma)} \gamma \frac{\Delta S}{S_t + \Delta S} X^i_{t+1}$. The right side of this inequality is the trade size in a market with no illiquidities where $\varphi_t(\Delta X^i) = \Delta X^i$.

(Step 2)

If $X^i_{t+1} < 0$ and $\beta^i_t > 0$. Let the price shock be $\Delta S > 0$. Then, by the previous theorem the minimum purchase $\Delta X^i > 0$ necessary to satisfy the margin constraint satisfies $\Delta X^i = \frac{1}{\left(\frac{\varphi_t(\Delta X^i)}{\Delta X^i} - (1+\gamma)\right)} (1+\gamma) \frac{\Delta S}{S_t + \Delta S} X^i_{t+1} > \frac{1}{\gamma} (1+\gamma) \frac{\Delta S}{S_t + \Delta S} X^i_{t+1}$. The right side of this inequality is the trade size in a market with no illiquidities where $\varphi_t(\Delta X^i) = \Delta X^i$.

In both steps above, the loss in wealth is larger than in the economy with liquidity costs.

(Step 3)

The change is wealth is computed by recognizing that the constraint is satisfied both before and after the trades. □

This corollary (18) can be explained as follows. After a price shock, a sale/purchase is needed to satisfy the trading constraints. If such a sale/purchase exists that is feasible and an equilibrium still exists, then the size of the sale/purchase exceeds that which would occur in a market with no illiquidities. This is intuitive because if there are liquidity costs to trading, then these liquidity costs must be subtracted from the proceeds, and the size of the trade must be larger to account for these costs. As such, this implies that the aggregate trades (aggregated by the percentage of agents that trade) will affect equilibrium prices more in an economy with liquidity costs than in an economy without liquidity costs. This increased trading will result in a secondary price change (of the same sign) that augments the original price shock of ΔS. This in turn will lead to another necessary sale/purchase to satisfy the binding trading constraints, leading to another subsequent price change to the second price shock, and so forth until the sequence either converges to a stable and feasible price change or default and market failure occurs. Consequently, the loss of wealth is larger in an economy with liquidity costs. This is the accelerator effect of funding illiquidity due to an exogenous price shock in the economy. This accelerator effect is analogous to that in Brunnermeier and Pedersen (2008)[14] and Brunnermeier and Sannikov (2014)[17].

We can quantify the probability of massive agent defaults and market failure, or systemic risk, in our economy due to an exogenous price shock of size ΔS.

Definition 19. *(Systemic Risk)*

The probability of market failure at time t for a random shock of size ΔS is

$$\begin{aligned}\mathbb{P}_{fail} = \mathbb{P}\{\omega \in \Omega : \exists\, i \; where \; &\left(\beta_t^i < 0, Y_{t+1}^i = -\gamma S_t X_{t+1}^i < 0, \Delta X^i < 0, \varphi_t(\Delta X^i) > \gamma \Delta X^i\right) \\ or \; &\left(\beta_t^i > 0, Y_{t+1}^i = -\left(1+\gamma\right) S_t X_{t+1}^i > 0, \Delta X^i > 0, \varphi_t(\Delta X^i) > (1+\gamma)\Delta X^i\right)\Big\}\end{aligned} \tag{11}$$

where ΔX^i is the solution to

$\varphi_t(\Delta X^i) - \gamma \Delta X^i = \gamma \frac{\Delta S}{S_t + \Delta S} X_{t+1}^i$ for $\Delta S<0$ and

$\varphi_t(\Delta X^i) - \left(1+\gamma\right) \Delta X^i = \left(1+\gamma\right) \frac{\Delta S}{S_t + \Delta S} X_{t+1}^i$ for $\Delta S>0$.

Note that this definition implicitly takes into account the probability of wealth loss as disequilibrium necessarily entails some agents' inability to satisfy the trading constraint, resulting in default and a loss of wealth (see Theorem 16). Thus, this definition is seen to be equivalent to the probability of a significant loss of wealth in the economy due to massive agent defaults and consequent market failure. By direct inspection of expression (11), the following theorem follows.

Theorem 20. *(Bubbles, Liquidity, Constraints, and Systemic Risk)*

(i) As $\mathbb{I}^+_{\beta_t}$ or $\mathbb{I}^-_{\beta_t}$ increase, all else constant, $\mathbb{P}_{fail}$ increases.

(ii) As the market becomes more illiquid, all else constant, $\mathbb{P}_{fail}$ increases.

(iii) As the constraints become more restrictive, all else constant, $\mathbb{P}_{fail}$ decreases.

Proof. By expression (11), we have three cases.

(i) As $\mathbb{I}_{\beta_t}$ increases, $|\beta^i_t|$ is increasing, which makes $\beta^i_t < 0$ more likely for $\Delta X^i < 0$ and it makes $\beta^i_t > 0$ more likely for $\Delta X^i > 0$. This gives the result.

(ii) As markets become more illiquid, $\varphi_t(\Delta X^i)$ increases for all $\Delta X^i \neq 0$. See Lemma (9). This makes $\varphi_t(\Delta X^i) > \gamma \Delta X^i$ more likely for $\Delta X^i < 0$ and it makes $\varphi_t(\Delta X^i) > (1+\gamma)\Delta X^i$ more likely for $\Delta X^i > 0$, due to monotonicity of probability measure. This gives the desired result.

(iii) As the constraints become more binding, this means that γ decreases for $X^i_{t+1} > 0$ and that $1+\gamma$ increases for $X^i_{t+1} < 0$. This makes $0 > \varphi_t(\Delta X^i) > \gamma \Delta X^i$ less likely for $\Delta X^i < 0$ and it makes $\varphi_t(\Delta X^i) > (1+\gamma)\Delta X^i > 0$ less likely for $\Delta X^i > 0$, due to monotonicity of probability measure. This gives the desired result. □

This theorem shows the effects that changing fundamentals have on systemic risk. It states that if the percentage of agents that believe bubbles exist increases or if the market becomes more illiquid or if trading constraints are relaxed, then systemic risk increases. We note that using Corollary (13), a secondary effect of making the trading constraints more restrictive is that for any agent, the absolute value of a bubble's magnitude increases, i.e. $|\beta^i_t|$ increases. This means that the percentage of agents seeing bubbles, $\mathbb{I}^+_{\beta_t}$ and $\mathbb{I}^-_{\beta_t}$, increases as well. Then, by implication (i) above, $\mathbb{P}_{fail}$ increases. This secondary effect, which involves staying in equilibrium, may or may not dominate the primary implication in (iii) above, generated under the market failure condition described in Theorem (16) where equilibrium does not exist, assuming all else constant, including the size of the asset price bubble.

All of these implications are intuitive and consistent with recent market experience during the financial crisis of 2007-09. U.S. households were highly levered during the pre-crisis periods and entered the crisis with huge amounts of debt. Indeed, U.S. household gross debt as a percent of personal income was 96% in 2000 and 128% in 2008.

Then the shock emanating from the financial sector eventually lead to the collapse of housing bubbles, stock market crash and consequently reduced borrowing limits in the economy drastically. It further lead to massive and rapid de-leveraging in the economy accompanied by a significant loss of wealth, see Eggertsson and Krugman (2012) [28] for further discussion on this debt and deleveraging mechanism. In our analysis of systemic risk, we have shown multiple effects of similar deleveraging mechanisms in a much more general set up through market failure and accelerator effects due to borrowing and short sale constraints.

5 Conclusion

This paper provides an equilibrium model with heterogeneous agents and trading constraints to study asset price bubbles, market illiquidities, and systemic risk. Systemic risk is defined as the risk of market failure due to an exogenous shock to the economy. This results in funding illiquidity, which is the conjunction of market illiquidity (i.e. liquidity risk) and binding trading constraints. To do this, we introduce a different framework for analyzing asset price bubbles and how they affect the macroeconomy. In our framework, asset price bubbles arise endogenously in a rational equilibrium due to the heterogeneous beliefs, heterogeneous preferences, and binding trading constraints. We show that: (i) positive price bubbles are larger in more illiquid markets, (ii) the percentage of agents in the economy who believe a positive price bubble exists decreases as a market becomes more liquid, (iii) a bubble's magnitude increases when trading constraints are more restrictive, and (iv) systemic risk increases as the percentage of agents seeing bubbles increases or as the market becomes more illiquid. The realization of systemic risk results in a large fraction of agents violating their trading constraints thereby defaulting, the non-existence of an equilibrium, and a large loss of wealth in the economy.

Our results also have policy implications because market liquidity, trading constraints and asset price bubbles affect systemic risk. We show that improved market liquidity decreases systemic risk, and it is well known that monetary and other regulatory policies can affect market liquidity. We also show that as the percentage of agents viewing price bubbles increase, systemic risk increases. Regulators can directly influence agents' beliefs with appropriate policy actions. To influence the impact of bubbles on systemic risk, policy makers should focus on determining the *percentage of* *agents that see bubble*s, and *not* whether a uniform price bubble exists, which is a difficult if not an impossible task. Alternatively stated, the percentage of agents that see price bubbles matters because this is the most relevant channel through which policy with respect to bubbles works, and not via the overall level of market prices. This is important because the realization of systemic risk results in a significant fraction of agents defaulting with a corresponding loss of wealth in the economy. Lastly, we show that as trading constraints become more binding, systemic risk declines. Regulators can certainly modify trading constraints. More work still needs to be done to fully understanding how asset price bubbles, trading constraints, and market liquidities are influenced by regulatory policies. In particular, it would be useful to introduce monetary policy directly into the model and explore its equilibrium implications. This is an open and fruitful future research area.

References

[1] Acharya V. and L. Pedersen, 2005, "Asset Pricing with Liquidity Risk," *Journal of Financial Economics*, 77, 375 - 410.

[2] Acharya, V., L. Pedersen, T. Philippon and M. Richardson. 2010, "Measuring Systemic Risk." Working Paper, New York University.

[3] Achdou, Y., J. Han, J. Lasry, P. Lions and B. Moll. 2015, "Heterogeneous Agent Models in Continuous Time." Princeton University Working Papers.

[4] Adrian, T. and M. Brunnermeier. 2014, "CoVaR." Federal Reserve Bank of New York Staff Reports, no. 348.

[5] Aiyagari, S.R., 1994. "Uninsured Idiosyncratic Risk and Aggregate Saving." *The Quarterly Journal of Economics*, 109(3): 659-84.

[6] Benhabib, J., A. Bisin and S. Zhu, 2011, "The Distribution of Wealth and Fiscal Policy in Economies With Finitely Lived Agents." *Econometrica*, 79(1): 123-157.

[7] Benhabib, J., A. Bisin and S. Zhu, 2011, "The Wealth Distribution in Bewley Models with Investment Risk." NYU Working Papers.

[8] Bernanke, B. and M. Gertler. 1989, 'Agency Costs, Net Worth, and Business Fluctuations." *American Economic Review*, 79 (1): 14-31.

[9] Bernanke, B., M. Gertler and S. Gilchrist. 1999, "The Financial Accelerator in a Quantitative Business Cycle Framework." *Handbook of Macroeconomics.* Vol. 1, edited by John B. Taylor and Michael Woodford, 1341-93. Amsterdam: Elsevier Science, North-Holland.

[10] Billio, M., M. Getmansky, A.W. Lo, and L. Pelizzon. 2010, "Measuring Systemic Risk in the Finance and Insurance Sectors." MIT Sloan School Working Paper 4774.

[11] Bisias, M. Flood, A.W. Lo, S. Valavanis. 2012, "A Survey of Systemic Risk Analytics," Office of Financial Research, Working Paper 0001, January, 2012.

[12] Brownlees, C. and R. Engle. 2016, "SRISK: A Conditional Capital Shortfall Measure of Systemic Risk." Working Paper, New York University.

[13] Brunnermeier, M., and Nagel, S., 2004. "Hedge funds and the technology bubble," *Journal of Finance*, 59 (5),2013-40.

[14] Brunnermeier, M. and L. Pedersen. 2008. "Market Liquidity and Funding Liquidity," *Review of Financial Studies*, 22(60), 2201-2238.

[15] Brunnermeier, M. and M. Oehmke. 2012. "Bubbles, Financial Crises, and Systemic Risk," NBER Working Paper No. 18398.

[16] Brunnermeier, M.K., T.M. Eisenbach, and Y. Sannikov. 2013. "Macroeconomics with Financial Frictions: A Survey." In *Advances in Economics and Econometrics: Tenth World Congress of the Econometric Society.* Vol. 2, edited by Daron Acemoglu, Manuel Arellano and Eddie Dekel, 3-94. New York: Cambridge University Press.

[17] Brunnermeier, M., and Y. Sannikov. 2014. "A Macroeconomic Model with a Financial Sector. *American Economic Review*, 104(2): 379-421.

[18] United States Census Bureau, *New Residential Sales*, 22 Sep. 2017. URL: https://www.census.gov/construction/nrs/index.html

[19] U. Cetin, R. Jarrow, and P. Protter, 2004, "Liquidity Risk and Arbitrage Pricing Theory," *Finance and Stochastics*, 8, 311 - 341.

[20] U. Cetin and L. C. B. Rogers, 2007, "Modeling Liquidity Effects in Discrete Time," *Mathematical Finance*, 17 (1), 15 - 29.

[21] S. Chjebbi and H. Soner, 2013, "Merton Problem in a Discrete Market with Frictions," *Nonlinear -Analysis: Real World Applications*, 14, 179 - 187.

[22] Christiano, L., M. Eichenbaum and C. Evans, 2005, "Nominal rigidities and the dynamic effects of a shock to monetary policy," *Journal of Political Economy*, 113(1), 1-45.

[23] Christiano, L., M. Trabandt and K. Walentin, 2010. "DSGE Models for Monetary Policy Analysis," NBER Working Paper No. 16074.

[24] Clark S., and Coggin, D., 2011. "Was there a US house price bubble? An econometric analysis using national and regional panel data," *Quarterly Review of Economics and Finance*, 51, 189-200.

[25] Cuoco, C. and H. He. 1994. "Dynamic Equilibrium in Infinite-Dimensional Economies with Incomplete Financial Markets," working paper, Wharton.

[26] Cuoco, C. and H. He, 2001, "Dynamic Aggregation and Computation of Equilibria in Finite-Dimensional Economies with Incomplete Financial Markets," *Annals of Economics and Finance*, 2, 265 - 296.

[27] Diamond, D., and P. Dybvig (1983), "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy*, 91, 401-419.

[28] Eggertsson, G. and P. Krugman, 2012. "Debt, Deleveraging, and The Liquidity Trap: A Fisher-Minsky-Koo Approach," *The Quarterly Journal of Economics*, 1469-1513.

[29] Board of Governors of the Federal Reserve System, *Financial Accounts of the United States - Z.1*, 25 Sep. 2017. URL: https://www.federalreserve.gov/releases/z1/.

[30] Gabaix, X., J.M. Lasry, P.L. Lions, and B. Moll. 2016. "The Dynamics of Inequality." *Econometrica*, 84(6): 2071-2111.

[31] Garber, P., 1989. "Tulipmania," *Journal of Political Economy*, 97 (3), 535-60.

[32] Garber, P., 1990. "Famous first bubbles," *Journal of Economic Perspective*, 4 (2), 35-54.

[33] Guerrieri, L. and M. Iacoviello, 2017. "Collateral constraints and macroeconomic asymmetries," *Journal of Monetary Economics*, 90, 28-49.

[34] Hall, R., "The Long Slump," *American Economic Review*, 101: 431-469.

[35] Harrison, M. and D. Kreps, 1978. "Speculative investor behavior in a stock-market with heterogeneous expectations," *Quarterly Journal of Economics*, 92, 323-336.

[36] Hayashi, F., 1982, "Tobin's marginal q and average q: A neoclassical interpretation," *Econometrica*, 50(1), 213-224.

[37] He, Z., and A. Krishnamurthy. 2012, "A Model of Capital and Crises," *Review of Economic Studies*, 79 (2): 735-77.

[38] He, Z., and A. Krishnamurthy. 2013, "Intermediary Asset Pricing," *American Economic Review*, 103 (2): 732-70.

[39] He, Z., and A. Krishnamurthy. 2014, "A Macroeconomic Framework for Quantifying Systemic Risk," National Bureau of Economic Research, NBER Working Papers 19885.

[40] R. Holmes, 1970, A *Course on Optimization and Best Approximation*, Lecture Notes in Mathematics 257, Springer.

[41] Huggett, M., 1993. "The risk-free rate in heterogeneous-agent incomplete-insurance economies." *Journal of Economic Dynamics and Control*, 17(5-6):953-969.

[42] International Monetary Fund. 2009. *Global Financial Stability Report: Responding to the Financial Crisis and Measuring Systemic Risk.* IMF, Washington, DC.

[43] R. Jarrow and P. Protter, 2008, "Liquidity Risk and Option Pricing Theory," *Handbooks in OR&MS, vol. 15*, eds J. R. Birge and V. Linetsky, Elsevier B. V.

[44] Jarrow, R., 2015. "Asset Price Bubbles." *Annual Review of Financial Economics*, 07: 201-18.

[45] Jarrow, R. and M. Larsson, 2015, "On Aggregation and Representative Agent Equilibria," forthcoming, *Journal of Mathematical Economics.*

[46] Jarrow, R., 2016. "On the Existence of Competitive Equilibrium in Frictionless and Incomplete Stochastic Asset Markets, " *Mathematics and Financial Economics,* 11 (4), 455 - 477.

[47] R. Jarrow, 2016, "Asset Market Equilibrium with Liquidity Risk," forthcoming, *Annals of Finance.*

[48] Jarrow, R., 2017a. *Continuous Time Asset Pricing Theory: A Martingale Based Approach*, Manuscript, Cornell University.

[49] Jarrow, R., 2017b. "Capital Asset Market Equilibrium with Liquidity Risk, Trading Constraints, and Asset Price Bubbles," Working Paper, Cornell University.

[50] Kashakari, N., 2017, "Monetary Policy and Bubbles," URL: https://www.minneapolisfed.org/news-and-events/messages/monetary-policy-and-bubbles.

[51] Kiyotaki, N., and J. Moore. 1997, "Credit Cycles," *Journal of Political Economy*, 105 (2): 211-48.

[52] Lamichhane, S. 2017, "Systemic Risk and Financial Innovation: A Macro-Finance Perspective", Working Paper, Cornell University.

[53] Nikolau, K. 2009. "Liquidity (Risk) Concepts: Definitions and Interactions", Working Paper No. 1008, European Central Bank.

[54] T. Pennanen, 2014, "Optimal Investment and Contingent Claim Valuation in Illiquid Markets," *Finance and Stochastics*, 18, 733 - 754.

[55] Protter, P., 2005. *Stochastic Integration and Differential Equations, Second Edition, Version 2.1*, Springer.

[56] R. Rockafellar, 1970, *Convex Analysis*, Princeton U. Press.

[57] R. Radner, 1982, "Equilibrium under Uncertainty," *Econometrica* 36 (1), 31–58.

[58] Ruszczynski, A., 2006, *Nonlinear Optimization*, Princeton U. Press.

[59] Tobin, J., 1969, "A general equilibrium approach to monetary theory," *Journal of Money, Credit and Banking*, 1(1), 15-29.

[60] H. Tuy, 1998, *Convex Analysis and Global Optimization*, Kluwer Academic Publishers, Norwell MA.

[61] V. Vath, M. Mnif, and H. Pham, 2007, "A Model of Optimal Portfolio Selection under Liquidity Risk and Price Impact," *Finance and Stochastics*, 11, 51 - 90.

[62] White, E., 1990. "The stock market boom and crash of 1929 revisited," *Journal of Economic Perspective*, 4 (2), 67-83.

[63] Xiong, W. 2013. "Bubbles, Crises, and Heterogeneous Beliefs," NBER Working Paper No. 18905.