Estimating Portfolio Risk for Tail Risk Protection Strategies*

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Abstract

We investigate appropriate forecasting models to estimate portfolio risk for timely managing the investment exposure of dynamic tail risk protection strategies. Specifically, we consider risk targeting and dynamic proportion portfolio insurance (DPPI) strategies to limit the downside risk of an investment. For forecasting (tail) risk, we examine various techniques based on extreme value theory, quantile/expectile regression and Copula-GARCH methods in order to forecast the risk of a global asset allocation portfolio. We find that more sophisticated models dominate more naive ones in modelling the tail of the portfolio return distribution, and that associated dynamic tail risk protection strategies are effectively protecting investors from downside risk. For risk targeting strategies, we document superior performance of the extreme value theory, the quantile/expectile regression, and the Copula-GARCH approaches; for DPPI strategies, the weakness of less elaborate risk forecasting models is alleviated by the mechanics of the portfolio insurance strategy that automatically reduces investment exposure when approaching the desired protection level.

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1 Introduction

Given extreme events, such as the global financial crisis in 2008/09, tail risk hedging strategies to protect investment portfolios against extreme negative market moves are of vital interest to many market participants. An obvious way of protection is the purchase of a put option. This protective put strategy ensures that the portfolio value will not fall below a pre-specified protection level at expiry. However, such a strategy can be expensive, since the option premium is payable each investment period, although the protection could prove unnecessary in the majority of periods. Moreover, it may be difficult to find option contracts that fit the needs of the given portfolio—particularly when it comes to complex investment vehicles (Figlewski, Chidambaran, and Kaplan, 1993). Given the shortcomings of option-based tail risk protection, a possible alternative might be to resort to dynamic asset allocation strategies. These strategies primarily aim to improve the downside risk profile of an investment strategy without jeopardizing its long-term return potential by dynamically shifting between a risky asset (or asset portfolio) and a risk-free asset. Of these, risk targeting strategies[1] are one possibility (Hocquard, Ng, and Papageorgiou, 2013; Perchet, de Carvalho, Heckel, and Moulin, 2015; Bollerslev, Hood, Huss, and Pedersen, 2017). A risk targeting strategy controls portfolio risk over time by taking advantage of the negative relationship between risk and return. Specifically, the investment exposure of the portfolio is adjusted according to updated risk forecasts in order to keep the ex-ante risk at a constant target level. A stricter way to limit downside risk is to apply portfolio insurance strategies, such as the constant proportion portfolio insurance (CPPI) strategy (Perold, 1986; Black and Jones, 1987, 1988; Perold and Sharpe, 1988). Herein, the investor defines a minimum capital level that should be preserved at the end of the investment period. Key element to determine the investment exposure to the risky asset is the so-called multiplier. It represents the inverse of the maximum sudden loss of the risky asset, so that a given risk capital (i.e. the spread between portfolio value and protection level) is not fully consumed and the portfolio value does not fall below the protection level. Initially, the multiplier was assumed to be static and unconditional (e.g. Bertrand and Prigent, 2002; Balder, Brandl, and Mahayni, 2009; Cont and Tankov, 2009). However, given the empirical characteristics of financial assets, such as time-varying volatility or volatility clustering (cf. Longin and Solnik, 1995; Andersen, Bollerslev, Christoffersen, and Diebold, 2006), various other studies (e.g. Hamidi, Maillet, and Prigent, 2014) propose to model the multiplier as time-varying and conditional. The corresponding strategy is known as dynamic proportion portfolio insurance (DPPI).[2]

The success of both tail risk protection strategies strongly depends on the success in forecasting

[1]Risk targeting strategies are also known as constant risk, target risk or inverse risk weighting strategies.

[2]For a comprehensive literature review on portfolio insurance and CPPI/DPPI multipliers, see Benninga (1990); Black and Perold (1992); Basak (2002); Dichtl and Drobetz (2011); Hamidi, Maillet, and Prigent (2014), among others.

tail risk (Perchet, de Carvalho, Heckel, and Moulin, 2015). Given the vast amount of available risk models, the question is to determine which forecasting approach is appropriate to protect a portfolio against downside risk. In this vein, we contribute to the existing literature on tail risk protection strategies by investigating suitable risk models for timely managing the investment exposure in dynamic tail risk protection strategies. At the same time, we contribute to the literature on risk model evaluation by not only assessing the statistical performance but also its economical relevance when testing the risk forecasts in a thorough portfolio application.

The risk targeting strategy, in particular the volatility targeting strategy, is extensively backtested using historical data, and is known to show superior performance compared to a simple buy-and-hold strategy (Cooper, 2010; Kirby and Ostdiek, 2012; Ilmanen and Kizer, 2012; Giese, 2012). Hallerbach (2012, 2013) demonstrates that the Sharpe ratio increases, even if the portfolio mean return is constant over time. Hocquard, Ng, and Papageorgiou (2013) show that not only do constant volatility portfolios deliver higher Sharpe ratios than their passive counterpart but also that drawdowns are reduced. Thus, they provide evidence that targeting constant volatility helps to reduce tail risk. Closely related to the risk targeting strategy that we apply is the dynamic VaR portfolio insurance strategy of Jiang, Ma, and An (2009) that also aims at controlling the exposure of a risky asset such that a specified Value at Risk is not violated. However, their strategy can only be applied to parametric location-scale models, while the one we apply is more flexible and allows use of any type of risk model. Similar to us, Bollerslev, Hood, Huss, and Pedersen (2017) use a risk targeting strategy to compare realized volatility models to more conventional procedures that do not incorporate the information in high-frequency intraday data.

In the literature on dynamic proportion portfolio insurance, there are various ways to model the conditional time-varying multiplier. While Ameur and Prigent (2006, 2014) employ GARCH-type models, Hamidi, Jurczenko, and Maillet (2009) and Hamidi, Maillet, and Prigent (2008) define the multiplier as a function of a dynamic autoregressive quantile model of the Value at Risk according to Engle and Manganelli (2004). In contrast, Chen, Chang, Hou, and Lin (2008) propose a multiplier framework based on genetic programming. More recently, Hamidi, Maillet, and Prigent (2014) employ a dynamic autoregressive expectile (DARE) model to estimate the conditional multiplier. In this framework, the multiplier is modelled as a function of the expected shortfall determined by a combination of quantile functions in order to reduce the potential model error. Specifically, they combine the historical simulation approach, three methods based on distributional assumptions and four methods based on expectiles and conditional autoregressive specifications into the DARE approach. All these papers provide evidence that the DPPI strategy, with a multiplier based on a time-varying conditional risk estimate, outperforms the traditional CPPI strategy.

We are particularly interested in comparing different ways to determine the risky investment exposure—by assessing various models to estimate portfolio risk. While the academic literature suggests a myriad of risk models—Andersen, Bollerslev, Christoffersen, and Diebold (2006, 2013), Kuester, Mittnik, and Paolella (2006), and Righi and Ceretta (2015) provide thorough summaries on risk modelling—practitioners still only use a limited number of them. There may be various reasons for this discrepancy such as complexity, (computational) efficiency or the perception that the additional benefit of implementing a highly sophisticated model could be minor. We therefore examine simple methods that are common among practitioners and more flexible methods to estimate portfolio risk. In particular, we perform an empirical study using a global multi-asset data set consisting of stock, bond, commodity and currency indices.

When using international daily return data, the problem of different market closing times arises. Ignoring this fact would lead to distorted portfolio risk estimates given that the degree of co-movement is considerably underestimated (see Scholes and Williams, 1977; Lo and MacKinlay, 1990; Burns, Engle, and Mezrich, 1998; Audrino and Bühlmann, 2004; Scherer, 2013). Hence, we first synchronize daily returns by extrapolating asset prices of closed markets based on information from markets that close later. Following Audrino and Bühlmann (2004), we thus employ a synchronization approach based on a VAR(1) model. Second, we estimate portfolio risk using the following models: historical simulation, RiskMetrics, Cornish-Fisher Approximation, quantile and expectile regression, extreme value theory and Copula-GARCH. To quantify risk, we resort to the classic downside risk measures Value at Risk (VaR) and Expected Shortfall (ES). Third, we employ the ensuing risk forecasts in dynamic tail risk protection strategies.

Our empirical findings indicate a superiority of sophisticated risk forecasts over simple approaches in terms of historical accuracy and statistical fit. Using the most prominent VaR and ES tests (Kupiec, 1995; Christoffersen, 1998; McNeil and Frey, 2000; Christoffersen and Pelletier, 2004), we document the RiskMetrics, quantile/expectile regression and the Copula-GARCH approach to be the most suitable methods. When feeding the risk forecasts in the tail risk protection framework, our findings are twofold. For the risk targeting strategy, we observe a clear outperformance of the more flexible methods, confirming the results from the statistical analysis. For the DPPI strategy, we likewise evidence that the use of more sophisticated risk models helps to protect investors from downside risk. Yet, more naive approaches do not fall short of providing downside protection. Given that the portfolio insurance strategy automatically reduces investment exposure when approaching the protection level, the less sophisticated methods' weaknesses seem to be compensated by this second line of defense.

The remainder of the paper is structured as follows: Section 2 discusses the employed tail risk protection strategies. Section 3 briefly presents the different models to estimate portfolio risk. In Section 4, we perform the empirical study using a global multi-asset data set to compare the performance of dynamic tail risk protection strategies based on the different risk models. Section 5 concludes.

2 Tail Risk Protection Strategies

We consider a risk-averse investor who primarily aims to limit the downside risk of his investment over an investment horizon of H time steps. Further, let $t = 1, 2, ..., T$ be the time index of portfolio rebalancing and $I(t) = t - \left(\lceil \frac{t}{H} \rceil - 1\right) H$ a subindex for each investment period $\lceil \frac{t}{H} \rceil$, so that the latter runs from 1 to H in each investment period. At the beginning of each investment period $\lceil \frac{t}{H} \rceil$, i.e. at $I(t) = 1$, the investor determines a risk target that should be achieved at the end of the period, i.e. at $I(t) = H$. The management of the protected portfolio follows a dynamic portfolio allocation. In particular, the value of the protected portfolio, denoted by V_t, is invested in a risky asset (or portfolio) and a non-risky asset in such a way that the given risk target will not be violated. Prices and returns are denoted by P_t and r_t for the risky asset and B_t and $r_{f,t}$ for the non-risky asset, respectively. To explicitly determine the exposure to the risky asset e_t, we need to forecast the downside risk of the risky asset, quantified by a risk measure $\rho(\cdot)$.

2.1 Risk Targeting Strategies

A risk targeting strategy's underlying principle is to systematically adjust exposure to a given asset (or portfolio) conditional to its current risk (forecast) in order to maintain a pre-specified target risk level. For example, if we target a 1% Value at Risk for a given asset portfolio and the asset portfolio's current Value at Risk is 1.5%, we would lower our investment exposure by shifting towards the risk-free asset, and vice versa if the current risk is lower than our target. The rationale for maintaining a constant risk level can be found in the negative relationship between risk and return (see French, Schwert, and Stambaugh, 1987). Empirical evidence suggests that asset returns tend to be greater during periods of low risk. Consequently, investors should maximize asset exposure during these periods, taking advantage of a favorable risk–reward tradeoff. As risk increases, they should, however, decrease asset exposure to maintain the desired risk level.

Given the level of ex-ante risk of the underlying risky asset $\rho_t(r_{t+1})$ and the predefined target risk $\overline{\rho}$, the allocation to the risky asset e_t is simply $\overline{\rho}/\rho_t(r_{t+1})$. As $\rho_t(r_{t+1})$ is unknown, we employ a forecast based on the information available at time t, $\mathcal{F}_t$:

$$e_t \equiv \frac{\overline{\rho}}{\hat{\rho}_t(r_{t+1}|\mathcal{F}_t)} \tag{1}$$

Correspondingly, the weight of the risk-free asset, $1 - e_t$, can be positive or negative, depending on whether the risky asset must be de-levered or levered to attain the constant target risk.

2.2 Constant and Dynamic Proportion Portfolio Insurance

The basic idea of the constant proportion portfolio insurance (CPPI) strategy (see Perold, 1986; Black and Jones, 1987, 1988; Perold and Sharpe, 1988) is a portfolio that dynamically shifts between the risky and non-risky asset to guarantee the investor to recover a given proportion of its initial capital. At the beginning of each investment period $\lceil \frac{t}{H} \rceil$, i.e. at $I(t) = 1$, the investor determines this minimum portfolio value, the so-called floor $F_{\lceil \frac{t}{H} \rceil}$, that should be preserved at the end of the period, i.e. at $I(t) = H$. The corresponding risk capital, called the cushion, is derived as the difference of (the investment in) portfolio value, V_t, and discounted floor, i.e. the net present value, $\text{NPV}(\cdot)$, of the floor:

$$C_t = V_t - \text{NPV}_t(F_{\lceil \frac{t}{H} \rceil}). \tag{2}$$

The cushion represents a certain amount of the portfolio value that allows potential market shocks to be absorbed before the manager can rebalance the portfolio. In order to avoid a breach of the discounted floor, the investment exposure to the risky asset, defined as $E_t = e_t V_t$, should be set such that the cushion at t is at least as high as the maximum sudden drop in the portfolio value between the rebalancing dates t and $t+1$, i.e.

$$C_t \geq V_t \left| \inf \left(\frac{V_{t+1} - V_t}{V_t} \right) \right|. \tag{3}$$

As the portfolio consists of the risky and the non-risky asset, (3) can be simplified to

$$C_t \geq e_t V_t \left| \inf \left(\frac{S_{t+1} - S_t}{S_t} \right) \right|. \tag{4}$$

Rearranging (4) then yields the (total) exposure to the risky asset as

$$E_t \leq C_t |\inf(r_{t+1})|^{-1} = C_t m, \tag{5}$$

where $r_{t+1} = \frac{S_{t+1} - S_t}{S_t}$ and $m \equiv |\inf(r_{t+1})|^{-1}$ is called the multiplier.[3] The latter indicates how often a given cushion can be invested in the risky asset without breaching the floor. Thus, it reflects the investor's risk tolerance. The higher the multiplier, the more the investor will participate in upward market movements of the underlying. But the higher the multiplier, the faster the portfolio will reach the floor when there is a sustained decrease in the underlying's price. In order to allow for the highest possible participation in the underlying risky asset, it is common to set E_t such that equality holds in (5). The remainder of the investor's wealth is invested in the risk-free asset.

If rebalancing were continuous and price movements sufficiently smooth, the CPPI allocation rule would ensure that the portfolio does not fall below the floor (Cont and Tankov, 2009; Balder, Brandl, and Mahayni, 2009; Hamidi, Hurlin, Kouontchou, and Maillet, 2014; Ardia, Boudt, and Wauters, 2016). However, with discrete rebalancing and jumps in prices, there is a non-negligible probability that the floor will be breached. This risk of losing more than the cushion between two

[3]We follow Benninga (1990) and restrict the participation ratio to vary between 0% and 100% of the risky asset in order to rule out short positions. This approach leads to a slightly different exposure definition: $E_t = \max[\min(mC_t, V_t), 0]$.

rebalancing dates and thus failing to ensure the protection at the end of the investment period is called gap risk. A common way to minimize the gap risk is to employ the minimum return of the risky asset over the whole sample history, i.e. $\inf(r_{t+1}) = \min(r_1, ..., r_T)$. Then, the CPPI strategy is based on a static unconditional multiplier—often reflecting a constant worst-case scenario. Although such a conservative stance would have meaningfully addressed catastrophic drawdowns during extreme market turmoil, it would also have capped upside potential over the long term. Dynamic proportion portfolio insurance (DPPI) is designed to introduce more flexibility. Instead of using a static multiplier, the risk budget adapts dynamically to changes in a risk forecast, measured by $\hat{\rho}(\cdot)$. Thus, the exposure changes to

$$E_t = C_t |\hat{\rho}_t(r_{t+1}|\mathcal{F}_t)|^{-1} = C_t m_t, \tag{6}$$

where the risk forecast $\hat{\rho}_{t+1}$ is based on information $\mathcal{F}_t$ and measures the risk when the risky asset price S evolves from t to $t+1$. The dynamic multiplier is therefore given by

$$m_t = |\hat{\rho}_t(r_{t+1}|\mathcal{F}_t)|^{-1}. \tag{7}$$

In this way, the portfolio's exposure to the risky asset reacts to changes in the risk forecast—ensuring that it does not remain artificially low as a result of a constant conservative risk assumption. For this to work in practice, the risk model must be capable of quickly homing in on volatility spikes, and just as quickly readjusting to a normalization of market volatility.

The advantage of the CPPI and DPPI strategy, respectively, is the simple practical implementation that does not call for forecasting the returns of the risky asset. Major drawbacks are the strategies' path dependencies as well as the lock-in effect. Depending on the underlying portfolio return path, the CPPI/DPPI strategy can deliver considerably different results. In general, the more volatile the risky asset, the lower the average participation ratio. While the CPPI strategy is fully exposed to the problem of path dependency, the DPPI strategy can mitigate this problem at least to some extent by quickly reacting to market changes. The lock-in effect occurs when the cushion is entirely consumed by losses. The CPPI/DPPI strategy is then fully invested in the risk-free asset until the end of the investment period and no participation in subsequent upward movements is possible.

The success of both described tail risk protection techniques strongly depends on the success in forecasting tail risk. For both academics and practitioners it is therefore of crucial interest to identify suitable risk models to model a portfolio's downside risk adequately.

3 Estimating Portfolio Risk

Given the vast amount of available risk models (see, e.g. Kuester, Mittnik, and Paolella, 2006; Nadarajah, Zhang, and Chan, 2014), we focus on a few yet distinct approaches, ranging from rather simple approaches that are widely used among practitioners to more flexible models that are standard in the academic literature. We consider both portfolio-level (aggregated, "top-down") and asset-level (disaggregated, "bottom-up") risk modelling. Following the description on how we measure downside risk, we briefly summarize these methods in this section.[4]

3.1 Conditional Risk Measurement

The literature suggests various ways to quantify market risk of financial assets. As we are particularly interested in protecting risky portfolios against extreme market losses, we resort to the most common downside risk measures, Value at Risk (VaR) and Expected Shortfall (ES). VaR measures the maximum potential loss of a given asset (portfolio) at a given confidence level.[5] Therefore, VaR is simply the negative p-quantile of the conditional return distribution, that is,

$$\mathrm{VaR}^{p}_{t+1|t} = -Q_p(r_{t+1}|\mathcal{F}_t) = -\inf_{x}\{x \in \mathbb{R} : P\left(r_{t+1} \leq x|\mathcal{F}_t\right) \geq p\}, \tag{8}$$

where $Q_p(\cdot)$ denotes the quantile function, r_t reflects the return of the asset (portfolio) in period t and $\mathcal{F}_t$ represents the information available at time t.

Although VaR is still the risk measure of choice in the financial industry, it has been criticized for disregarding outcomes beyond the specified VaR-quantile. Moreover, VaR is not a subadditive risk measure. This property posits that the total portfolio risk should not be greater than the sum of the risks of its constituents (see Artzner, Delbaen, Eber, and Heath, 1999; Acerbi and Tasche, 2002; Taylor, 2008). Expected shortfall, also known as conditional VaR or expected tail loss, is a risk measure that overcomes these weaknesses by aggregating information about the tail of the portfolio return distribution. It is defined as the conditional expectation of the return given that VaR is exceeded (see Yamai, Yoshiba, et al., 2002), specifically

$$\mathrm{ES}^{p}_{t+1|t} = -\mathbb{E}_t\left[r_{t+1}|r_{t+1} < -\mathrm{VaR}^{p}_{t+1|t}, \mathcal{F}_t\right] = -p^{-1}\int_0^p \mathrm{VaR}^{s}_{t+1|t}\,ds. \tag{9}$$

3.2 Conditional Portfolio-Level Risk Models

Generally, there are two ways of risk modelling classes, depending on the aggregation level. Portfolio-level analysis, as discussed in this section, requires only a univariate model based on aggregated portfolio returns. The latter can be easily constructed using portfolio holdings $\mathbf{w}_t = (w_{1,t}, w_{2,t}, ..., w_{N,t})$

[4] For a rigorous discussion of the analyzed risk models, see Kuester, Mittnik, and Paolella (2006), Andersen, Bollerslev, Christoffersen, and Diebold (2013) and Righi and Ceretta (2015).

[5] As more common in the academic literature, we refer to VaR and ES as positive numbers using low-probability terminology (e.g. a Var of 1%).

and the individual asset returns $\mathbf{r}_t = (r_{1,t}, r_{2,t}, ..., r_{N,t})$:

$$r_{PF,t} = \sum_{i=1}^{N} w_{i,t} r_{i,t} = \mathbf{w}_t' \mathbf{r}_t, \quad t = 1, 2, ..., T. \tag{10}$$

While aggregation generally comes with the loss of information, Andersen, Bollerslev, Christoffersen, and Diebold (2013) argue that there is no reason why a parsimonious dynamic model should not be estimated for portfolio-level returns. If the distribution of portfolio returns is of interest, then this distribution can be modelled directly rather than via aggregation based on a larger, and almost inevitably less well-specified, multivariate model.

3.2.1 Historical Simulation

The simplest way to estimate VaR and ES is to use the sample quantile function based on historic return data, which is referred to as historical simulation (HS). Let $r_{PF,(1)} \leq r_{PF,(2)} \leq ... \leq r_{PF,(t)}$ denote the order statistics in ascending order corresponding to the original historical pseudo portfolio returns $r_{PF,1}, r_{PF,2}, ..., r_{PF,t}$. Then, the HS-VaR for $t+1$ is simply the empirical $100p$-th quantile or the tp-th order statistic, i.e.

$$\text{VaR}^p_{t+1|t} = -r_{PF,(\lceil tp \rceil)}. \tag{11}$$

Correspondingly, the ES estimate for $t+1$ can be computed as

$$\text{ES}^p_{t+1|t} = -\left(\sum_{i=\lceil tp \rceil}^{t} r_{PF,(i)} \right) (t - \lceil tp \rceil)^{-1}. \tag{12}$$

The main advantage of the HS approach is its computational simplicity and non-parametric dimension, i.e. VaR and ES do not rely on any distributional assumptions. In contrast, the most pertinent disadvantage[6] is its inability to properly incorporate conditionality (see Andersen, Bollerslev, Christoffersen, and Diebold, 2006). This deficiency of the conventional HS approach is forcefully highlighted by the clustering of the corresponding VaR violation in time, reflecting a failure to properly account for persistent changes in market volatility. The only source of dynamics in the HS risk estimates is the evolving window used to construct historical pseudo portfolio returns. Nevertheless, the choice of the window size is crucial: too few observations will lead to sampling error, whereas too many observations will slow down estimates when reacting to changes in the true distribution of financial returns. Moreover, the risk estimates can exhibit jumps when large negative returns either enter or exit the estimation window.

[6]For a rigorous discussion of several serious issues of the HS approach, see Pritsker (2006).

3.2.2 Cornish-Fisher Approximation

Another simple approach is the Cornish-Fisher Approximation (CFA) method (Zangari, 1996), where the VaR is modelled as a Taylor-series type expansion (cf. Cornish and Fisher, 1938) around the VaR of a normal distribution. Specifically, the CFA-VaR is an extension of the normal quantile function by accounting for skewness γ and kurtosis δ, and is calculated as

$$\text{VaR}^p_{t+1|t} = -\mu_t - \sigma_t F^{-1}_{CF}(p), \tag{13}$$

where

$$F^{-1}_{CF}(p) \equiv \Phi^{-1}_p + \left([\Phi^{-1}_p]^2 - 1\right)\frac{\gamma}{6} + \left([\Phi^{-1}_p]^3 - 3\Phi^{-1}_p\right)\frac{\delta - 3}{24} - \left(2[\Phi^{-1}_p]^3 - 5\Phi^{-1}_p\right)\frac{\gamma^2}{36}$$

and $\Phi(\cdot)$ is the standard normal cdf. Moreover, μ_t and σ_t are computed by the sample mean and sample standard deviation, respectively.

Boudt, Peterson, and Croux (2008) show how the Edgeworth and Cornish-Fisher expansions of the density and quantile functions can be used to obtain an estimator for ES that delivers accurate downside risk estimates even in the presence of non-normal returns. The modified or Cornish-Fisher ES is thus computed as

$$\text{ES}^p_{t+1|t} = -\mu_t - \sigma_t \mathbb{E}_{F_{CF}}\left[z | z \leq F^{-1}_{CF}(p)\right] \tag{14}$$

where

$$\begin{aligned}\mathbb{E}_{F_{CF}}\left[z | z \leq F^{-1}_{CF}(p)\right] = -\frac{1}{p}\Bigg(&\phi(F^{-1}_{CF}(p)) + \frac{\delta}{24}\left[I^4 - 6I^2 + 3\phi(F^{-1}_{CF}(p))\right] + \frac{\gamma}{6}\left[I^3 - 3I^1\right] \\ &+ \frac{\gamma^2}{72}\left[I^6 - 15I^4 + 45I^2 - 15\phi(F^{-1}_{CF}(p))\right]\Bigg)\end{aligned}$$

with

$$I^q = \begin{cases} \sum_{i=1}^{q/2}\left(\frac{\prod_{j=1}^{q/2} 2j}{\prod_{j=1}^{i} 2j}\right) g_p^{2i}\phi(g_p) + \left(\prod_{j=1}^{q/2} 2j\right)\phi(g_p) & \text{for } q \text{ even} \\ \sum_{i=0}^{q^*}\left(\frac{\prod_{j=0}^{q^*}(2j+1)}{\prod_{j=0}^{i}(2j+1)}\right) g_p^{2i+1}\phi(g_p) - \left(\prod_{j=0}^{q^*} 2(j+1)\right)\phi(g_p) & \text{for } q \text{ odd} \end{cases}$$

and $q^* = (q-1)/2$, $g_p = F^{-1}_{CF}(p)$. $\phi(\cdot)$ denotes the standard normal pdf.

The main advantage of the CFA method is its ability to account for fat tails. However, an issue is that the CFA-VaR is not necessarily monotone, i.e. the 1% VaR might be smaller than the 5% VaR. Martin and Arora (2017) also document that the CFA-VaR and CFA-ES suffer in terms of statistical efficiency.

3.2.3 Quantile/Expectile Regression

As VaR and ES are directly linked to quantiles and expectiles, a natural approach to risk modelling employs the concepts of quantile and expectile regressions. The basic idea of quantile regression is to directly model the conditional quantile rather than the whole distribution of portfolio returns. More precisely, the conditional p-quantile, $Q_p(r_{PF,t}|\mathcal{F}_{t-1}) = -\text{VaR}_{t|t-1}$, is modelled as a parametric function of the information $\mathcal{F}_{t-1}$:

$$\text{VaR}^p_{t|t-1} \equiv -g_p\left(\mathcal{F}_{t-1}; \beta_p\right), \tag{15}$$

where $g(\cdot,\cdot)$ and the parameter vector β explicitly depend on p. Following Koenker and Bassett (1978), the conditional sample p-quantile can be found as the solution to

$$\min_{\beta_p} \left\{ \sum_{r_{PF,t} \geq \text{VaR}^p_{t|t-1}} p|r_{PF,t} + \text{VaR}^p_{t|t-1}| + \sum_{r_{PF,t} < -\text{VaR}^p_{t|t-1}} (1-p)|r_{PF,t} + \text{VaR}^p_{t|t-1}| \right\}, \tag{16}$$

where we determine VaR^p_t by the conditional autoregressive Value at Risk (CAViaR) specification of Engle and Manganelli (2004). In particular, we adopt their asymmetric slope CAViaR model[7] that is given by

$$\text{VaR}^p_{t|t-1} = \beta_0 + \beta_1 \text{VaR}^p_{t-1|t-2} + \beta_2 \max\left[r_{PF,t-1}, 0\right] + \beta_3 \max\left[-r_{PF,t-1}, 0\right]. \tag{17}$$

In a similar fashion, we can use expectile regression to estimate ES. In particular, we employ the conditional autoregressive expectile (CARE) model of Taylor (2008). First, we consider that the population τ_p expectile of $r_{PF,t}$ is the parameter μ_{τ_p} that minimizes the function $\mathbb{E}\left[|\tau_p - \mathbb{1}(r_{PF,t} - \mu_{\tau_p})|(r_{PF,t} - \mu_{\tau_p})^2\right]$. Hence, we can represent the conditional expectile as a parametric function of past information, i.e. $\mu_{\tau_p}(r_{PF,t}) \equiv h_{\tau_p}\left(\mathcal{F}_{t-1}; \gamma_{\tau_p}\right)$. The parameters γ_{τ_p} can be estimated using asymmetric least squares (cf. Newey and Powell, 1987), i.e.

$$\min_{\gamma_{\tau_p}} \left\{ \sum_{r_{PF,t}} |\tau_p - \mathbb{1}(r_{PF,t} < h_{\tau_p}\left(\mathcal{F}_{t-1}; \gamma_{\tau_p}\right))| \left(r_{PF,t} - h_{\tau_p}\left(\mathcal{F}_{t-1}; \gamma_{\tau_p}\right)\right)^2 \right\}, \tag{18}$$

where $\mathbb{1}(\cdot)$ denotes the indicator function. Similar to the asymmetric slope CAViaR model, we assume the conditional expectile to have an asymmetric slope specification. The ES can then be computed

[7] For the sake of simplicity, we focus on one CAViaR model. Particularly, we choose the asymmetric slope specification because of its ability to accommodate the leverage effect.

as the product of a correction term and the conditional expectile, i.e.

$$\mathrm{ES}^{p}_{t|t-1} = \left(1 + \frac{\tau_p}{(1-2\tau_p)\,p}\right)\left(\gamma_0 + \gamma_1 \mu_{\tau_p}(r_{PF,t-1}) + \gamma_2 \max\left[r_{PF,t-1}, 0\right] + \beta_3 \max\left[-r_{PF,t-1}, 0\right]\right). \tag{19}$$

The attractiveness of the quantile and expectile regression approach is that no explicit distributional assumption for the time series behaviour of returns is needed, thus reducing the risk of model misspecification. The main drawback of the CAViaR modelling strategy is that it might generate out-of-order quantiles similar to the CFA method. Also, estimation of model parameters is challenging.[8]

3.2.4 Extreme Value Theory

As we are primarily interested in the tails of the portfolio distribution, a natural way is to resort to extreme value theory (EVT) which makes it possible to meaningfully estimate the tails based on extrapolating from available observations. McNeil and Frey (2000) propose a semi-parametric framework based on extreme value theory to describe the tail of the conditional distribution. In a first step, the authors employ pseudo-maximum-likelihood fitting of AR(1)-GARCH(1,1) models to estimate conditional volatility forecasts $\hat{\sigma}_{t+1}$ and conditional mean forecasts $\hat{\mu}_{t+1}$. In a second step, they resort to EVT for estimating the tail of the innovation distribution of the AR(1)-GARCH(1,1) model. In particular, they use the peak-over-threshold method where a Generalized Pareto Distribution (GPD) is fitted to the negative portfolio returns over a specified threshold.[9] The quantile $\hat{z}_p$ can then be estimated as

$$\hat{z}_p = u + \frac{\hat{\beta}}{\hat{\xi}}\left[\left(\frac{1-q}{n_u/n}\right)^{-\hat{\xi}} - 1\right], \tag{20}$$

where $\hat{\beta}$ and $\hat{\xi}$ are the GPD estimates and n_u is the number of observations above threshold u. Consequently, the VaR and ES forecasts can be computed as

$$\mathrm{VaR}^{p}_{t+1|t} = -\left(\hat{\mu}_{t+1} + \hat{\sigma}_{t+1}\hat{z}_p\right), \tag{21}$$

$$\mathrm{ES}^{p}_{t+1|t} = -\left(\hat{\mu}_{t+1} + \hat{\sigma}_{t+1}\hat{z}_p\left(\frac{1}{1-\hat{\xi}} + \frac{\hat{\beta} - \hat{\xi}u}{(1-\hat{\xi})\hat{z}_p}\right)\right). \tag{22}$$

The ARMA-GARCH fitting in the first step makes it possible to capture certain stylized facts such as time-varying volatility, fat tails and volatility clustering. Then, EVT is particularly suitable to estimate the tails of the distribution. The crucial assumption of EVT is, however, that one is in the tails of the distribution. Hence, the difficulty is the determination of the threshold. If the threshold

[8] We thank James Taylor for providing the Gauss code for his CARE models.

[9] We follow McNeil and Frey (2000) when determining the thresholds. See their paper for details.

is too low, then the approximation can hardly be justified and the associated risk estimates may be biased. Vice versa, if the threshold is too high, there are too few observations over the threshold resulting in highly volatile estimates.

3.3 Conditional Asset-Level Risk Models

The models discussed so far have focused on dynamic risk modelling for univariate returns. In contrast, conditional asset-level risk analysis is based on a multivariate model that additionally makes it possible to account for the dependence structure among the portfolios' assets.

3.3.1 The RiskMetrics Approach

The RiskMetrics (RM) model is arguable the most simple and commonly applied approach among finance practitioners for estimating time-varying covariance matrices. It utilizes an exponentially weighted moving average filter that implicitly assumes a very tight parametric specification by incorporating conditionality via the exponential smoothing of individual squared returns and cross products. The estimate for the $N \times N$ covariance matrix at time $t+1$, $\hat{\Sigma}_{t+1}$, is then defined by

$$\hat{\Sigma}_{t+1} = \lambda \hat{\Sigma}_t + (1-\lambda)\mathbf{r}_t \mathbf{r}_t^{'}, \tag{23}$$

where $\lambda < 1$ is known as the decay factor.[10] The VaR and ES are then simply obtained as

$$\mathrm{VaR}_{t+1|t}^{p} = -\left(\mathbf{w}_t^{'} \hat{\Sigma}_{t+1} \mathbf{w}_t\right)^{-1/2} \Phi_p^{-1}, \tag{24}$$

$$\mathrm{ES}_{t+1|t}^{p} = -\left(\mathbf{w}_t^{'} \hat{\Sigma}_{t+1} \mathbf{w}_t\right)^{-1/2} \frac{\phi\left(\Phi_p^{-1}\right)}{p}. \tag{25}$$

The RM model is appealing because no parameters need to be estimated—which is due to the implicit assumption of zero mean returns, a fixed smoothing parameter and conditional normality. At the same time, however, the RM approach is very restrictive, imposing the same degree of smoothness on all elements of the covariance matrix. Moreover, the RM model tends to underestimate VaR and ES under the normality assumption. We therefore employ a t-distribution instead.

3.3.2 The Copula-GARCH Approach

The Copula-GARCH (CG) approach proposed by Jondeau and Rockinger (2006) and Patton (2006) is based on the concept of inference from margins, i.e. dependencies between the marginal distributions are captured by a copula.

In the first step, univariate GARCH(1,1)-models are fitted to the underlying return series.

[10] In practice, λ is typically fixed at a preset value of 0.94 when using daily returns.

Assuming a return process $(r_{i,t})_{i\in\mathbb{N},t\in\mathbb{Z}}$, the mean and variance equations are given by

$$r_{i,t} = \mu_i + \varepsilon_{i,t}, \tag{26}$$

$$\varepsilon_{i,t} = z_{i,t}\sqrt{\sigma_{i,t}^2}, \tag{27}$$

$$z_{i,t} \sim \mathcal{D}_i(0, 1, \xi_i, \nu_i), \tag{28}$$

$$\sigma_{i,t}^2 = \omega_i + \alpha_i\varepsilon_{i,t-1}^2 + \beta_i\sigma_{i,t-1}^2, \tag{29}$$

where $\omega_i > 0, \alpha_i \geq 0$ and $\beta_i \geq 0$, $i = 1, ..., N$. Moreover, r_i are the returns of the i-th portfolio asset, and $\mathcal{D}_i$ reflects the skewed standardized-t distribution with skewness parameter ξ_i and shape parameter ν_i.

In the second step, we use a time-varying copula to estimate the marginal distributions of the asset returns together with the dependence structure. In particular, the joint distribution of the N GARCH return processes can be expressed depending on an N-dimensional copula C:

$$F_t\left(\mathbf{r}_t|\boldsymbol{\mu}_t, \boldsymbol{\sigma}_t\right) = C_t\left(F_{1,t}(r_{1,t}|\mu_{1,t}, \sigma_{1,t}), ..., F_{N,t}(r_{N,t}|\mu_{N,t}, \sigma_{N,t})|\mathcal{F}_{t-1}\right), \tag{30}$$

where $F_1(\cdot), ..., F_N(\cdot)$ are the conditional marginal distributions of the return processes. The dependence structure of the margins is assumed to follow a Student-t copula with conditional correlation $\mathbf{R}_t$ and constant shape parameter η. We opt for the Student-t copula for modelling the dependence of financial assets since the normal copula cannot account for tail dependence. The conditional density of the Student-t copula at time t is given by:

$$c_t\left(u_{i,t}, ..., u_{N,t}|\mathbf{R}_t, \eta\right) = \frac{f_t\left(F_{i,t}^{-1}(u_{i,t}|\eta), ..., F_{i,t}^{-1}(u_{N,t}|\eta)|\mathbf{R}_t, \eta\right)}{\prod_{i=1}^{n} f_i\left(F_{i,t}^{-1}(u_{i,t}|\eta)|\eta\right)}, \tag{31}$$

where $u_{i,t} = F_{i,t}(r_{i,t}|\mu_{i,t}, \sigma_{i,t}, \xi_i, \nu_i)$ is the probability integral transformation of each series by its conditional distribution $F_{i,t}$ estimated via the first stage GARCH process, $F_{i,t}^{-1}(u_{i,t}|\eta)$ represents the quantile transformation of the uniform margins subject to the common shape parameter of the multivariate density, $f_t(\cdot|\mathbf{R}_t, \eta)$ is the multivariate density of the Student-t distribution with conditional correlation $\mathbf{R}_t$ and shape parameter η and $f_i(\cdot|\eta)$ is the univariate margins of the multivariate Student-t distribution with common shape parameter η. Furthermore, we allow the parameters of the conditional copula to vary through time in a manner analogous to a GARCH model for conditional variance (e.g. Patton, 2006). Specifically, we assume the dynamics of $\mathbf{R}_t$ to follow an Asymmetric Generalized Dynamic Conditional Correlation (AGDCC) model according to Cappiello, Engle, and Sheppard (2006).

Based on the copula estimates, we then generate N sets of random pseudo-uniform variables and transform these into corresponding realizations of the error processes by using the quantile function of the margins. These simulated numbers are then used together with the conditional volatility forecast of the GARCH models to derive a Monte Carlo set of returns for each asset. By means of the portfolio's weight vector we can then compute a distribution of portfolio returns for $t + 1$ which allows us to calculate VaR and ES forecasts.

The Copula-GARCH model has several advantages over more simplistic approaches. The GARCH models with skewed standardized-t distribution applied in the first stage make it possible to capture the main empirical characteristics of financial asset returns. Moreover, as correlation coefficients can capture the dependency between random variables correctly only for elliptical distributions, and asset returns hardly follow elliptic distributions, dependencies between the portfolios' assets cannot be measured correctly in most cases. Using the concept of copulas enables us to separate the marginal distributions and the dependence structure so that dependencies between the portfolios' assets can be incorporated in the VaR estimation. Given the associated computational effort and complexity, however, most practitioners prefer to resort to simpler methods.

4 Empirically Validating Risk Models for Tail Risk Protection

In this section, we describe the design and the results of the empirical study that compares the various methods for portfolio risk modelling using tail risk protection strategies. For this, we use a global multi-asset data set consisting of the four main factors of market risk: (i) equity, (ii) fixed income, (iii) commodities and (iv) exchange rates. In particular, we utilize the following representative assets: the equity futures for *Nikkei 225, EURO STOXX 50, FTSE 100, S&P 500, MSCI EM*, the bond futures for *JGB 10Y, Euro Bund, UK Gilt, US 10Y*, the total return indices for the commodities *Oil, Gold, Copper* and the spot market foreign exchange rates *JPY/USD, EUR/USD, GBP/USD*. The money market investment is based on the *3-month U.S. Treasury Bill*. We retrieve all data from Bloomberg. All asset prices are in local currency. Portfolio returns (and associated portfolio risk figures) are computed from the perspective of a U.S. investor who is hedging any currency exposure. The sample spans from 2/1/1991 to 31/3/2017, giving rise to 6,847 daily return observations for each series.

To calculate the portfolio risk figures, we assume a given static strategic allocation of portfolio weights. Alternatively, we could consider a dynamic weight structure driven by a tactical asset allocation component. However, then it would not be possible to determine whether an increase in performance is due to superior risk forecasts or due to predictability of the tactical component.

Although the static weights of the multi-asset portfolio are chosen reasonably from a practitioners' point of view, they are still to some extent arbitrary. Therefore, we also investigate four other allocations: a pure equity portfolio, a pure bond portfolio, a 30/70 equity/bond portfolio and 60/40 equity/bond portfolio. Table 1 reports the corresponding allocation of portfolio weights as well as the descriptive statistics of the log returns of each asset and portfolio: all time series exhibit the typical features of financial assets such as fat tails and non-normality.

[Table 1 about here]

The empirical study considers one day-ahead estimation of the 1% and 5% conditional VaR and ES. We choose these quantiles since they are appropriate for modelling tail risk. Our focus on one day-ahead estimation is consistent with the portfolio rebalancing frequency of the considered tail risk protection strategies. Like Kuester, Mittnik, and Paolella (2006) and Taylor (2008), we use a moving window of 1,000 observations to re-estimate parameters for the various methods repeatedly. Thus, we get an out-of-sample estimation period from 3/11/1994 to 31/3/2017 consisting of 5,846 daily VaR and ES forecasts for each method and portfolio. We use these out-of-sample predictions as the basis of our comparison of methods.

As common in financial applications, we implement the tail risk protection strategies without short sales or leverage and assume round-trip transaction costs of 10 basis points. In addition, to avoid portfolio shifts triggered by rather small market movements, we apply a trading filter of 2%, i.e. we only act on exposure changes in excess of 2% (cf. Dichtl, Drobetz, and Wambach, 2017).

4.1 Synchronizing Returns

When modelling risk using international daily return data, one has to properly account for the fact that markets have different closing times.[11] Even worse, for some markets trading times do not overlap at all, as is the case for the U.S. and Japan. Obviously, these differences will make equity markets appear less (cor)related than they actually are. As a result, portfolio risk estimates will overstate the diversification benefit attached to investing across these assets (see Scholes and Williams, 1977; Lo and MacKinlay, 1990; Burns, Engle, and Mezrich, 1998). Ideally, daily returns can be computed for all series using the same time stamp. This approach, however, is hardly feasible, even when using high-frequency data. Instead, the academic literature suggests synchronizing daily returns by extrapolating asset prices for those markets that close earlier, based on information from markets that close latest. While Burns, Engle, and Mezrich (1998) use a first-order vector moving average model with a multivariate GARCH covariance matrix to estimate synchronized returns, Audrino and Bühlmann (2004) employ a simple first-order vector autoregressive model (see Appendix A for details on the return synchronization methodology). We follow the latter approach due to its computational efficiency and performance advantages.

Based on our sample, we compare the synchronized daily returns to the original ones. Table 2 shows the descriptive statistics of the original and synchronized daily returns. We observe that differences in the mean are only marginal, whereas volatilities are slightly higher when synchronizing. Thus, the return characteristics of the original data are maintained.

[11] The opening times of the markets in our sample are as follows: Japanese markets are open from 19:00(-1) to 1:00 ET, EU/UK markets from 3:00 to 11:30 ET, and U.S. markets open from 09:30 to 16:15 ET.

[Table 2 about here]

To check the effectiveness of synchronization, Figure 1 shows the correlation matrices of both return types. For the synchronized daily returns, the chosen VAR(1) model is successfully re-correlating the within-asset class correlations. While equity correlations are no longer underestimated, equity-bond correlations tend to be more negative when using synchronized returns. Hence, the improved equity-bond diversification could mitigate the equity risk pick-up. However, we learn that the latter effect dominates and unreported results evidence an increase of portfolio risk figures that average to 15% for the conservative multi-asset portfolio. These findings are in line with Scholes and Williams (1977) and Lo and MacKinlay (1990).

[Figure 1 about here]

4.2 Statistical Validity of Risk Forecasts

To test the statistical validity of the risk forecasts, we perform various backtests for VaR and ES. The objective of this backtest is to consider the ex ante portfolio risk forecasts from a specific model and compare them with the ex post realized portfolio returns.

In principle, backtesting VaR forecasts boils down to evaluating the distribution of VaR violations. That is, counting the number of realized returns that fall below the predicted VaR-level for a given estimation period. Hence, for example, in a set of 252 forecasts of daily 1% VaRs per year, there should be 2.52 violations in theory. The test for unconditional coverage of Kupiec (1995) assesses whether the frequency of violations is consistent with the quantile of loss a VaR measure is intended to reflect. However, this test does not account for serial independence of the number of violations. The conditional coverage test of Christoffersen (1998) offers a remedy by jointly testing the frequency as well as the independence of violations, assuming that VaR violations are modelled with a first order Markov chain. This test is therefore able to reject a VaR model that generates too many clustered violations. An alternative way to account for clustering of extremes is the duration test of Christoffersen and Pelletier (2004), which examines the duration between violations by testing the null hypothesis that the duration between violations is exponentially distributed against a Weibull alternative.

Backtesting ES is more complicated. Hence, we resort to the most established test in the literature, the zero mean test of McNeil and Frey (2000). It is based on the excess conditional shortfalls, i.e. $(r_t - \mathrm{ES}_t)_{r_t < -\mathrm{VaR}_t}$. Standardized by the conditional volatility, these should be i.i.d. and have zero mean under the null hypothesis of a correctly specified risk model.

Figure 2 presents the predicted 1% VaR and ES figures of the Historical Simulation approach (HS), RiskMetrics (RM), Cornish-Fisher Approximation (CFA), Quantile Regression (QR)/Expectile Regression (ER), Extreme Value Theory (EVT) and Copula-GARCH (CG) model for the multi-asset portfolio over the whole out-of-sample period. Panel (a) shows the HS forecasts. As expected, the majority of realized returns were higher than the predicted VaRs. In the sample period from November 1994 to March 2017, there are only 60 violations (red dots)—which is very close to the number of expected violations (=1% of 5,846). An analysis of VaR violations throughout time, however, calls into doubt the validity of the HS-VaR—given a latent underestimation of risk with most violations occuring during the 2008 Financial Market Crisis. Subsequently, the HS forecasts were overly conservative, and there were no violations in the following five years. Thus, it seems that a related portfolio insurance strategy would have too small investment exposure over time. This conjecture is confirmed by rigorous statistical testing (cf. Table 3). Using the unconditional coverage test, the HS-VaR does indeed deliver a conclusive number of violations over the entire period. But, based on the test for correct coverage and independence and the duration test, it is clear that the violations are not occurring independently, but rather appear in clusters. As a correctly specified VaR model is the basis of estimating ES, the positive results of the ES test of McNeil and Frey are practically useless.

Regarding the Cornish-Fisher Approximation approach in Panel (c) we can draw similar conclusions. Although it accounts for fat tails by incorporating skewness and kurtosis, it still remains sluggish over time. Like the HS-VaR, the CFA-VaR passes the test for unconditional coverage but fails when accounting for clustering of returns again invalidating the subsequent ES test.

[Figure 2 about here]

The remaining risk models are more sensitive and quicker to react to the prevailing risk environment (see Panels (b), (d), (e) and (f)). Moreover, the occurrence of violations is markedly less clustered—as confirmed by the statistical tests. All four risk methods pass the conditional coverage and duration test. The EVT-VaR, however, fails the unconditional coverage test which is mostly due to the high deviation of the realized to the expected number of violations (77 vs. 58). The same observation holds true for the ES test. In contrast, the RM, the QR/ER and the CG model pass all tests for VaR and ES. In a nutshell, more sophisticated risk modelling techniques are found to dominate more naive ones in terms of statistically fitting the left tail of the portfolio return distribution.

[Table 3 about here]

4.3 The Economic Relevance of Risk Forecasting for Tail Risk Protection

We consider two steps when evaluating the various risk models in our tail risk protection framework. First, we analyze the historical path of each strategy. That is, assessing how each strategy would have performed when implemented over the whole out-of-sample period. For this, we assume an investment horizon of one calendar year—a typical choice of institutional and private investors alike (see Benartzi and Thaler, 1995). For the DPPI strategy, the floor is then adjusted to the current portfolio value at the start of each year to initialize the cushion. This procedure helps to mitigate the lock-in effect.

Due to the problem of path dependency that arises when analyzing the historical path, we additionally conduct a block-bootstrap approach[12] in the second step. Following Annaert, Van Osselaer, and Verstraete (2009); Bertrand and Prigent (2011); Dichtl and Drobetz (2011); Dichtl, Drobetz, and Wambach (2017), we draw blocks of 250 subsequent daily portfolio and risk-free returns on a rolling window basis and implement the tail risk protection strategies in each draw. Thus, we obtain 5,597 overlapping yearly returns as basis for the comparison of our methods. Intuitively, this approach enables us to assess a strategy's robustness with respect to alternative entry dates. Moreover, it enables the available data to be used in the most efficient way while preserving all dependency effects in the series, such as autocorrelation and conditional heteroskedasticity (see Dichtl and Drobetz, 2011).

As the objective of tail risk protection strategies is twofold—providing downside protection while still enjoying the upside potential of the risky portfolio—the performance should be evaluated accordingly. Alongside standard measures like the Sharpe Ratio and maximum drawdown we therefore employ specific downside risk measures commonly used in the literature of portfolio insurance such as the Calmar, Sortino or Omega ratios (see Bertrand and Prigent, 2011).[13]

4.3.1 Tail Risk Protection via Risk Targeting

Figure 3 illustrates the performance of the ES targeting strategy for the historical path and the block-bootstrap. Exposure is calculated based on the Copula-GARCH 1%-ES. The underlying is the synchronized multi-asset portfolio and we target an ES level of 1.5%, which is a reasonable assumption given the conservative underlying. Panel (a) shows the evolution of the protected portfolio, the underlying multi-asset strategy and a money market investment over the out-of-sample period from 1994 to 2017. We notably observe a decrease in exposure of the ES targeting strategy during the financial market crisis in 2008, at least avoiding the huge drawdowns of the underlying but reducing returns at the end of the sample period.

[Figure 3 about here]

[12]This method is sometimes referred to as historical simulation, see Dichtl and Drobetz (2011).

[13]While the *Calmar ratio* is defined as the ratio of annualized return over the absolute value of the maximum drawdown, the *Sortino ratio* is the difference of mean return and minimum acceptable return (here: zero) divided by downside deviation (that measures the variability of underperformance below a minimum target rate). The *Omega ratio* is calculated by dividing the upper partial moment of degree one by the lower partial moment of degree one. Lower (upper) partial moments indicate the return potential below (above) a predefined threshold return (here: zero). See Bertrand and Prigent (2011) for details on these performance risk measures.

Panel (b) shows the distribution of block-bootstrapped yearly returns of the protected portfolio in comparison with a pure buy-and-hold portfolio investment strategy. We see that the distribution of the ES targeting strategy is shifted to the right, thus reducing the mass in the left tail. However, this reduction comes at the cost of some return potential in the upper right tail.

Table 4 complements the previous chart with the estimation results of the ES targeting strategy based on all different 1%-ES forecasts for the historical path and block-bootstrap. Panel A reports the results for the historical path. We find similar size of risk-adjusted returns (cf. Sharpe ratio), but lower maximum drawdown and thus higher Calmar ratio for all risk methods compared to the underlying. These figures confirm the ability of the ES targeting strategy to reduce downside risk. Comparing across the risk methods, we obtain the best risk-adjusted performance of the extreme value theory (EVT) and Copula-GARCH (CG) approach measured in Sharpe ratio. With regard to downside risk measures like the Calmar ratio, both methods are joined by the RiskMetrics (RM) and the expectile regression (ER) approach that perform equally well. Thus, our results indicate that the ES targeting strategy is more profitable when using the more flexible methods RM, EVT, ER and CG. This finding is confirmed by the finding of the block-bootstrap method shown in Panel B. We reveal a higher Omega ratio for the EVT, ER and CG approach (5.63, 5.56 and 5.76) compared to the one for the historical simulation (HS) method (4.87), for example. Also in terms of the Sharpe and Sortino ratios, those three approaches outperform all other risk methods.

[Table 4 about here]

As the results of the ES targeting strategy may be sensitive to the static allocation of portfolio weights and choice of the risk target, we also investigate the strategy using different underlying portfolios—a pure equity, a pure bond, a 30/70 equity/bond and a 60/40 equity/bond portfolio in addition to the multi-asset portfolio—and various ES target levels (1%, 1.25%, 1.5%, 1.75%, 2%). Table 5 reports the corresponding results. Assuming a risk target level of 1.5%, we see that for all portfolios the more flexible methods show a higher Calmar ratio (cf. Panel A1 and A2). The same holds for the robustness to the risk target level (cf. Panel B1 and B2). For almost all levels (except 1%) and for both, historical path and block-bootstrap, we find superior performance of the EVT and CG forecasting approach measured by the mean of the yearly Calmar ratios.

[Table 5 about here]

4.3.2 Tail Risk Protection via DPPI

We have seen that the ES targeting strategy is able to mitigate downside risk to some extent, but that it fails to clearly reduce measures such as the maximum drawdown. A stricter way to limit downside risk is the DPPI strategy. Panel (a) of Figure 4 illustrates how the mechanism of a DPPI strategy generally works. The chart shows the performance of the conservative multi-asset portfolio using the DPPI strategy in relation to the floor over time. The investment exposure is mainly driven by two components: the floor and the multiplier. If the portfolio value of the underlying approaches the floor line, i.e. the cushion shrinks, the exposure is reduced and parts of the investment are shifted into the risk-free asset. Similarly, the exposure is reduced if risk estimates predict too high (overnight) risk, i.e. the multiplier decreases given that the distance to the floor is not excessive. In this example, the conditional multiplier is based on the CG-ES at 1% confidence.[14]

[Figure 4 about here]

Examining the whole sample period, we learn that the DPPI strategy did indeed prevent catastrophic drawdowns. With the onset of the global financial crisis, investment exposure drops to zero, so that the portfolio value at the end of 2008 is equal to the floor. Then, even with the V-shaped market evolution (sudden decline followed by a rapid recovery) in early 2009—a major impediment for portfolio insurance—the DPPI portfolio does not end up in a "cash lock". It participates in at least part of the subsequent recovery. On the whole, the DPPI portfolio has an average investment exposure of approximately 60% to 90%, depending on the chosen risk method, and delivers slightly lower returns compared to the pure multi-asset portfolio (cf. Table 6). However, risk-adjusted the results are clearly in favour of the DPPI portfolio. This relative advantage remains when considering downside risk measures. The lower maximum drawdown of the DPPI portfolio evidences that downside protection is effective—irrespective of the choice of the risk method. Comparing the performance of the DPPI portfolio across risk models yields less clear-cut results. Panel A of Table 6 shows the corresponding results. Analyzing returns we find a 76bp-difference between the best performing risk model, the CG model and the weakest model, the RM approach. However, when risk adjusting returns, this spread diminishes so that we observe only marginal differences across the different models. In particular, the Sharpe ratio ranges from 0.61 to 0.68. The same conclusions can be drawn in terms of maximum drawdown. Evaluating the risk models on the basis of the Calmar, Sortino and Omega ratios shows only marginal differences. The ranges from 0.38 to 0.46, around 0.10, and from 1.20 to 1.21 indicate that not only sophisticated but also more naive approaches are able to provide downside protection in the context of DPPI. The best performing model is the Copula-GARCH approach (Calmar ratio of 0.46). This finding can be rationalized as follows: in general, few allocation changes are necessary to protect from downside risks if the DPPI strategy is reasonably calibrated. In particular, the investment exposure will be decreased when approaching the floor, irrespective of the underlying risk forecast. This embedded line of defense is most likely

[14] In order to reflect the preferences of risk-averse investors, we follow Soupé, Heckel, and de Carvalho (2014) and scale the risk forecast by a term consisting of an investor's risk aversion parameter and the expected Sharpe ratio given a Constant Relative Risk Aversion (CRRA) utility function of the investor. Specifically, assuming a risk-averse investor we set the risk aversion parameter to 0.15 and the expected Sharpe ratio to 0.6.

preventing less accurate risk forecasts from inhibiting overall performance. As a result, any DPPI strategy dominates the underlying risky portfolio when evaluating Calmar, Sortino and Omega ratios.

[Table 6 about here]

Similar to several studies (Bertrand and Prigent, 2002; Ameur and Prigent, 2006; Hamidi, Jurczenko, and Maillet, 2009; Ameur and Prigent, 2014; Hamidi, Maillet, and Prigent, 2014), we also benchmark the DPPI performance with multipliers based on the different risk models against the CPPI performance based on a static unconditional multiplier (FM). In particular, the latter is calculated as the maximum loss of the underlying over the whole sample period, resulting in a multiplier of 8. In terms of downside measures, FM shows slightly better results than the competing DPPI strategies owing to a more defensive investment exposure (approx. 60%). However, there is a severe performance drag relative to the DPPI strategies: the static multiplier underperforms in terms of returns (5.5% vs. approx. 6.5%).

The analysis of the block-bootstrap method confirms our findings. Panel (b) of Figure 4 shows the distribution of the block-bootstrapped yearly returns of the DPPI strategy. For comparison, we also include the return distribution of a pure buy-and-hold portfolio investment strategy. The chart clearly highlights the effect of portfolio insurance. The left tail of the return distribution is shifted towards the floor level such that downside risk is reduced—albeit at the expense of some return potential in the right tail. Panel B in Table 6 reports the corresponding performance statistics. Compared to the historical path, we obtain slightly different results. Concerning the performance of the underlying, we learn that the Sortino and Omega ratios increase substantially for all strategies. This finding can be explained by the fact that the massive drawdown year 2008 loses weight when performing the block-bootstrap. In other words, the crisis year 2008 is "averaged out" to some extent. Again, we find only marginal differences across risk methods. In essence, the results support the conclusion drawn from the historical path analysis. Dynamic proportion portfolio insurance strategies building on sophisticated risk models do a good job in protecting investors from downside risk. Given that the mechanics of the portfolio insurance strategy automatically reduce investment exposure when approaching the protection level, a less sophisticated risk forecast is mostly profiting from this second line of defense.

Again, we provide robustness checks with respect to the allocation of portfolio weights and the level of the floor. We use the same portfolios as in the robustness check of the ES targeting strategy and employ the following floor levels: 93%, 94%, 95%, 96% and 97%. The corresponding results are shown in Table 7. Assuming a floor level of 95% we find no significant differences across the risk methods for all portfolios (cf. Panel A1 and A2). The same holds for the robustness to the floor level (cf. Panel B1 and B2).

[Table 7 about here]

5 Conclusion

Tail risk protection strategies are an effective way to limit downside risk of a given investment portfolio while maintaing most of its upside return potential. Given the limitations of option-based hedging strategies, dynamic asset allocations strategies such as the risk targeting and the dynamic proportion portfolio insurance (DPPI) strategy are popular choices among practitioners. As the success of both strategies strongly depends on the success in forecasting (tail) risk, this paper investigates a number of forecasting models to generate portfolio risk estimates that are especially suitable in timely managing the investment exposure of these strategies. To this end, we analyze risk models both prominent in the academic literature and popular among practitioners—from simple historical simulation, the RiskMetrics approach and the Cornish-Fisher Approximation, to quantile/expectile regression, extreme value theory and the Copula-GARCH approach. Empirically, we build our analysis on a global multi-asset return data set including stocks, bonds, commodities and foreign exchange rates. To take account of different market closing times we apply a return synchronization technique by extrapolating prices of closed markets, based on information from markets which close later. It turns out that the risk forecasts of the more flexible methods, such as the quantile/expectile regression approach or the Copula-GARCH method, dominate the more naive approaches in terms of statistical fit. The same holds when feeding these risk forecasts into the risk targeting strategy. For the DPPI strategy, however, we find less clear-cut results. We evidence that dynamic portfolio insurance strategies building on sophisticated risk models are capable of protecting investors from downside risk. However, more naive approaches are also able to provide downside protection. Given that portfolio insurance only leads to few allocation changes, simple risk models might have simply been lucky. Going forward a more accurate quantile/expectile regression or Copula-GARCH approach appears to be more likely to help mitigate the next downturn.

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Appendix A Return Synchronization

In this section, we describe the return synchronization methodology that we apply to the global multi-asset data set (see Audrino and Bühlmann, 2004).

Let $S_{t_i,i}$ denote the continuous time price of asset i $(i = 1, ..., N)$, where time t_i is the closing time of market i measured in local time of the base market, i.e. the market with which to synchronize to. The corresponding synchronized price $S^s_{t,i}$ is then defined as

$$\log\left(S^s_{t,i}\right) = \mathbb{S}\left[\log\left(S_{t,i}\right)|\mathcal{F}_t\right] = \mathbb{E}\left[\log\left(S_{t_i+1,i}\right)|\mathcal{F}_t\right], \quad t_i \leq t \leq t_i + 1 \ (t \in \mathbb{N}), \tag{32}$$

where $t = t_1$ and $\mathcal{F}_t$ is the complete information of all recorded prices up to time t. The logarithms are used to be consistent with continuously compounded returns. Clearly, if the closing price S is observed at time $t \in \mathbb{N}$, its conditional expectation given $\mathcal{F}_t$ is the observed price. This is the case for the assets from the base market. If the market closes before t, its past prices and all the other markets may be useful in predicting S at time t. As a simplifying approximation, the authors therefore assume that, given the information $\mathcal{F}_t$, the best predicted log-prices at t and at the nearest succeeding closing time $t_i + 1$ remain the same, saying that future changes up to $t_i + 1$ are unpredictable.

Then, we denote r_t as the vector of log-returns in different markets using the multi index $t = (t_1, t_2, \cdots, t_N)$ and define the synchronized returns r^s_t as the change in the logarithms of the synchronized prices:

$$r_t = \begin{bmatrix} \log\left(\frac{S_{t_1,1}}{S_{t_1-1,1}}\right) \\ \vdots \\ \log\left(\frac{S_{t_N,N}}{S_{t_N-1,N}}\right) \end{bmatrix}, \qquad r^s_t = \begin{bmatrix} \log\left(\frac{S^s_{t,1}}{S^s_{t-1,1}}\right) \\ \vdots \\ \log\left(\frac{S^s_{t,N}}{S^s_{t-1,N}}\right) \end{bmatrix}. \tag{33}$$

In order to estimate the relationship between the individual asset markets, the authors employ a simple "auxiliary" VAR(1) model:

$$r_t = \mathbf{A} r_{t-1} + \varepsilon_t, \tag{34}$$

where the innovations ε_t are i.i.d.$\sim \mathcal{N}(\mathbf{0}, \Sigma)$, independent from $\{r_s; s < t\}$, and $\mathbf{A}$ is the matrix of VAR coefficients. We can then derive the synchronized returns as follows

$$\begin{aligned} r^s_t &= \log\left(\mathbf{S}^s_t\right) - \log\left(\mathbf{S}^s_{t-1}\right) \\ &= \mathbb{E}\left[\log\left(\mathbf{S}_{t+1}\right)|\mathcal{F}_t\right] - \mathbb{E}\left[\log\left(\mathbf{S}_t\right)|\mathcal{F}_{t-1}\right] \\ &= \mathbb{E}\left[\log\left(\mathbf{S}_{t+1}\right) - \log\left(\mathbf{S}_t\right)|\mathcal{F}_t\right] - \mathbb{E}\left[\log\left(\mathbf{S}_t\right) - \log\left(\mathbf{S}_{t-1}\right)|\mathcal{F}_{t-1}\right] + \log\left(\frac{\mathbf{S}_t}{\mathbf{S}_{t-1}}\right) \\ &= \mathbb{E}\left[r_{t+1}|\mathcal{F}_t\right] - \mathbb{E}\left[r_t|\mathcal{F}_{t-1}\right] + r_t \end{aligned}$$

$$= \mathbf{A}r_t - \mathbf{A}r_{t-1} + r_t$$
$$= r_t + \mathbf{A}(r_t - r_{t-1}). \tag{35}$$

That is, any synchronized return r_t^s is still anchored in the actual realized return r_t plus an anticipated innovation according to the estimated VAR-relation as captured in matrix $\mathbf{A}$. The "missing" dynamics of markets closing early in the day are thus proxied according to the short-term relationship with respect to those markets closing later that day.

Sorting markets according to their closing times makes it possible to readily formulate a restriction matrix for the VAR model such that markets are explained only by those markets with a later closing time. Given that U.S. markets are the last to close in our sample, we anchor our synchronization of daily returns in U.S. markets. Thus, the U.S. time series remain unchanged but are still included in the VAR model to serve as explanatory markets, i.e. the final set of synchronized daily returns does not build on forecasted time series for the U.S. but uses their original daily returns. Non-U.S. data is, however, forecasted to the closing time of the U.S. market by the VAR(1).

Table 1

Descriptive Statistics and Test Portfolio Allocations

This table reports the descriptive statistics of the daily log-returns of the individual assets and test portfolios over the period 2/1/1991 to 31/3/2017 (including 6,847 observations). The following statistics are reported: mean, median (Med), minimum (Min), maximum (Max), standard deviation (Sd), skewness (Skew) and kurtosis (Kurt). All statistics are given in percentage, except skewness and kurtosis. In addition, we provide the static weights of the test portfolio allocations (Multi-Asset (MA), Equity (EQ), Bond (BO), 30/70 equity/bond (30-70), 60/40 equity/bond (60-40)) in percentage in the last five columns.

								Portfolio weights				
	Mean	Med	Min	Max	Sd	Skew	Kurt	MA	EQ	BO	30-70	60-40
Individual assets												
A. Stocks												
Nikkei 225	-0.00	0.00	-14.0	18.82	1.51	-0.20	119.32	5	9.8	0	2.94	5.88
Euro STOXX 50	0.03	0.05	-9.44	11.38	1.39	-0.12	88.46	5	12.5	0	3.75	7.50
MSCI EM	0.02	0.08	-9.99	10.07	1.14	-0.52	108.25	5	15.2	0	4.56	9.12
FTSE 100	0.02	0.00	-9.70	9.58	1.13	-0.15	86.28	5	9.9	0	2.97	5.94
S&P 500	0.02	0.03	-10.4	13.20	1.13	-0.15	142.71	15	52.6	0	15.78	31.56
B. Bonds												
JGB 10Y	0.01	0.00	-1.55	2.18	0.25	-0.28	82.85	10	0	10	7	4
Euro Bund	0.02	0.01	-1.73	1.96	0.33	-0.19	48.7	10	0	20	14	8
UK Gilt	0.01	0.00	-2.34	3.65	0.41	0.06	62.65	10	0	10	7	4
US 10Y	0.01	0.00	-2.63	3.53	0.37	-0.10	62.63	10	0	40	28	16
C. Commodities												
Oil	0.00	0.00	-38.4	13.34	2.16	-0.95	206.1	5	0	0	0	0
Gold	0.02	0.00	-9.81	8.84	1.01	-0.17	112.25	5	0	0	0	0
Copper	0.02	0.00	-11.7	11.65	1.61	-0.19	76.55	5	0	0	0	0
D. Foreign exchange rates												
EUR/USD	-0.00	0.00	-3.38	3.93	0.62	0.04	52.18	15	0	20	14	8
GBP/USD	-0.01	0.00	-7.94	5.24	0.60	-0.49	113.7	15	0	10	7	4
JPY/USD	0.00	0.00	-4.07	7.06	0.68	0.46	83.9	15	0	10	7	4
Asset portfolios												
Multi-asset	0.02	0.03	-3.83	3.67	0.46	-0.27	93.07	-	-	-	-	-
Equity	0.02	0.06	-8.42	10.24	0.93	-0.32	130.9	-	-	-	-	-
Bond	0.02	0.02	-1.93	1.98	0.36	-0.12	47.99	-	-	-	-	-
30/70	0.02	0.03	-2.72	2.76	0.34	-0.15	79.18	-	-	-	-	-
60/40	0.02	0.04	-5.12	6.04	0.55	-0.24	124.77	-	-	-	-	-
3-M US T-Bill	0.01	0.01	-0.00	0.02	0.01	0.06	13.86	-	-	-	-	-

Table 2

Synchronized vs. Original Daily Returns

This table reports daily return and daily volatility of the synchronized and original daily returns based on the international multi-asset data set. All figures are given in percentage.

	Nikkei 225	JGB10Y	Euro Bund	UK Gilt	EURO STOXX 50	FTSE 100
Mean original returns	-0.0035	0.0136	0.0165	0.0149	0.0305	0.0178
Mean synchronized returns	-0.0034	0.0136	0.0165	0.0149	0.0307	0.0179
Volatility original returns	1.5109	0.2486	0.3341	0.4098	1.3875	1.1306
Volatility synchronized returns	1.6338	0.2561	0.3581	0.4386	1.5829	1.2916

Table 3

Results of VaR and ES Testing

This table shows the results of the unconditional coverage test, the conditional coverage test, the duration test, and the ES test of McNeil and Frey to evaluate the risk forecasts of the different risk models—Historical Simulation (HS), RiskMetrics (RM), Cornish-Fisher Approximation (CFA), Quantile Regression (QR)/Expectile Regression (ER), Extreme Value Theory (EVT) and Copula-GARCH (CG). We report the number of realized VaR violations, the p-value and the test decision using the synchronized multi-asset portfolio over the period 3/11/1994 to 31/3/2017. The test decision is positive (✓) if the p-value is greater than the confidence level of 5%, and is conversely negative (x), if not. We calculate the VaR and ES at 1% confidence level and expect 58 violations over the whole out-of-sample period.

	HS	RM	CFA	QR/ER	EVT	CG
Realized violations	60	67	46	66	77	70
VaR tests						
a) Test for unconditional coverage (H_0: correct violations)						
p-Value	0.84	0.27	0.09	0.33	0.02	0.14
Test decision	✓	✓	✓	✓	x	✓
b) Test for conditional coverage (H_0: correct & independent violations)						
p-Value	0.00	0.08	0.00	0.30	0.07	0.33
Test decision	x	✓	x	✓	✓	✓
c) Duration test (H_0: duration between violations have no memory)						
p-Value	0.00	0.57	0.00	0.92	0.86	0.57
Test decision	x	✓	x	✓	✓	✓
ES tests						
a) Test of McNeil and Frey (H_0: zero mean of excess conditional shortfalls)						
p-Value	0.22	0.99	0.06	0.84	0.03	0.61
Test decision	✓	✓	✓	✓	x	✓

Table 4

Risk Targeting for Multi-Asset Portfolio

This table shows the estimation results of the risk targeting strategy based on different 1%-ES forecasts for the historical path (Panel A) and block-bootstrap (Panel B) over the sample period 1994 to 2017 using a synchronized international multi-asset portfolio and a money market investment. The analyzed risk models are the Historical Simulation approach (HS), RiskMetrics (RM), Cornish-Fisher Approximation (CFA), Expectile Regression (ER), Extreme Value Theory (EVT) and Copula-GARCH (CG). For comparison, we include the performance of the underlying multi-asset portfolio (PF) and the money market investment (Cash). We target an ES of 1.5% over the whole out-of-sample period. For the historical path, all performance measures are calculated based on the daily returns resulting from the strategy. For the block-bootstrap, the performance measures are based on the bootstrapped yearly returns, except for *Mean Participation* and *Mean Turnover*. Those are based on the daily risky asset exposure of the corresponding draw and show the yearly mean of the specific measure.

			Risk targeting					
	PF	Cash	HS	RM	CFA	ER	EVT	CG
Panel A: historical path								
Return p.a. (in %)	7.04	2.57	6.13	5.36	6.24	6.33	6.57	6.48
Volatility p.a. (in %)	7.55	1.09	6.25	5.43	6.17	6.45	6.60	6.43
Sharpe ratio	0.59	0	0.57	0.51	0.59	0.58	0.61	0.61
Maximum drawdown (in %)	-31.80	-0.00	-26.38	-17.64	-26.17	-21.41	-24.11	-21.48
Calmar ratio	0.22	-	0.23	0.30	0.24	0.30	0.27	0.30
Sortino ratio	0.09	-	0.09	0.09	0.09	0.09	0.09	0.09
Omega ratio	1.18	-	1.18	1.18	1.19	1.18	1.18	1.18
Participation (in %)	100	0	88.48	82.39	87.99	93.79	96.26	95.12
Turnover (in %)	0.04	0	0.04	1.29	0.07	2.48	0.64	0.98
1% Value at Risk (in %)	1.28	0.00	1.08	0.94	1.04	1.10	1.11	1.06
1% Expected shortfall (in %)	1.81	0.00	1.45	1.18	1.41	1.35	1.39	1.32
Panel B: block-bootstrap								
Return p.a. (in %)	4.53	3.50	5.81	4.99	5.88	6.02	6.23	6.14
Volatility p.a. (in %)	8.97	3.27	8.81	7.97	8.60	8.38	8.39	8.18
Sharpe ratio	0.12	0	0.26	0.19	0.28	0.30	0.33	0.32
Mean calmar ratio	1.02	-	1.38	1.14	1.41	1.26	1.31	1.29
Sortino ratio	0.86	-	1.24	1.43	1.31	1.70	1.60	1.71
Omega ratio	3.45	-	4.87	4.74	5.18	5.56	5.63	5.76
Mean participation (in %)	100	0	89.50	81.72	87.47	93.75	96.11	94.92
Mean turnover (in %)	0	0	0.37	1.63	0.40	2.79	1.03	1.37
1% Value at Risk (in %)	26.07	-0.03	23.85	15.55	22.92	16.23	19.64	16.92
1% Expected shortfall (in %)	29.28	-0.02	25.66	16.25	24.58	19.01	21.58	18.73

Table 5

Risk Targeting: Various Portfolios and Target Levels

This table shows the estimation results of the risk targeting strategy based on different risk forecasts for the historical path and the block-bootstrap over the sample period 1994 to 2017 using various portfolios and various target levels. The analyzed risk models are the Historical Simulation approach (HS), RiskMetrics (RM), Cornish-Fisher Approximation (CFA), Expectile Regression (ER), Extreme Value Theory (EVT) and Copula-GARCH (CG). For comparison, we include the performance of the underlying (PF). To benchmark the results of the ES targeting strategy based on the multi-asset portfolio we estimate the ES targeting strategy also for four different test portfolio allocations. For this, we choose a pure equity, a pure bond, a 30/70 equity/bond, and a 60/40 equity/bond portfolio as underlying for the ES targeting strategy and an ES target level of 1.5%. The corresponding results are given in Panel A1 and A2. Moreover, we check the robustness of the results of the ES targeting strategy based on the multi-asset portfolio with respect to the chosen ES target level, here: 1%, 1.25%, 1.5%, 1.75% and 2%. The corresponding results are presented in Panel B1 and B2. We base our comparison on the Calmar ratio for the historical path and on the mean of the yearly Calmar ratios for the block-bootstrap.

	PF	HS	RM	CFA	ER	EVT	CG
Portfolios							
Panel A1: historical path							
Multi-asset	0.22	0.23	0.30	0.24	0.30	0.27	0.30
Equity	0.14	0.16	0.19	0.18	0.19	0.19	0.19
Bond	0.64	0.62	0.73	0.61	0.65	0.67	0.72
30-70	0.38	0.36	0.41	0.38	0.39	0.34	0.42
60-40	0.21	0.20	0.25	0.23	0.24	0.25	0.28
Panel A2: block-bootstrap							
Multi-asset	1.02	1.38	1.14	1.41	1.26	1.31	1.29
Equity	1.06	1.50	1.23	1.53	1.39	1.16	1.20
Bond	1.03	1.69	1.69	1.69	1.60	1.69	1.68
30-70	1.39	2.08	1.99	2.07	1.90	2.05	2.01
60-40	1.31	1.76	1.56	1.79	1.53	1.57	1.58
Target levels							
Panel B1: historical path							
1%	0.22	0.28	0.36	0.29	0.32	0.30	0.34
1.25%	0.22	0.25	0.32	0.26	0.30	0.28	0.32
1.5%	0.22	0.23	0.30	0.24	0.30	0.27	0.30
1.75%	0.22	0.22	0.30	0.23	0.29	0.26	0.29
2%	0.22	0.21	0.30	0.23	0.29	0.24	0.28
Panel B2: block-bootstrap							
1%	1.25	1.98	1.73	2.02	1.63	1.73	1.77
1.25%	1.25	1.87	1.71	1.90	1.67	1.74	1.74
1.5%	1.25	1.84	1.69	1.85	1.73	1.79	1.77
1.75%	1.25	1.83	1.70	1.84	1.79	1.81	1.80
2%	1.25	1.84	1.73	1.84	1.82	1.82	1.82

Table 6

DPPI for Multi-Asset Portfolio

This table shows the estimation results of the DPPI strategy with conditional multipliers based on different 1%-ES forecasts for the historical path (Panel A) and block-bootstrap (Panel B) over the sample period 1994 to 2017 using a synchronized international multi-asset portfolio and a money market investment. The analyzed risk models are the Historical Simulation approach (HS), RiskMetrics (RM), Cornish-Fisher Approximation (CFA), Expectile Regression (ER), Extreme Value Theory (EVT) and Copula-GARCH (CG). For comparison, we include a static multiplier (FM) based on the maximum loss of the portfolio returns (resulting in $m = 8$) as well as the performance of the underlying multi-asset portfolio (PF) and the money market investment (Cash). In each calendar year, a floor of 95% of the initial portfolio value is targeted. For the historical path, all performance measures are calculated based on the daily returns resulting from the strategy. For the block-bootstrap, the performance measures are based on the bootstrapped yearly returns, except for *Mean Participation* and *Mean Turnover*. Those are based on the daily risky asset exposure of the corresponding draw and show the yearly mean of the specific measure.

			CPPI	DPPI					
	PF	Cash	FM	HS	RM	CFA	ER	EVT	CG
Panel A: historical path									
Return p.a. (in %)	7.04	2.57	5.52	6.44	5.95	6.51	6.51	6.54	6.71
Volatility p.a. (in %)	7.55	1.09	4.33	5.83	5.56	5.83	6.10	6.16	6.14
Sharpe ratio	0.59	0	0.68	0.66	0.61	0.68	0.65	0.64	0.67
Maximum drawdown (in %)	-31.80	-0.00	-9.94	-14.38	-13.90	-14.36	-16.91	-14.96	-14.65
Calmar ratio	0.22	-	0.55	0.45	0.43	0.45	0.38	0.44	0.46
Sortino ratio	0.09	-	0.12	0.10	0.10	0.10	0.10	0.10	0.10
Omega ratio	1.18	-	1.25	1.21	1.21	1.21	1.20	1.20	1.21
Participation (in %)	100	0	62.22	85.53	82.16	85.69	88.16	90.12	89.56
Turnover (in %)	0.91	0	0.91	0.86	1.07	0.88	1.69	0.68	0.86
1% Value at Risk (in %)	1.28	-0.00	0.73	0.98	0.95	0.98	1.04	1.05	1.05
1% Expected shortfall (in %)	1.81	0.00	0.93	1.26	1.24	1.26	1.32	1.34	1.33
Panel B: block-bootstrap									
Return p.a. (in %)	4.53	3.50	5.75	6.40	5.88	6.40	6.35	6.38	6.38
Volatility p.a. (in %)	8.97	3.27	7.06	8.06	7.82	8.05	8.13	8.06	8.06
Sharpe ratio	0.12	0	0.32	0.36	0.30	0.36	0.35	0.36	0.36
Mean calmar ratio	1.25		1.64	1.74	1.67	1.74	1.73	1.74	1.74
Sortino ratio	0.86	-	3.10	2.41	2.31	2.44	2.36	2.34	2.39
Omega Ratio	3.45	-	8.89	6.38	6.05	6.43	6.09	6.16	6.23
Mean participation (in %)	100	0	74.49	90.33	86.73	89.99	90.61	91.11	90.82
Mean turnover (in %)	0.00	0	1.13	0.81	1.09	0.85	1.29	0.79	0.89
1% Value at Risk (in %)	26.07	-0.03	8.24	9.30	8.48	9.31	9.01	8.73	8.60
1% Expected shortfall (in %)	29.28	-0.02	8.44	9.61	8.67	9.56	9.21	9.01	8.84

Table 7

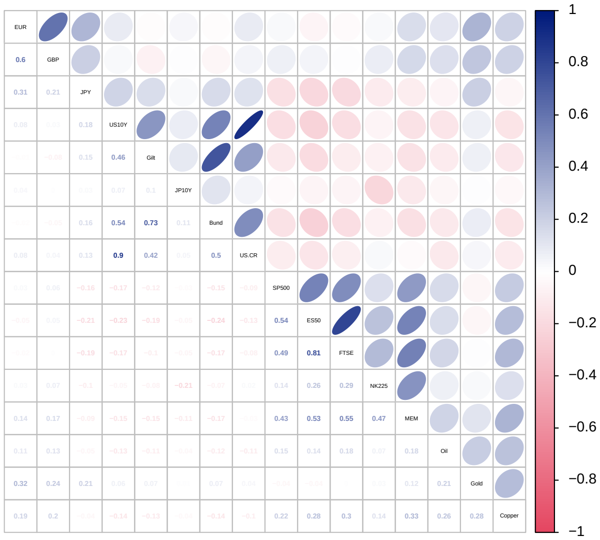
DPPI: Various Portfolios and Floors

This table shows the estimation results of the DPPI strategy with conditional multipliers based on different risk forecasts for the historical path and the block-bootstrap over the sample period 1994 to 2017 using various portfolios and various floor levels. The analyzed risk models are the Historical Simulation approach (HS), RiskMetrics (RM), Cornish-Fisher Approximation (CFA), Expectile Regression (ER), Extreme Value Theory (EVT) and Copula-GARCH (CG). For comparison, we include the performance of the underlying (PF). To benchmark the results of the DPPI strategy based on the multi-asset portfolio we also estimate the DPPI strategy for four different test portfolio allocations. To this end, we choose a pure equity, a pure bond, a 30/70 equity/bond and a 60/40 equity/bond portfolio as underlying for the DPPI strategy and a floor level of 95%. The corresponding results are given in Panel A1 and A2. Moreover, we check the robustness of the results of the DPPI strategy based on the multi-asset portfolio with respect to the chosen floor level, here: 93%, 94%, 95%, 96%, and 97%. The corresponding results are presented in Panel B1 and B2. We base our comparison on the Calmar ratio for the historical path and on the mean of the yearly Calmar ratios for the block-bootstrap.

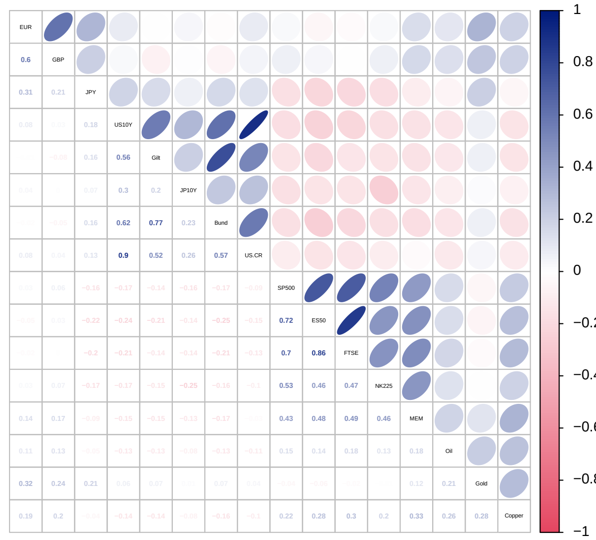
	PF	HS	RM	CFA	ER	EVT	CG
Portfolios							
Panel A1: historical path							
Multi-asset	0.22	0.45	0.43	0.45	0.38	0.44	0.46
Equity	0.14	0.37	0.39	0.37	0.34	0.43	0.41
Bond	0.64	0.56	0.61	0.54	0.52	0.61	0.64
30-70	0.38	0.58	0.67	0.60	0.56	0.52	0.59
60-40	0.21	0.34	0.32	0.36	0.36	0.34	0.36
Panel A2: block-bootstrap							
Multi-asset	1.25	1.74	1.67	1.74	1.73	1.74	1.74
Equity	1.06	1.15	1.03	1.18	1.09	0.94	0.94
Bond	1.03	1.59	1.58	1.57	1.51	1.60	1.60
30-70	1.39	2.05	2.05	2.05	2.04	2.05	2.05
60-40	1.31	1.55	1.47	1.56	1.53	1.55	1.56
Floors							
Panel B1: historical path							
93%	0.22	0.40	0.39	0.40	0.40	0.39	0.41
94%	0.22	0.42	0.41	0.42	0.42	0.42	0.43
95%	0.22	0.45	0.43	0.45	0.38	0.44	0.46
96%	0.22	0.52	0.45	0.52	0.41	0.41	0.45
97%	0.22	0.62	0.50	0.61	0.43	0.42	0.52
Panel B2: block-bootstrap							
93%	1.25	1.77	1.74	1.77	1.78	1.78	1.78
94%	1.25	1.76	1.71	1.76	1.76	1.76	1.76
95%	1.25	1.74	1.67	1.74	1.73	1.74	1.74
96%	1.25	1.71	1.63	1.71	1.70	1.70	1.70
97%	1.25	1.66	1.58	1.66	1.65	1.65	1.66

Figure 1. The Effects of Return Synchronization

This figure shows the correlation matrices of the original and synchronized returns based on the international multi-asset data set including stocks, bonds, commodities and foreign exchange rates. Blue shade indicates positive correlations, red shade negative correlations. The more straight the circles, the higher the correlation.



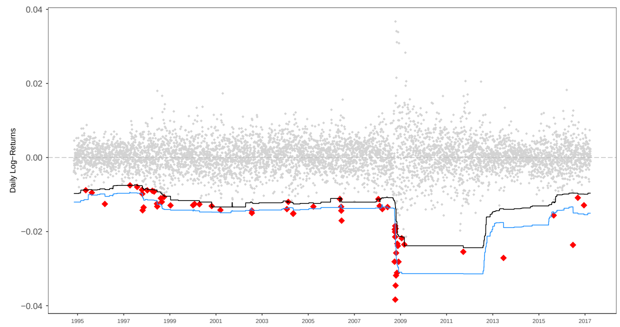
(a) Original returns



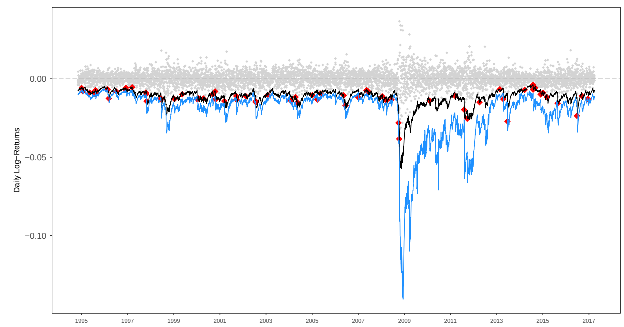
(b) Synchronized returns

Figure 2. VaR and ES Forecasts over Time

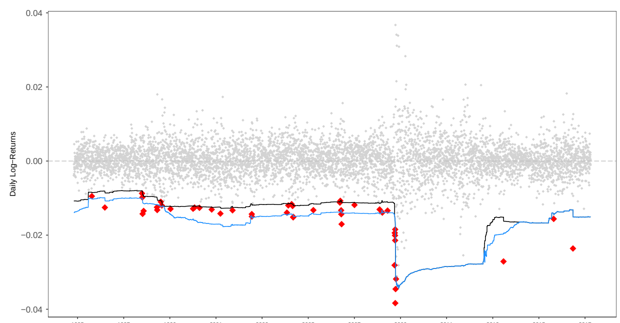
This figure shows the (negative) daily 1% VaR forecasts (in black) and associated ES forecasts (in blue) of the different risk models and the realized returns of the multi-asset portfolio (grey dots) over the period 3/11/1994 to 31/3/2017. VaR violations are marked in red. At a confidence level of 1%, a total of 58 violations are expected over the model period.



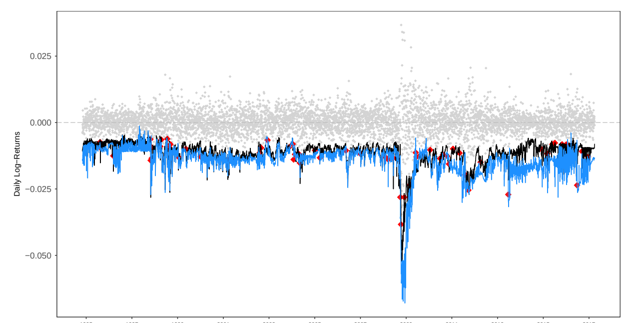
(a) HS



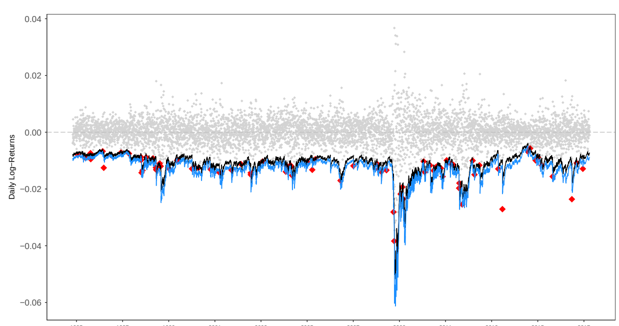
(b) RM



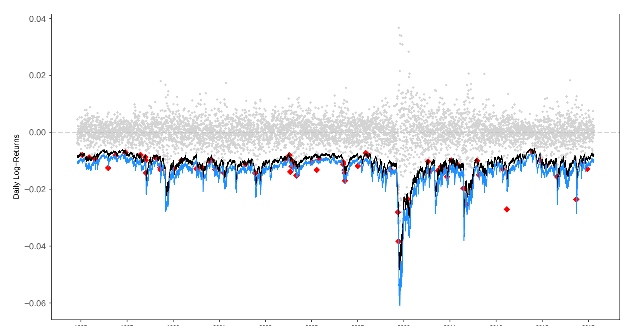
(c) CFA



(d) QR/ER



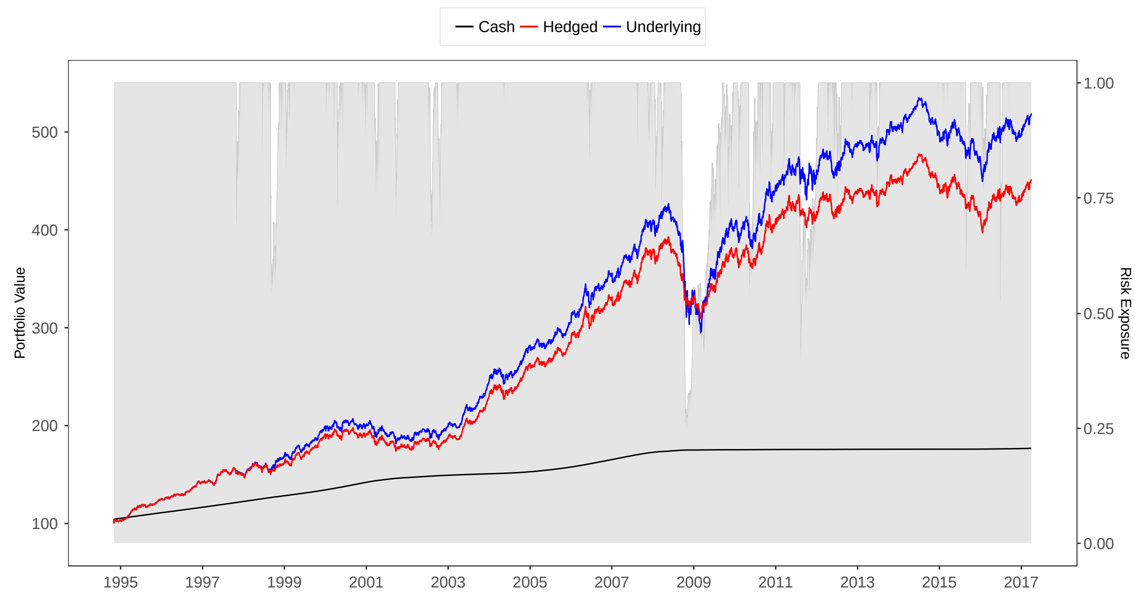
(e) EVT



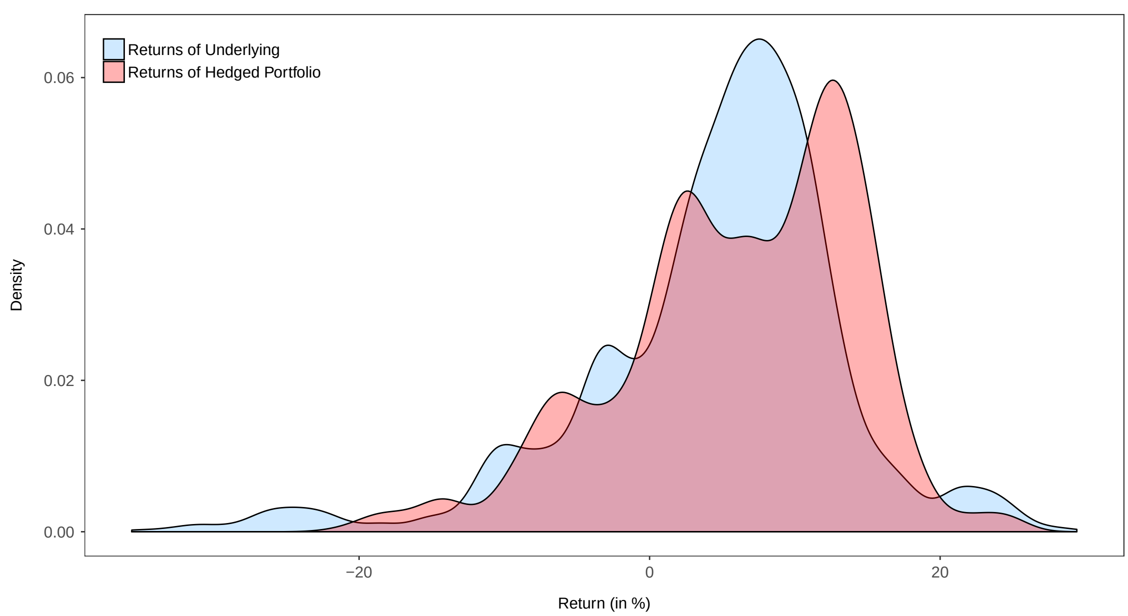
(f) Copula-GARCH

Figure 3. Historical Path and Block-Bootstrap of Risk Targeting

This chart illustrates the performance of the multi-asset portfolio (35% equities, 40% fixed income, 15% commodities, 45% currencies) using the ES targeting strategy. Panel (a) shows the historical path of the protected portfolio (red line) over the sample period 1994 to 2017. Exposure is calculated based on the Copula-GARCH 1%-ES. For comparison, we include the performance of the underlying multi-asset strategy (blue line) and a money market investment (black line). Panel (b) shows the distribution of block-bootstrapped yearly returns of the protected portfolio (red shade) and a pure buy-and-hold portfolio investment strategy (blue shade). For both approaches we target a 1.5% ES.



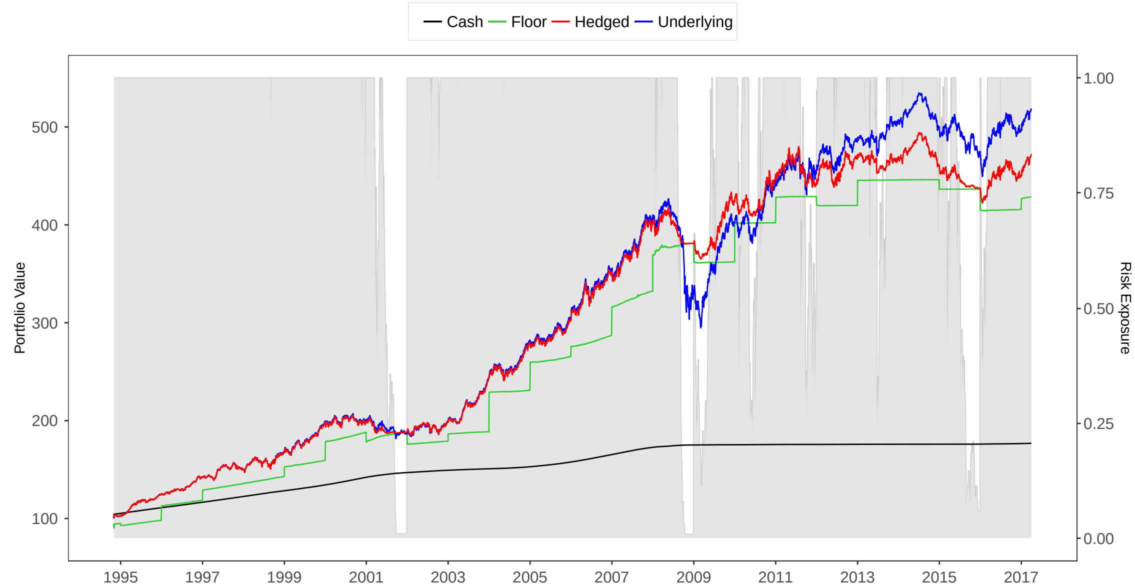
(a) Historical path



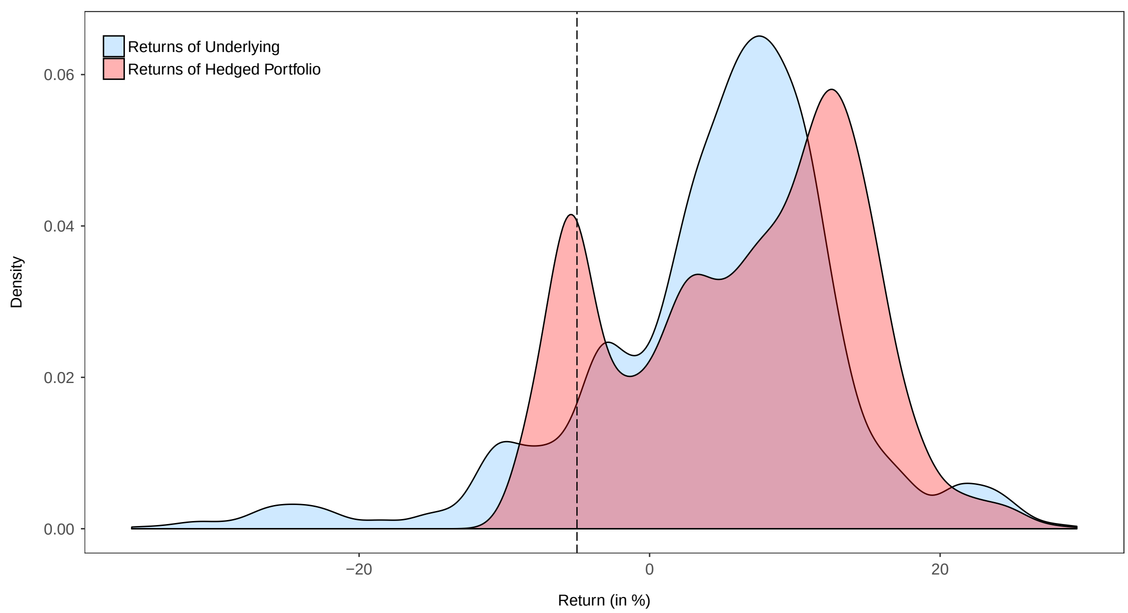
(b) Block-bootstrap

Figure 4. Historical Path and Block-Bootstrap of DPPI

This chart illustrates the performance of the multi-asset portfolio (35% equities, 40% fixed income, 15% commodities, 45% currencies) using the DPPI strategy based. Panel (a) shows the historical path of the protected portfolio (red line) in relation to the floor (green line) over the sample period from 1994 to 2017. Exposure is calculated based on the Copula-GARCH 1%-ES. We assume a floor level of 95%. For comparison, we include the performance of the underlying multi-asset strategy (blue line) and a money market investment (black line). Panel (b) shows the distribution of block-bootstrapped yearly returns of the protected portfolio (red shade) and the pure buy-and-hold portfolio investment strategy (blue shade). The black dashed line indicates the floor level.



(a) Historical path



(b) Block-bootstrap