Tail Risks, Asset prices, and Investment Horizons*

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June 15, 2018

Abstract

We examine how extreme market risks are priced in the cross-section of asset returns at various horizons. Based on the frequency decomposition of covariance between indicator functions, we define the quantile cross-spectral beta of an asset capturing tail-specific as well as horizon-, or frequency-specific risks. Further, we work with two notions of frequency-specific extreme market risks. First, we define tail market risk that captures dependence between extremely low market as well as asset returns. Second, extreme market volatility risk is characterized by dependence between extremely high increments of market volatility and extremely low asset return. Empirical findings based on the datasets with long enough history, 30 Fama-French Industry portfolios, and 25 Fama-French portfolios sorted on size and book-to-market support our intuition. Results suggest that both frequency-specific tail market risk and extreme volatility risks are significantly priced and our five-factor model provides improvement over specifications considered by previous literature.

Keywords: Asset pricing, downside risk, frequency-specific risk, tail risk
JEL: C21; C58; G12

1 Introduction

Classical result of asset pricing literature states that price of an asset should be equal to its expected discounted payoff. In the Capital Asset Pricing Model (CAPM) introduced by Sharpe (1964), Lintner (1965), Black (1972), we assume that stochastic discount factor can be approximated by return on market portfolio and thus expected excess returns can be fully described by their market betas based on covariance between asset return and market return. Yet, decades of the consequent research show that we are unable to sufficiently explain the cross-section of asset returns with this notion. Instead, literature calls for more accurate characterization of risks associated with assets that will better reflect preferences of investors. We aim to show that in order to understand formation of expected returns, one has to look into some special features of asset returns that are crucial in terms of preferences of a representative investor. We argue that the two important features are risk related to tail events, and frequency-specific risk.

*Note that this is the first draft of the paper.

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To characterize the risks, we derive novel quantile cross-spectral representation of beta. Our work nests classical representation that simply averages beta with equal weights over different quantile levels, as well as frequencies.

Economists has long recognized that decisions under risk are more sensitive to changes in probability of possible extreme events compared to probability of a typical event. The expected utility might not reflect this behavior since it weights probability of outcomes linearly. Consequently, CAPM beta as an aggregate measure of risk may fail to explain the cross-section of asset returns. Several alternative notions emerged in the literature. Mao (1970) presents survey evidence showing that decision makers tend to think of risk in terms of the possibility of outcomes below some target. For an expected utility maximizing investor, Bawa and Lindenberg (1977) has provided a theoretical rationale for using lower partial moment as a measure of portfolio risk. Based on the rank-dependent expected utility due to Yaari (1987), Polkovnichenko and Zhao (2013) introduce utility with probability weights and derive corresponding pricing kernel. More recently, Ang et al. (2006); Lettau et al. (2014) argue that downside risk – risk of negative returns – is priced across asset classes and is not captured by CAPM betas. Further, Farago and Tédongap (2017) extend the results using general equilibrium model based on generalized disappointment aversion and shows that downside risks in terms of market return and market volatility are priced in the cross-section of asset returns.[1]

The results described above leads us to question appropriateness of the expected utility maximizers in asset pricing. A recent strand of literature solves the problem by considering quantile of the utility instead of expectation. This literature complements the literature focusing on downside risk as it highlights the notion of economic agents particularly averse to outcomes below some threshold compared to outcomes above this threshold. The concept of a quantile maximizer and its features was proposed by Manski (1988), and later axiomatized by Rostek (2010). Most recently, de Castro and Galvao (2017) develop a model of quantile optimizer in a dynamic setting. A different approach to emphasizing investor's aversion towards least favorable outcomes defines theory based on Choquet expactations. This approach is based on distortion function that alters probability distribution of future outcomes by accentuating probabilities associated with least desirable outcomes. This approach was utilized in finance, for example, by Bassett Jr et al. (2004).

Whereas aggregating linearly weighted outcomes may not reflect the sensitivity of investors to tail risk, aggregating linearly weighted outcomes over various frequencies, or economic cycles may not reflect risk specific to different investment horizons. One can suspect that an investor cares differently about short-term and long-term risk according to their preferred investment horizon. Distinguishing between long-term and short-term dependence between economic variables was proven to be an insightful approach since the introduction of co-integration (Engle and Granger, 1987). Frequency decomposition of risk thus uncovers another important feature of risk which cannot be captured solely by market beta which captures risk averaged over all frequencies. This recent approach to asset pricing enables to shed light on asset returns and investor's behaviour from a different point of view highlighting heterogeneous preferences.

[1] In addition, it is interesting to note that equity and variance risk premium are also associated with compensation for jump tail risk (Bollerslev and Todorov, 2011). More general exploration of asymmetry of stock returns is provided by Ghysels et al. (2016), who propose a quantile-based measure of conditional asymmetry and show that stock returns from emerging markets are positively skewed. Conrad et al. (2013) use option price data and find a relation between stock returns and their skewness. Another notable approach uses high frequency data to define realized semivariance as a measure of downside risk (Barndorff-Nielsen et al., 2008). From a risk-measure standpoint, dealing with negative events, especially rare events, is highly discussed theme in both practice and academics. The most prominent example is Value-at-Risk (Adrian and Brunnermeier, 2016; Engle and Manganelli, 2004).

Empirical justification is brought by Boons and Tamoni (2015) and Bandi and Tamoni (2017) who show that exposure in long-term returns to innovations in macroeconomic growth and volatility of matching half-life is significantly priced in variety of asset classes. The results are based on decomposition of time series into components of different persistence proposed by Ortu et al. (2013). Piccotti (2016) further sets portfolio optimization problem into frequency domain using matching of utility frequency structure and portfolio frequency structure, and Chaudhuri and Lo (2016) present approach to constructing mean-variance-frequency optimal portfolio. This optimization yields mean-variance optimal portfolio for a given frequency band, and thus optimizes portfolio for a given investment horizon.

From a theoretical point of view, preferences derived by Epstein and Zin (1989) enables to study frequency aspects of investor's preferences, and quickly became a standard in the asset pricing literature. With the important results of Bansal and Yaron (2004), long-run risk started to enter asset pricing discussions. Dew-Becker and Giglio (2016) investigate frequency-specific prices of risk for various models and conclude that cycles longer than business cycle are significantly priced in the market. Other papers utilizes frequency domain and Fourier transform to facilitate estimation procedures for parameters hard to estimate using conventional approaches. Berkowitz (2001) generalizes band spectrum regression and enables to estimate dynamic rational expectations models matching data only in particular ways, for example, forcing estimated residuals to be close to white noise. Dew-Becker (2016) proposes spectral density estimator of long-run standard deviation of consumption growth, which is a key component for determining risk premiums under Epstein-Zin preferences, and shows its superior performance compared to the previous approaches. Crouzet et al. (2017) develop model of multi-frequency trade set in frequency domain and show that restricting trading frequencies reduces price informativeness at those frequencies, reduces liquidity and increases return volatility.

The debate clearly indicates that the standard assumptions leading to classical asset pricing models do not correspond with reality. In this paper, we suggest that more general pricing models have to be defined and they should take into consideration both asymmetry of dependence structure among stock market, and different behavior of investors at various investment horizons.

The main contribution of this paper is twofold. First, based on the frequency decomposition of covariance between indicator functions, we define the quantile cross-spectral beta of an asset capturing tail-specific as well as frequency-specific risks. The newly defined notion of beta can be viewed as disaggregation of a classical beta to a frequency-, and tail- specific beta. With this notion, we examine how extreme market risks are priced in the cross-section of asset returns at various horizons. We define frequency-specific tail market risk that captures dependence between extremely low market and asset returns, as well as extreme market volatility risk that is characterized by dependence between extremely high increments of market volatility and extremely low asset return. Second, based on the quantile cross-spectral betas, we define five-factor model that provides considerable improvements in explaining cross-section of asset returns. Results on a 30 Fama-French Industry portfolios, and 25 Fama-French portfolios sorted on size and book-to-market suggest extreme market risk is priced in cross section of asset returns and it is differently priced for long and short horizon. This extreme market risk is characterized by the risk of extremely low returns or extremely high volatility.

The rest of the paper has the following structure. Section 2 introduces concept of quantiles cross-spectral betas later employed in defining empirical models. Section 3 defines the empirical models used for testing significance of extreme risks. Section 4 conducts the empirical analysis of the extreme risks and provides definition of tested robustness checks. Section 5 concludes. In Appendix we report some robustness checks and give details on estimating quantile cross-spectral betas.

2 A notion of quantile-frequency risks

Our goal is to show that extreme risk is priced in cross-section of asset returns. Specifically, we focus on two types of extreme risk: tail market risk, and extreme volatility risk. Further, we are interested to decompose the tail risks into frequencies to be able to define short- and long-run extreme risks. We start the discussion with theoretical motivation followed by definition of quantile risk measure based on covariance between indicator functions, which has natural economic interpretation in terms of probabilities. Finally, we introduce frequency decomposition, and combine these two frameworks into quantile cross-spectral risk measure, which is the building blocks for our empirical model.

2.1 Quantile-specific (tail) risk

We propose to generalize the concept of Gul (1991) and his rational disappointment aversion utility function in the way that the strength of investor's preferences vary through the whole distribution of future wealth W, thus the utility can be characterized by the following functional form

$$U_{\mathbf{A},\mathbf{Q}}(W) = \frac{1}{K}\sum_{i=0}^{n} A_i \int_{Q_i}^{Q_{i+1}} u(w)dF(w) \tag{1}$$

where $U(W)$ is utility of a random variable W, $\mathbf{Q} \equiv (-\infty, Q_1, \ldots, Q_n, \infty)'$ is a column vector of predefined threshold values which characterizes different parts of distribution over which the preferences vary, $\mathbf{A} \equiv (1, A_1, \ldots, A_n)'$ is a column vector of preference weights, $u(w)$ is constant relative risk aversion utility function in the form $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$, $F(w)$ is a distribution function of wealth random variable W, and K is a standardizing constant obtained as

$$K = Pr(w \leq Q_1) + \sum_{i=2}^{n-1} A_i Pr(Q_i \leq W < Q_{i+1}) + A_n Pr(W \geq Q_n)$$

with $0 < A_i \leq 1$, $i = 1, \ldots, n$. Gul (1991) considers an investor who posses larger aversion to losses, relative to her attraction to gains, where losses are defined as future values of wealth below certainty equivalent and gains as outcomes of future wealth above certainty equivalent. We further weight the distribution of future wealth with different strength of aversion. Note that in case the weights are equal to 1, $\mathbf{A} = \mathbf{1}$, the utility $U(W)$ is in the form of classical expected utility $\mathbb{E}[U(W)]$ and is given by CRRA utility function $u(\mu_f)$ where μ_f is a certainty equivalent. On the other hand, it is reasonable to assume that investor is strongly averse to adverse outcomes of the future wealth, thus the weights she places decrease with the specified thresholds, i.e. $A_i > A_2 > \ldots > A_n$.

Thresholds $\mathbf{Q}$ may be set, for example, that they corresponds to predefined quantiles of the distribution of a random variable W, or as quantiles of some other reference random variable such as relative value of market portfolio. If the first threshold value is τ quantile and its corresponding weight is equal to 1 and the other weights are zero then we deal with quantile optimizer of Manski (1988). We can illustrate this notion on an example in which investor puts the most weight to the worst 5 percents of the cases of tis future wealth, less weight is put to the outcomes characterized by the 5 percent to 25 percent quantile of the future wealth, etc.

CAPM beta captures an average riskiness of an asset, but the behavior of an asset in extreme situations to which is an investor particularly averse is not fully captured. Thus if the investor posses this type of aversion, CAPM beta cannot in cross-section sufficiently explain average returns by its own. Thus we have to define broader risk measure which can capture behavior in these extreme situations.

Asset pricing theory assumes that risk premium of an asset or portfolio can be explained by its dependence structure with some reference economic or financial variable such as consumption growth or return on market portfolio. As discussed in Ang et al. (2006), if the investor's decisions are characterized by the rational disappointment utility function, classical covariance-based measure of dependence cannot fully explain asset prices. Hence, the most widely used measure of dependence between two variables $r_{t,i}$ and $r_{t,j}$, cross-covariance,

$$\gamma_k^{r_i,r_j} = \mathbb{C}ov\big(r_{t+k,i}, r_{t,j}\big) \equiv \mathbb{E}[(r_{t+k,i} - \bar{r}_i)(r_{t,j} - \bar{r}_j)], \tag{2}$$

is due to its averaging nature unable to describe asymmetry features of dependence structure between two variables unless the variables are jointly normal. If we want to measure dependence separately in different parts of a distribution - and obtain dependence measure in various parts of joint distribution, we have to employ more flexible measures. Since we are interested in pricing extreme negative events, we want to measure dependence and risk in lower quantiles of the joint distribution. We propose to use quantity of the following form

$$\gamma_k^{r_i,r_j}(\tau_{r_i}, \tau_{r_j}) \equiv \mathbb{C}ov\big(I\{r_{t+k,i} \leq q_{r_i}(\tau_{r_i})\}, I\{r_{t,j} \leq q_{r_j}(\tau_{r_j})\}\big), \tag{3}$$

where $r_{t,i}$ and $r_{t,j}$ are two time series of strictly stationary random variables, $q_X(\tau)$ is a quantile function of random variable X, $\tau_i, \tau_j \in (0,1)$, and $I\{A\}$ is indicator function of event A. The measure is given by the covariance between two indicator functions and can fully describe joint distribution of the pair of random variables r_i and r_j. If distribution functions of r_i and r_j are continuous, the quantity is essentially difference between copula of pair r_i and r_j and independent copula, thus the following quantity $Pr\big(r_{t+k,i} \leq q_{r_i}(\tau_{r_i}), r_{t,m} \leq q_{r_m}(\tau_{r_m})\big) - \tau_{r_i}\tau_{r_i m}$. Thus, covariance between indicators measures additional information from the copula over independent copula about how likely is that the series are jointly less or equal to their given quantiles. It enables to flexibly measure both cross-sectional structure and time-series structure of the pair of random variables.

The quantity introduced in Eq. 3 can be further generalized in the way that one can replace quantiles of respective variables by some general threshold values

$$\gamma_k^{r_i,r_j}(\tau_{r_i}, \tau_{r_j}) \equiv \mathbb{C}ov\big(I\{r_{t+k,i} \leq Q_{r_i})\}, I\{r_{t,j} \leq Q_{r_j}\}\big) \tag{4}$$

where Q_{r_i} and Q_{r_j} are general threshold values, which do not necessary need to be equal. These threshold values may be derived from distribution of reference variable. In our model we set threshold values to be equal and are derived from distribution of market returns, although we note that it is possible to work with different thresholds.

Since we are interested in explaining risk premiums of assets, we follow the usual setting and denote returns of some asset or portfolio i as $r_{i,t}$, and returns of market portfolio denoted as $r_{m,t}$.

2.2 Frequency-specific risk

It is natural to think that economic agents care differently about long-, medium-, and short-term investment horizon in terms of expected returns and associated risk. Investors may instead be interested in long-term profitability of their portfolio and do not concern with short-term fluctuations. Frequency-dependent features of an asset return play an important role for an investor. Bandi and Tamoni (2017) argues that covariance between two returns can be decomposed into covariance between disaggregated components evolving over different time scales, and thus the risk on these components can vary. Hence, market beta can be decomposed into linear combination of betas measuring dependence at various scales, i.e.dependence between fluctuations with various half-lives. Frequency specific risk at given time plays an important role for determination of asset prices, and the price of risk in various frequency bands may differ, i.e. the expected return can be decomposed into linear combination of risks in various frequency bands.

The most simple and natural way how to decompose covariance between two assets is via its Fourier and inverse Fourier transform. Frequency domain counterpart of cross-covariance is obtained as Fourier transform of the cross-covariance functions. Conversely, cross-covariance can be obtained from inverse Fourier transform of its cross-spectrum in the following way

$$S^{r_i,r_m}(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_k^{r_i,r_m}e^{-ik\omega}$$

$$\gamma_k^{r_i,r_m} = \int_{-\pi}^{\pi} S^{r_i,r_m}(\omega)e^{ik\omega}d\omega$$

where $S^{r_i,r_m}(\omega)$ is cross-spectral density of random variables r_i and r_m, $\gamma_k^{r_i,r_m}$ is cross-covariance function given by equation 2. It is important to note that cross-covariance can be decomposed into frequencies, more specifically, for $k = 0$, we can decompose covariance between two time series into the covariance components at each frequency ω

$$\mathbb{C}ov(r_i, r_m) = \int_{-\pi}^{\pi} S^{r_i,r_m}(\omega)d\omega.$$

and following the same logic decomposition of variance follows as

$$Var(r_i) = \int_{-\pi}^{\pi} S^{r_i}(\omega)d\omega.$$

where $S^{r_i}(\omega)$ is spectrum of r_i.

Since we can decompose cross-covariance between two returns into covariances at each frequency, we can disentangle the dependence at short- and long-term components. Then, beta for an asset i and factor m can be decomposed to as

$$\beta^{i,m} \equiv \frac{Cov(r_i, r_m)}{Var(r_m)} = \int_{-\pi}^{\pi} w(\omega)\frac{S^{r_i,r_m}(\omega)}{S^{r_m}(\omega)}d\omega = \int_{-\pi}^{\pi} w(\omega)\beta^{i,m}(\omega)d\omega \tag{5}$$

where $w(\omega) = \frac{S^{r_m}(\omega)}{Var(r_m)}$ represent weights. The decomposition is important step since it provides decomposition of classical beta into the weighted frequency-specific betas. Using this approach, Bandi and Tamoni (2017) estimate price of risk for different investment horizons and show that investors posses heterogeneous preferences over various economic cycles instead of looking only on averaged quantities over the whole frequency spectrum.

2.3 Quantile-frequency specific risk

Since we argue that both tail risk as well as frequency-specific risk are important in explaining formation of asset returns, we aim to combine the risks into a single model. We start by defining measure of risk associated with various combinations of quantile and frequency in order to determine the most important combination priced across assets.

Our measures of risk in the quantile-frequency domain are based on the dependence measures recently introduced by Baruník and Kley (2015). To quantify risk premium across frequencies and across the joint distribution, we use the quantile cross-spectral densities to build a quantile cross-spectral beta. Both these points are explained in more detail in Section 3.

2.4 Quantile cross-spectral beta

The cornerstone of the new beta representation lies in quantile cross-spectral density kernels which are defined as

$$f(\omega;\tau_{r_i},\tau_{r_m}) \equiv \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_k^{r_i,r_m}(\tau_{r_i},\tau_{r_m})e^{-ik\omega} \tag{6}$$

where $\tau_{r_i},\tau_{r_m}\in[0,1]$. A quantile cross-spectral density kernel is obtained as a Fourier transform of covariances of indicator functions defined in Equation 3, and will allow us to define beta that will capture the tail risks as well as frequency specific risks.

A quantile cross-spectral (QS) betas are defined as

$$\beta^{r_i,r_m}(\omega;\tau_{r_i},\tau_{r_m}) \equiv \frac{f^{r_i,r_m}(\omega;\tau_{r_i},\tau_{r_m})}{f^{r_m}(\omega,\tau_{r_m})} \equiv \frac{\sum_{k=-\infty}^{\infty}\gamma_k^{r_i,r_m}(\tau_{r_i},\tau_{r_m})e^{-ik\omega}}{\sum_{k=-\infty}^{\infty}\gamma_k^{r_m}(\tau_{r_m})e^{-ik\omega}}. \tag{7}$$

QS betas for given asset quantify the dependence between asset i and market factor m for a given frequency ω at chosen quantiles τ_{r_i} and τ_{t_m} of the joint distribution. We can also construct beta for a given frequency band, accordingly

$$\beta^{r_i,r_m}(\Omega;\tau_{r_i},\tau_{r_m}) \equiv \int_{\Omega}\frac{f^{r_i,r_m}(\omega;\tau_{r_i},\tau_{r_m})}{f^{r_m}(\omega,\tau_{r_m})}d\omega \tag{8}$$

where $\Omega\equiv[\omega_1,\omega_2)$, $\omega_1,\omega_2\in[-\pi,\pi]$, $\omega_1<\omega_2$ is a frequency band. This definition is important since it allows to define short-run, or long-run bands covering corresponding frequencies, and hence disaggregate beta based on the specific demands of a researcher.

2.5 Quantile cross-spectral beta under Gaussianity

Before we continue and use the new beta representation, it is important to note how newly defined quantity relates to a classical beta under the assumption of Gaussian distribution, as commonly assumed by many asset pricing models. Assuming that returns of an asset and returns of market portfolio are jointly normal random variables independently distributed through the time (correlated Gaussian white noises), QS betas would be in the following form

$$\beta^{r_i,r_m}_{Gauss}(\omega;\tau_{r_i},\tau_{r_m}) = \frac{C_{Gauss}(\tau_{r_i},\tau_{r_m};\rho)-\tau_{r_i}\tau_{r_m}}{\tau_{r_m}(1-\tau_{r_m})} \tag{9}$$

where C_{Gauss} is Gaussian copula with correlation coefficient ρ. This stems from the fact that quantile cross-spectral density corresponds to a difference of probabilities $Pr\big(r_{t,i}\leq q_{r_i}(\tau_{r_i}), r_{t,m}\leq q_{r_m}(\tau_{r_m})\big)-\tau_{r_i}\tau_{r_im}$, where $\{\tau_{r_i},\tau_{r_im}\}$ are probability levels under Gaussian distribution.

QS betas are constant over frequencies under Gaussian white noise assumption, and depend only on chosen quantiles and correlation coefficient between asset and market return. Hence

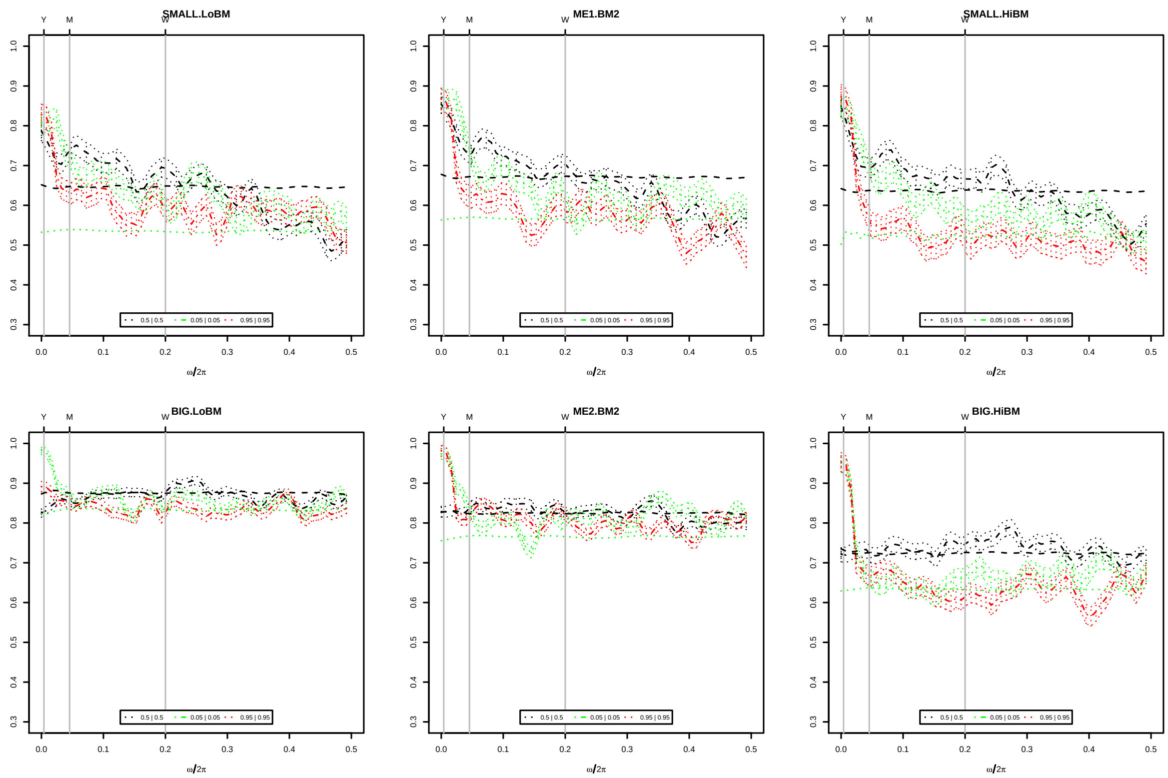

Figure 1: Quantile-spectral (QS) betas and their 95% confidence bands for 6 F-F portfolios sorted on size and book-to-market. Horizontal lines represent QS betas under Gaussian white noise assumption corresponding to CAPM beta.

Eq. 9 provides the quantile cross-spectral counterpart to classical CAPM beta as these are equivalent. We will use this fact to construct our model later. In the spirit of Ang et al. (2006) and Lettau et al. (2014), we define relative QS betas which capture additional information not contained in the classical CAPM beta.

In Figure 1 we illustrate both estimated QS betas and their confidence bands, and estimated QS betas under Gaussian white noise assumption (horizontal lines) corresponding to the CAPM beta. The illustration is on 6 Fama-French portfolios sorted on size and book-to-market. In case the QS betas would be constant over frequencies, and would moreover match the QS betas under Gaussian distribution corresponding to CAPM, then we could conclude that CAPM is sufficient model to describe the relation.

For majority of portfolios Gaussian white noise assumption does not hold and its degree of deviation varies over portfolios, and thus should be taken into consideration when assessing risk of a given portfolio.

Finally, we note that for serially uncorrelated variables (no matter of their joint or marginal distributions), the Frećhet/Hoeffding bounds gives the limits that QS beta can attain

$$\frac{\max\{\tau_{r_i}+\tau_{r_m}-1,0\}-\tau_{r_i}\tau_{r_m}}{\tau_{r_m}(1-\tau_{r_m})} \leq \beta^{r_i,r_m}(\omega;\tau_{r_i},\tau_{r_m}) \leq \frac{\min\{\tau_{r_i},\tau_{r_m}\}-\tau_{r_i}\tau_{r_m}}{\tau_{r_m}(1-\tau_{r_m})}.$$

3 Pricing extreme risks across frequency domain

Quantile cross-spectral betas defined in the previous section will be the cornerstone of the empirical model defined in this section. Given utility function from Equation 1, we assume that QS betas for low values of threshold values will be significant determinants of risk. Using QS betas, we define pricing model encompassing tail market risk and extreme volatility risk. Both these risks are further decomposed into long- and short-term components in order to obtain their prices of risk separately.

Tail market risk (TR) represents dependence between extreme negative events of both market as well as asset return. It differs from downside risk used in Ang et al. (2006); Lettau et al. (2014) since downside betas are computed based on covariates of asset return with a market return being under some threshold value. In contrast, QS betas captures risk that both market as well as asset return will be extremely unfavorable. In other words, it captures joint probability that market as well as asset returns will be below some threshold level.

Extreme market volatility risk (EVR) captures unpleasant situations in which extremely high increments of market volatility are linked to the extremely low asset asset returns. We argue that both these risks are significant determinants of risk of an asset and thus should be priced in cross-section of asset returns.

Values of τ_{r_i}, percentage value for the quantiles for asset thresholds, are not explicitly fixed to quantile of their returns because we do not explicitly care about dependence between quantile values in the cross-section. We rather care about dependence in extreme market situations. Thus the threshold values for asset returns are given by values of quantile of market returns; these threshold values are same for all the assets, which corresponds to different quantiles for each asset. Formally, for each portfolio we obtain threshold values as a τ_{r_i} quantile of its distribution where $\tau_{r_i} = F_{r_i}(q_{r_m}(\tau_{r_m}))$. Let's consider a model in which we set threshold value to be equal to 5% quantile of market return. Value of τ_{r_m} in Equation 7 is equal to 5% but τ_{r_i} must be estimated. First, this 5% market quantile must be transformed using empirical cumulative distribution functions into probability that given asset return is below this value for each asset, and then the QS betas are computed as $\beta^{r_i,r_m}(\omega; \tau^i_{r_i}, \tau_{r_m})$ where $\tau^i_{r_i}$ differs across assets (for one asset 5% quantile of market return may correspond to 1% quantile of its distribution, for another asset it may correspond to 8% quantile of its distribution). Same logic is applied to both tail market risk betas and extreme volatility risk betas. By setting market return and portfolio threshold equal, we avoid problem of potential data-mining. Potentially better fit could be obtained by finding threshold values with the best model fit for a specific dataset, but may not be robust across datasets.

Regarding the frequency decomposition of the risks, we specify our models to include disaggregation of risk into two horizons - long and short. Long horizon is defined by corresponding frequencies of cycles of 1.5 year and longer, and short horizon by frequencies of cycles shorter than 1.5 year. Procedure how to obtain these betas is explained in Section 4.

In each of the models defined in the paper we control for CAPM beta as a baseline measure of risk. This ensures that if the QS betas are significant determinants of risk premium, they do not simply duplicate information contained in CAPM beta. Moreover, in case of tail market risk, we define relative betas that explicitly capture only the additional information over CAPM beta. Throughout the paper we impose the restriction that market price of risk is correctly priced implying that it is equal its average return.

3.1 Tail market risk

We assume that dependence between market return and asset return during extreme negative events is priced across assets. The stronger the relationship between market and asset during an extreme events is, the bigger the risk premium investors demand. Tail market risk is a direct extension of downside risk discussed above. Whereas downside risk captures risk of negative events, tail risk is connected to negative events with more severe impact.

Because we want to quantify risk which is not captured by CAPM beta, we propose to test significance of tail market risk via differences of the estimated QS beta and QS beta implied by the Gaussian white noise assumption. We call this difference *relative* QS betas. For a given frequency ω and given quantiles τ_{r_i} and τ_{r_m}, the relative beta is defined as follows

$$\beta_{rel}^{r_i,r_m}(\Omega_j;\tau_{r_i},\tau_{r_m}) \equiv \beta_i(\Omega_j;\tau_{r_i},\tau_{r_m}) - \beta_i^{Gauss}(\Omega_j,\tau).$$

Relative QS betas measure additional information not captured by classical CAPM beta. In case the CAPM beta captures all information, and returns are Gaussian, the relative QS beta will be zero at all frequencies and quantiles.

Our first model is a three-factor market model which contains only tail market risk, and is defined as

$$\mathbb{E}[r_i^e] = \sum_{j=1}^{2} \beta_{rel}^{r_i,r_m}(\Omega_j;\tau_{r_i},\tau_{r_m})\lambda^{TR}(\Omega_j;\tau_{r_i},\tau_{r_m}) + \beta_i^{CAPM}\lambda^{CAPM}, \qquad (10)$$

where β_i is classical CAPM beta, λ^{CAPM} is price of risk for market risk captured by the classical beta, and $\lambda^{TR}(\Omega_j,\tau)$ is price of tail risk (TR) for given quantile and given frequency band. We impose restriction that market risk is correctly priced, i.e. λ^{CAPM} is equal to average market return, and portfolio threshold is the same as market threshold and, $\tau_{r_i} = F_{r_i}(q_{r_m}(\tau_{r_m}))$. If asset returns do not posses features of deviations from assumptions mentioned above, then the relative betas will be equal to zero and thus all the information about dependency during extreme events is captured by CAPM betas. On the other hand, if there is a significant difference between information captured by CAPM beta and QS betas, then the difference will be nonzero and may be priced in cross section of asset returns, which will be assessed based on significance of related prices of risk.

3.2 Extreme volatility risk

Volatility risk is important risk priced across assets. Ang et al. (2006) document that assets with high sensitivities to innovations in aggregate volatility have low average returns. Because of the fact that time of high volatility within the economy is perceived as a time with high uncertainty, investors are willing to pay more for the assets that yield high returns during these turmoils and thus positively covary with innovations in market volatility. This drives the prices of these assets up and decreases expected returns. In addition, decomposition of volatility into short-run and long-run when determining asset premium was proven to be useful as well (Adrian and Rosenberg, 2008). Moreover, Bollerslev et al. (2016) incorporated notion of downside risk into concept of volatility risk and showed that stocks with high negative realized semivariance yield higher returns. Farago and Tédongap (2017) examine downside volatility risk in their five-factor model and obtain model with negative prices of risk of volatility downside factor yielding low returns for assets that positively covary with innovations of market volatility during disappointing events.

We assume that assets that yield highly negative returns during times of large innovations of volatility are less desirable for investors and thus should be rewarded by holding these assets. For simplicity reasons, we estimate market volatility using basic GARCH(1,1) model [2] and obtain estimates of squared volatility. Then the changes in squared volatility are calculated as

$$\Delta\sigma_t^2 \equiv \sigma_t^2 - \sigma_{t-1}^2. \tag{11}$$

Because of the nature of covariance between indicator functions, we work with negative differences of the volatility, $-\Delta\sigma_t^2$, then the high volatility increments correspond to low quantiles of distribution of the negative differences. We investigate whether dependence between extreme market volatility and tail events of assets is priced across assets. Threshold values for portfolio returns are obtained in the same manner as for tail market risk and are derived from distribution of market returns, $\tau_{r_i} = F_{r_i}(q_{r_m}(\tau_{r_m}))$. For example, for model with $\tau_{r_m} = 0.05$, extreme market volatility beta is computed using threshold for innovations of market squared volatility as 5% quantile of its distribution of negative values (corresponding to 95% quantile of the original distribution), and threshold for portfolio returns is computed as 5% quantile of distribution of market returns.

Three-factor model containing extreme volatility risk betas solely is defined as

$$\mathbb{E}[r_i^e] = \sum_{j=1}^{2} \beta^{r_i,\Delta\sigma^2}(\Omega_j;\tau_{r_i},\tau_{r_m})\lambda^{EV}(\Omega_j;\tau_{r_i},\tau_{r_m}) + \beta_i^{CAPM}\lambda^{CAPM}, \tag{12}$$

where we also impose restriction that market risk is correctly priced, i.e. λ^{CAPM} is equal to average market return.

3.3 Full five-factor model

Finally, we combine the risks into a single five-factor model that includes both tail market risk and extreme volatility risk for both short- and long-run horizons, as well as market risk associated with classical CAPM beta. Model posses the following form

$$\begin{aligned}\mathbb{E}[r_i^e] = \sum_{j=1}^{2} \beta_{rel}^{r_i,r_m}(\Omega_j;\tau_{r_i},\tau_{r_m})\lambda^{TR}(\Omega_j;\tau_{r_i},\tau_{r_m}) + \beta_i^{CAPM}\lambda^{CAPM} \\ + \sum_{j=1}^{2} \beta^{r_i,\Delta\sigma^2}(\Omega_j;\tau_{r_i},\tau_{\Delta\sigma^2})\lambda^{EV}(\Omega_j;\tau_{r_i},\tau_{\Delta\sigma^2})\end{aligned} \tag{13}$$

where we restrict λ^{CAPM} to be equal to the average market return. We remind that the market threshold is equal to portfolio threshold. This means that τ_{r_m} is given and τ_{r_i} is computed for each portfolio from its respective empirical distribution. Threshold value for extreme volatility risk is given by quantile of distribution of differences of market volatility, and for given model, tau for extreme volatility risk is the same as tau for tail market risk, $\tau_{\Delta\sigma^2} = \tau_{r_m}$, and portfolio threshold is the same as for tail market risk.

Throughout the paper we focus on results for τ_{r_m} equal to 5% and 10%. In addition, we report various results for additional 1%, 15%, and 25% quantiles. Moreover, root mean squared pricing error of the fitted models is reported for continuum of quantiles between 1% and 50% for completeness. The choice of 5% and 10% quantiles is natural and arises in many economic and finance applications. Probably the most prominent example is Value-at-Risk, which a benchmark measure of risk widely used in practice and studied among academics.

[2] As a robustness check, we compute volatility as realized volatility from daily data

3.4 Simplified three-factor model

As an intermediary step, we define model which contains both tail market risk and extreme market volatility risk but does not take into consideration frequency decomposition. It posses the following form

$$\mathbb{E}[r_i^e] = \beta_{rel}^{r_i,r_m}(\tau_{r_i},\tau_{r_m})\lambda^{TR}(\tau_{r_i},\tau_{r_m}) + \beta_i^{CAPM}\lambda^{CAPM} + \beta^{r_i,\Delta\sigma^2}(\tau_{r_i},\tau_{\Delta\sigma^2})\lambda^{EV}(\tau_{r_i},\tau_{\Delta\sigma^2}) \quad (14)$$

where we define quantile betas as

$$\beta^{r_i,r_m}(\tau_{r_i},\tau_{r_m}) \equiv \frac{\gamma_0^{r_i,r_j}}{\gamma^{r_m}} = \frac{\mathbb{C}ov\big(I\{r_{t,i} \leq q_{r_i}(\tau_{r_i})\}, I\{r_{t,m} \leq q_{r_m}(\tau_{r_m})\}\big)}{\mathbb{V}ar\big(I\{r_m \leq q_{r_m}(\tau_{r_m})\}\big)} \quad (15)$$

where r_m stands either for return on market portfolio, or changes in negative of squared market volatility. Relative beta in case of TR is defined as difference between quantile beta and beta defined under normality assumption

$$\beta_{rel}^{r_i,r_j}(\tau_{r_i},\tau_{r_m}) \equiv \beta^{r_i,r_m}(\tau_{r_i},\tau_{r_m}) - \beta_{Gauss}^{r_i,r_m}(\omega;\tau_{r_i},\tau_{r_m}) \quad (16)$$

where beta under normality assumption is the same as in Equation 9 since it does not depend on frequency. Threshold values are obtained in the same way as in case of 5-factor model.

4 Testing for quantile-frequency specific risk

4.1 Estimation of QS betas

Estimation of QS betas (for both TR and EVR) relies on proper estimation of quantile cross-spectral densities using rank-based copula cross-periodograms which are then smoothed in order to obtain consistency of the estimator. Technical details are provided in the Appendix A. Betas from the simplified model defined in Equation 14 are simply estimated using empirical distribution function of the market return distribution.

4.2 Fama-MacBeth regressions

To test our model, we employ procedure of Fama and MacBeth (1973). In the first stage, we estimate all required QS betas, relative QS betas, and CAPM betas for all portfolios. We define two non-overlapping horizons: short and long. Horizon is specified by the corresponding frequency band. We specify long horizon by frequencies with corresponding cycles 1.5 year and longer, and short horizon by frequencies with corresponding cycles below 1.5 year. QS betas for these horizons are obtained by averaging QS betas over these frequency bands

$$\begin{aligned} \beta^{r_i,r_j}(\Omega_L;\tau_{r_i},\tau_{r_j}) &\equiv \frac{1}{n_L}\sum_{i=1}^{n_L} \beta^{r_i,r_j}(\omega_i^L;\tau_{r_i},\tau_{r_j}) \\ \beta^{r_i,r_j}(\Omega_S;\tau_{r_i},\tau_{r_j}) &\equiv \frac{1}{n_S}\sum_{i=1}^{n_S} \beta^{r_i,r_j}(\omega_i^S;\tau_{r_i},\tau_{r_j}) \end{aligned} \quad (17)$$

where Ω_L (Ω_S) is frequency band for long (short) horizon, and $\omega_i^L \in \Omega_L$ ($\omega_i^S \in \Omega_S$). In the second stage, we use these betas as explanatory variables and regress average portfolio returns on them. We assess significance of a given risk by significance of corresponding price of risk.

Thus, in the second stage in case of the five-factor full model, we estimate model of the following form

$$\bar{r}_i^e = \sum_{j=1}^{2} \hat{\beta}_{rel}^{r_i,r_m}(\Omega_j;\tau_{r_i},\tau_{r_m})\lambda^{TR}(\Omega_j;\tau_{r_i},\tau_{r_m}) + \hat{\beta}_i^{CAPM}\lambda^{CAPM} + \sum_{j=1}^{2} \hat{\beta}^{r_i,\Delta\sigma^2}(\Omega_j;\tau_{r_i},\tau_{\Delta\sigma^2})\lambda^{EV}(\Omega_j;\tau_{r_i},\tau_{\Delta\sigma^2}) + e_i. \tag{18}$$

The same estimation logic applies to the simplified three-factor model.

We compare the results for our model with i) classical CAPM ii) downside risk model by Ang et al. (2006) (DR1) iii) GDA3 and GDA5 models by Farago and Tédongap (2017). Performance of all models is assessed based on their root mean squared pricing error (RMSPE), which is widely used metric for assessing model fit in asset pricing literature. All the competing models are estimated for comparison purposes without any restrictions except that the market price of risk is correctly priced (equal to the average market return over the observed period) using OLS. Thus, GDA3 and GDA5 are despite their theoretical background estimated without setting any restriction to their coefficients and are also estimated in two stages. If anything, performance of the competing models can only deteriorate using restricted estimation procedures.

4.3 Data

For illustrating the main findings, we use 30 Fama-French industry portfolios data monthly sampled between July 1926 and November 2017 (1097 observations). These data satisfy the need of our model to posses long enough history in order to obtain reliable results. In Appendix C we report also results for 25 Fama-French portfolios sorted on size and book-to-market over the same time span. Regarding market data, instead of using consumption data, we follow Campbell (1993) and use data on broad market index to avoid problems connected to the consumption data. Excess market return is computed using value-weighted average return on all CRSP stocks and Treasury bill rate from Ibbotson Associates. Data were obtained from Kenneth French's online data library.

4.4 Estimation results

As a preliminary investigation, we conduct an analysis in which we examine tail risk and extreme volatility risk without taking into consideration the frequency aspect. Estimated coefficients can be found in left panel of Table 1. To take into account multiple hypothesis testing, we follow Harvey et al. (2016) and report t-statistics of estimated parameters. We can observe that tail risk is significantly priced across low quantiles with expected positive sign. Extreme volatility risk is significantly priced for 10%, 15%, and 25% quantiles suggesting that investors price dependence between assets and market volatility, but focus on more probable market situations. RMSPE of the model for various market threshold defined as τ quantile of market return is depicted in left panel of Figure 3. We can deduce that better fit is obtained for lower values of thresholds and for very low τ it can even outperform GDA5 model which is a 5-factor model. For higher values of τ, RMSPE of our simple model exceeds RMSPE of GDA5 model suggesting that indeed extreme risks of the assets are priced factor.

Estimated parameters of the full model can be found in the right panel of Table 1. We observe that significant determinants of the risk are short tail risk and long extreme volatility risk, both significantly priced across portfolios with expected signs. Tail risk is more significant for

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.56	0.35	0.66	15.57	-0.12	1.14	0.22	0.02	0.66	13.74
	(10.96)	(0.80)			(-0.71)	(7.01)	(1.88)	(0.06)		
0.05	2.41	0.48	0.66	20.78	-0.45	1.27	0.50	-2.93	0.66	15.87
	(4.72)	(0.55)			(-0.96)	(2.49)	(2.86)	(-2.71)		
0.1	2.07	1.61	0.66	19.69	-0.15	0.81	0.51	-2.23	0.66	15.94
	(3.48)	(2.34)			(-0.29)	(1.25)	(3.52)	(-1.82)		
0.15	3.67	1.58	0.66	19.85	0.61	0.72	0.46	-3.51	0.66	13.30
	(4.80)	(1.88)			(1.60)	(1.13)	(3.50)	(-3.35)		
0.25	4.50	3.26	0.66	25.44	0.69	0.71	0.68	-3.86	0.66	15.61
	(4.15)	(2.34)			(1.35)	(0.85)	(4.37)	(-2.83)		

Table 1: Estimated coefficients. Reported are estimated prices of risk of simple 3-factor and full 5-factor model on 30 Fama-French equal-weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds. Market price of risk is not estimated but imposed to be equal to the average market return.

lower values of τ meaning that dependence between market return and portfolio return during extremely negative events is a significantly determinant of risk premium. On the other hand, long-run extreme volatility risk is significantly priced across all values of τ_{r_m}, but becomes more prominent for higher values of the quantile. We can deduce that price of long-run risk of Bansal and Yaron (2004) is hidden in this coefficient. Coefficients of the prices of risk for long tail risk and short extreme volatility risk posses negative sign, which may seem counterintuitive. This may suggests that investors are extremely averse to long-run dependence between extremely negative returns and high volatility but at the same time exposure to the extreme volatility risk in the short run is desirable as the prices will adjust to the market turmoil quickly. Tail risk in the long run for lower quantiles is also negative but the coefficients are not significant.

In Figure 2, we compare performance of our QS models, QS05 ($\tau_{r_m} = 0.05$) and QS10 ($\tau_{r_m} = 0.10$), with various other models. It is notable that CAPM, and DR1 model completely fail to price the portfolios, better fit and lower RMSPE is obtained by GDA3 and GDA5 models. Finally, better fit is provided by our QS model since returns lie closer to the 45 degree line. Right panel of Figure 3 depicts performance of the QS model against market thresholds given by τ quantile of market distribution. We observe better performance of our model in comparison to GDA5 model for all threshold values below 30% market quantile, and generally very good performance for low values of threshold suggesting that extreme risks are significant determinants of risk premium.

4.5 Robustness checks

As a robustness checks, we first report results based on 30 Fama-French industry portfolio data which are value weighted. Results are summarized in Appendix B.1. We report estimated coefficients for both simple and full model, RMSPE for continuum of τ_{r_m} and comparison with competing models. We also conduct the same analysis with volatility being computed from daily data as a realized volatility for each month in the sample. It is obvious from estimated simple models that both tail market risk and extreme volatility risk are priced in cross-section. Estimated full models suggest that short tail risk is the driving force of aggregated tail risk, and although coefficients for long extreme volatility risk are not significant, they posses the right sign and are numerically close to their counterparts computed on volatility from GARCH

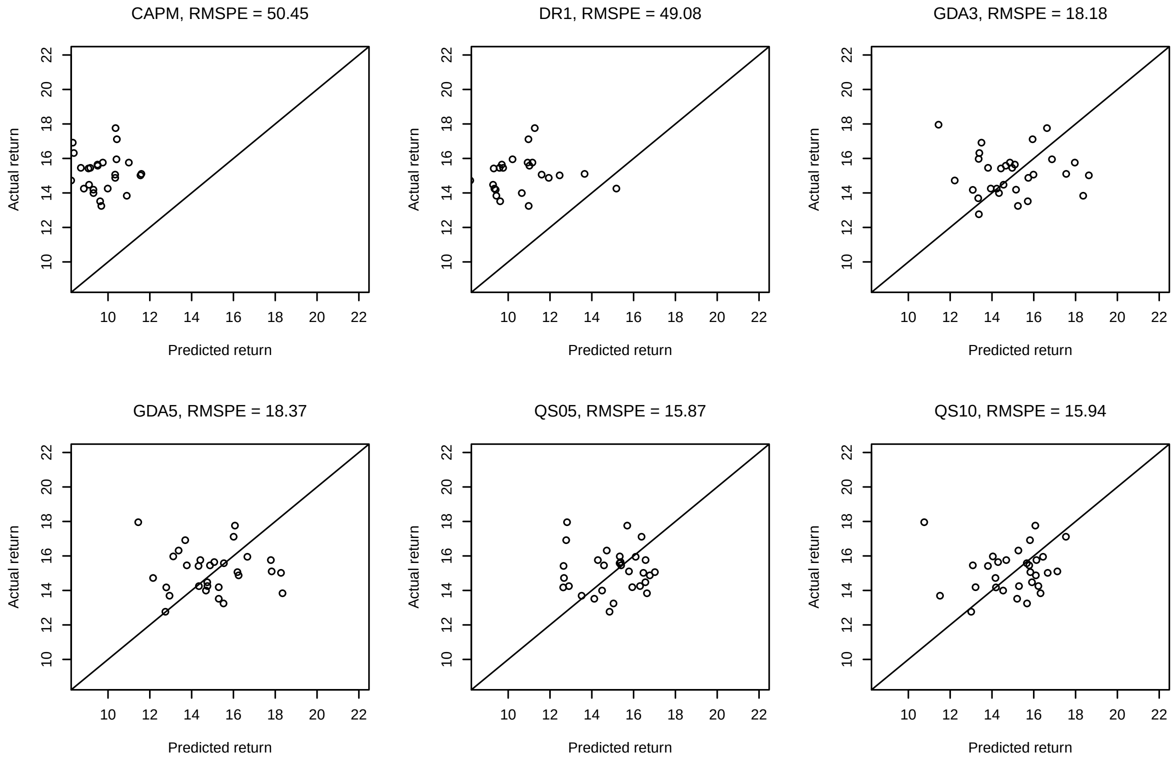

Figure 2: Predicted returns. Plots of predicted versus actual returns for competing models.

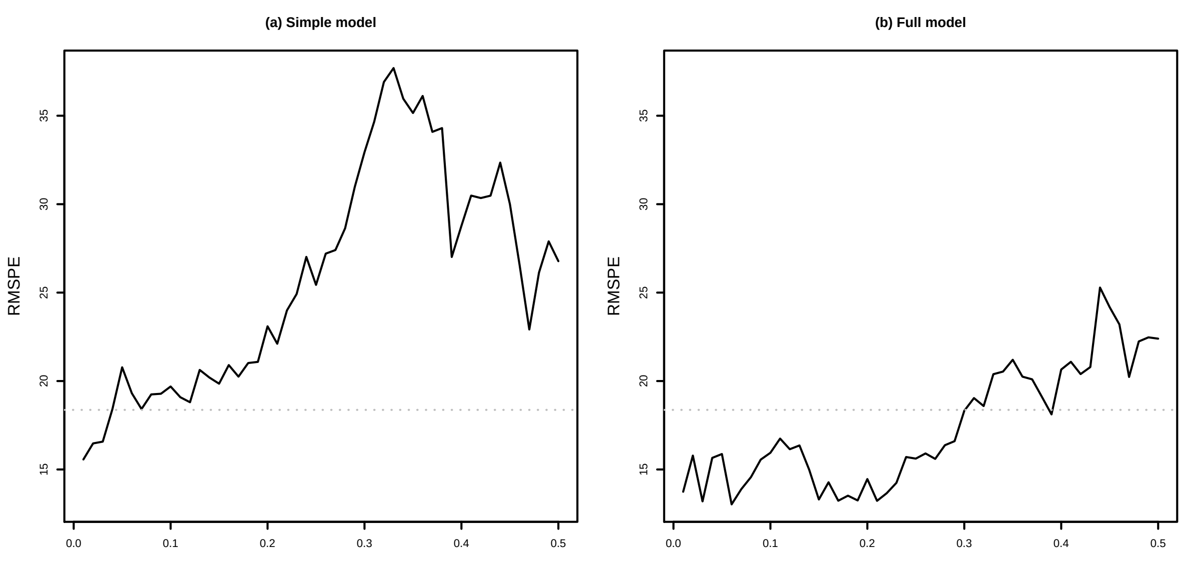

Figure 3: RMSPE for simple and full model. Horizontal line represents RMSPE of GDA5 model.

model. We argue that this is due to highly non-smoothed nature of the volatility computed as a sum over respective months.

In Appendix C, we perform the same analysis on 25 Fama-French portfolios sorted on size and book-to-market. We report results based on both equal and value weighted portfolios, and volatility is computed using GARCH model and as a realized volatility from daily data. In case of models with volatility computed from GARCH model, our model performs comparable to GDA5 model but slightly worse, but outperforms all the other competing models, and moreover all the features observed in the case of 30 industry portfolios are present in this case also with values of the coefficients being similar. In case of volatility computed from daily data, our model outperforms all the competing models including GDA5 model.

5 Conclusion

We have shown that extreme risks are priced in cross-section of asset returns. In the paper, we distinguish between tail market risk and extreme volatility risk. Tail market risk is characterized by the dependence between highly negative market and asset events. Extreme volatility risk is defined as cooccurrence of extremely high increases of market volatility and highly negative asset returns. Negative events are derived from distribution of market returns and its respective quantile is used for determining threshold values for computing quantile cross-spectral betas. We define two empirical models for testing associated risk premium. Simple model, which does not take into consideration frequency aspect, confirms that investors require premium for bearing both tail market risk and extreme volatility risk. Full model further identifies that premium for tail market risk is mostly featured in its short-term component, and premium for extreme volatility risk is mostly associated with its long-term component. In order to consistently estimate the model, data with long enough history has to be employed. But if the data are available, our model is able to outperform competing models and its performance is best for low threshold values suggesting that investors require risk premium for holding assets susceptible to extreme risks.

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A Estimation of quantile cross-spectral betas

Estimation of QS betas defined in our paper is based on the smoothed quantile cross-periodograms studied in Baruník and Kley (2015). For a strictly stationary time series $X_{0,j}, \dots, X_{n-1,j}$, we define $I\{\hat{F}_{n,j}(X_{t,j}) \leq \tau\} = I\{R_{n;t,j} \leq n\tau\}$ where $\hat{F}_{n,j}(x) \equiv n^{-1} \sum_{t=0}^{n-1} I\{X_{t,j} \leq x\}$ is the empirical distribution function of $X_{t,j}$ and $R_{n;t,j}$ denotes the rank of $X_{t,j}$ among $X_{0,j}, \dots, X_{n-1,j}$. We have seen that the cornerstone of quantile cross-spectral beta is quantile cross-spectral density defined in Equation 6. Its population counterpart is called rank-based copula cross-periodogram, CCR-periodogram, and is defined as

$$I_{n,R}^{j_1,j_2}(\omega; \tau_1, \tau_2) \equiv \frac{1}{2\pi n} d_{n,R}^{j_1}(\omega; \tau_1) d_{n,R}^{j_2}(-\omega; \tau_2) \tag{19}$$

where

$$d_{n,R}^{j}(\omega; \tau) \equiv \sum_{t=0}^{n-1} I\{\hat{F}_{n,j}(X_{t,j}) \leq \tau\} e^{-i\omega t} = \sum_{t=0}^{n-1} I\{R_{n;t,j} \leq n\tau\} e^{-i\omega t}, \quad \tau \in [0, 1]. \tag{20}$$

As discussed in Baruník and Kley (2015), CCR-periodogram is not a consistent estimator of quantile cross-spectral density. Consistency can be achieved by smoothing CCR-periodogram across frequencies. Following Baruník and Kley (2015), we employ the following

$$\hat{G}_{n,R}^{j_1,j_1}(\omega; \tau_1, \tau_2) \equiv \frac{2\pi}{n} \sum_{s=0}^{n-1} W_n(\omega - 2\pi s/n) I_{n,R}^{j_1,j_2}(2\pi s/n, \tau_1, \tau_2) \tag{21}$$

where W_n is defined in Section 3 of Baruník and Kley (2015). Estimator of quantile cross-spectral beta is defined as

$$\hat{\beta}_{n,R}^{j_1,j_2}(\omega; \tau_1, \tau_2) \equiv \frac{\hat{G}_{n,R}^{j_1,j_2}(\omega; \tau_1, \tau_2)}{\hat{G}_{n,R}^{j_2}(\omega; \tau_2)}. \tag{22}$$

Consistency of the estimator can be proven using exactly same logic as in Theorem 3.4 in Baruník and Kley (2015) by replacing quantile coherency with quantile cross-spectral beta.

B Robustness checks

B.1 Realized volatility

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.02	0.43	0.66	12.94	-0.08	0.85	0.07	0.55	0.66	12.30
	(5.36)	(3.67)			(-0.48)	(3.66)	(0.46)	(1.87)		
0.05	1.22	0.57	0.66	18.66	-0.21	1.40	0.42	0.11	0.66	18.20
	(2.12)	(2.67)			(-0.38)	(2.22)	(0.98)	(0.17)		
0.1	0.72	0.91	0.66	16.28	0.05	0.80	0.25	0.54	0.66	16.43
	(1.18)	(4.58)			(0.09)	(1.01)	(0.57)	(0.73)		
0.15	1.59	0.91	0.66	15.80	0.73	0.70	-0.07	0.87	0.66	15.41
	(2.05)	(4.67)			(1.64)	(0.75)	(-0.15)	(1.03)		
0.25	1.32	1.43	0.66	15.84	0.23	1.02	0.49	0.45	0.66	15.27
	(1.65)	(7.64)			(0.41)	(1.15)	(1.56)	(0.87)		

Table 2: Estimated coefficients. Reported are estimated prices of risk of simple 3-factor and full 5-factor model on 30 Fama-French value-weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds. Market price of risk is not estimated but imposed to be equal to the average market return. Volatility is computed as realized volatility from daily data.

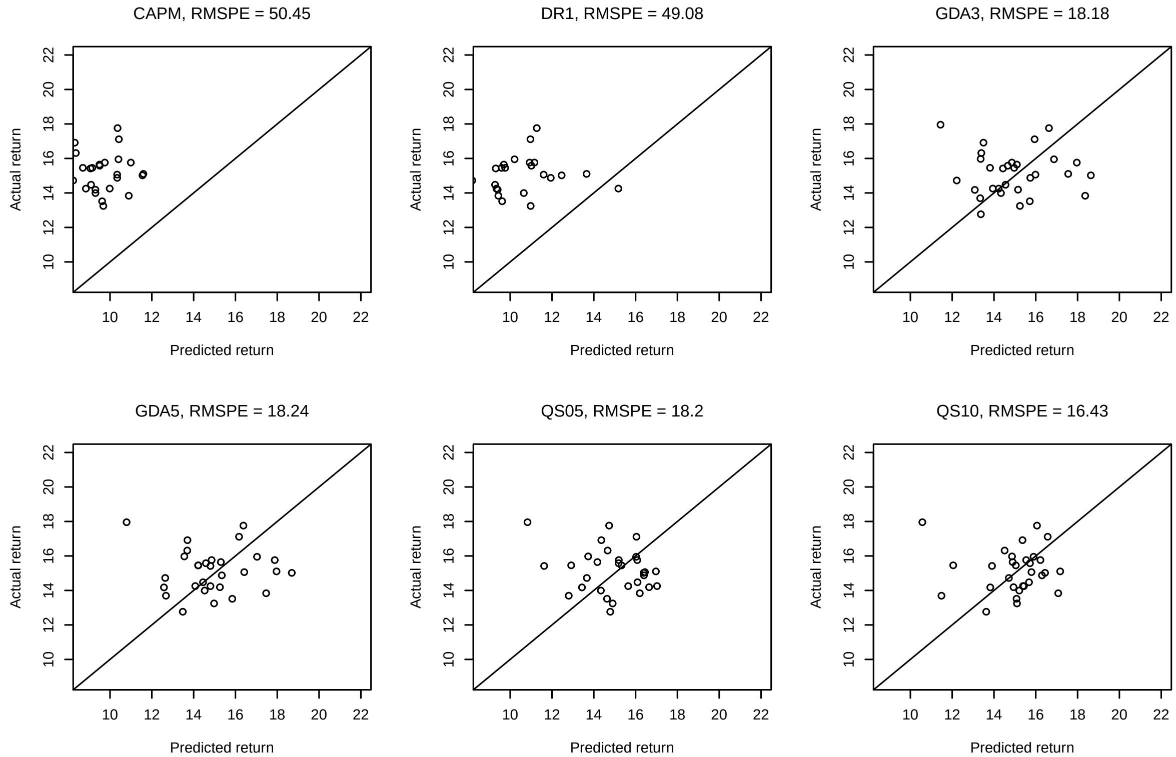

Figure 4: Predicted returns. Plots of predicted versus actual returns for competing models.

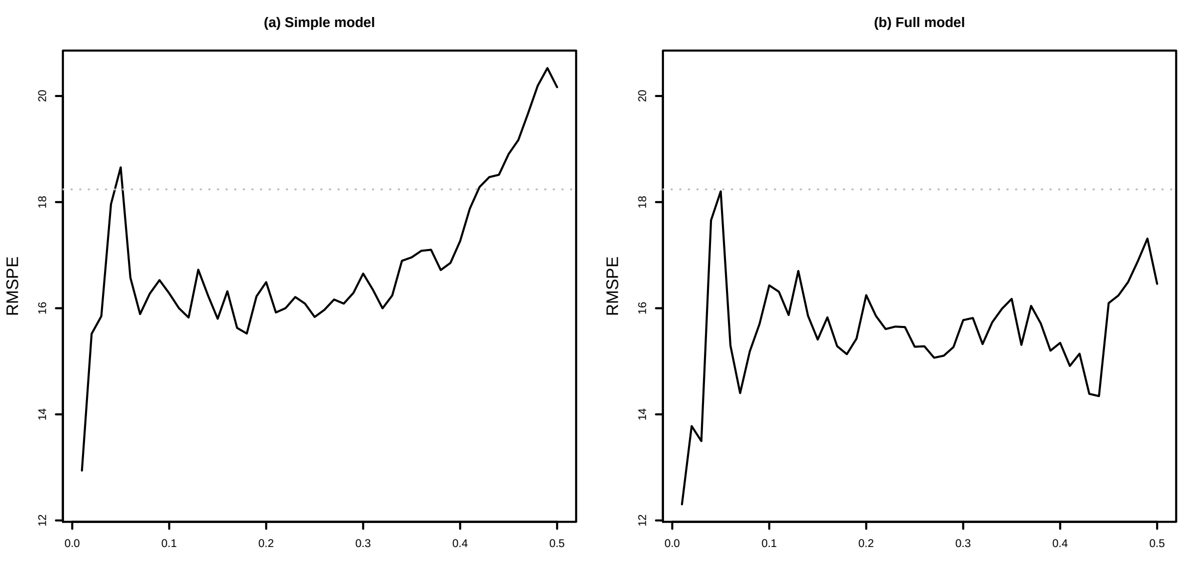

Figure 5: RMSPE for simple and full model. Horizontal line represents RMSPE of GDA5 model.

B.2 Value weighted portfolios

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.36	-0.50	0.66	16.72	-0.03	1.06	0.06	-0.34	0.66	16.52
	(9.88)	(-1.01)			(-0.15)	(6.17)	(0.34)	(-0.73)		
0.05	2.21	0.29	0.66	15.65	-0.12	2.12	0.18	0.62	0.66	14.92
	(5.58)	(0.57)			(-0.27)	(5.25)	(1.22)	(0.67)		
0.1	1.69	1.77	0.66	19.84	-0.54	1.93	0.39	-0.57	0.66	18.01
	(2.84)	(3.09)			(-1.11)	(3.16)	(3.51)	(-0.37)		
0.15	1.79	3.13	0.66	19.56	0.05	1.42	0.36	0.32	0.66	18.09
	(3.10)	(4.09)			(0.12)	(2.00)	(2.95)	(0.18)		
0.25	2.37	3.93	0.66	28.48	0.08	1.40	0.57	-2.39	0.66	18.19
	(2.33)	(2.52)			(0.17)	(1.85)	(4.88)	(-1.71)		

Table 3: Estimated coefficients. Reported are estimated prices of risk of simple 3-factor and full 5-factor model on 30 Fama-French value-weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds. Market price of risk is not estimated but imposed to be equal to the average market return.

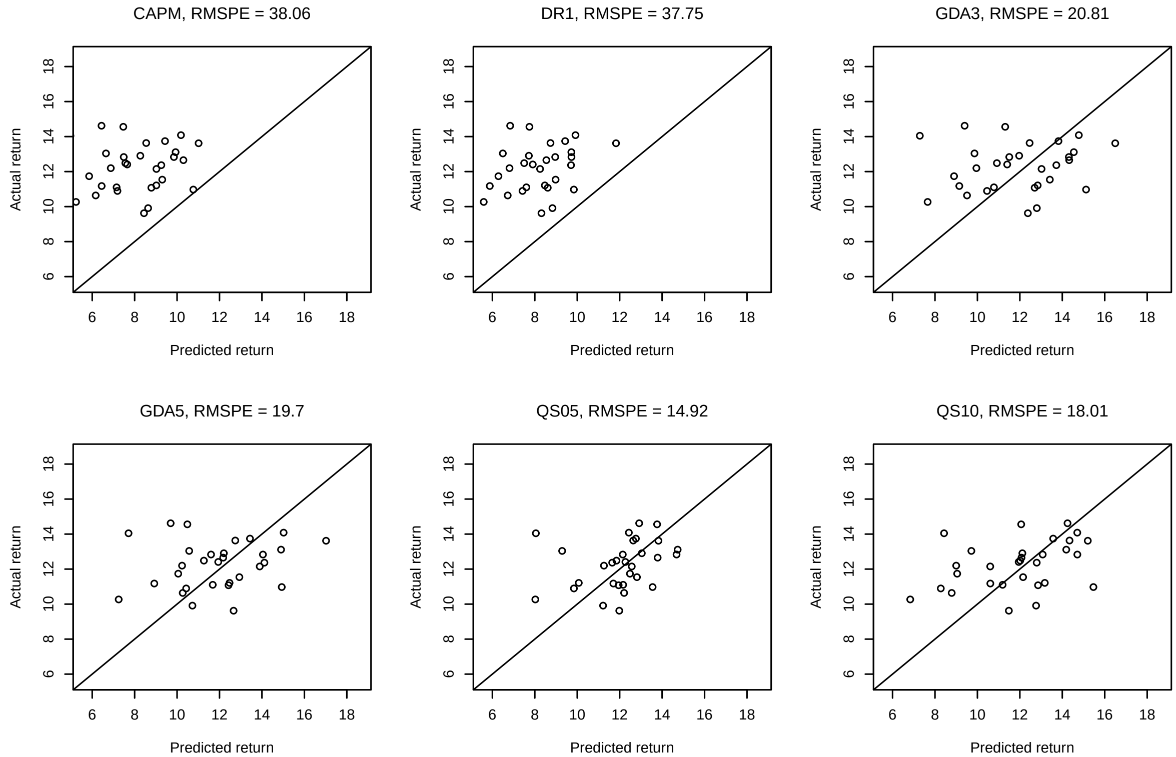

Figure 6: Predicted returns. Plots of predicted versus actual returns for competing models.

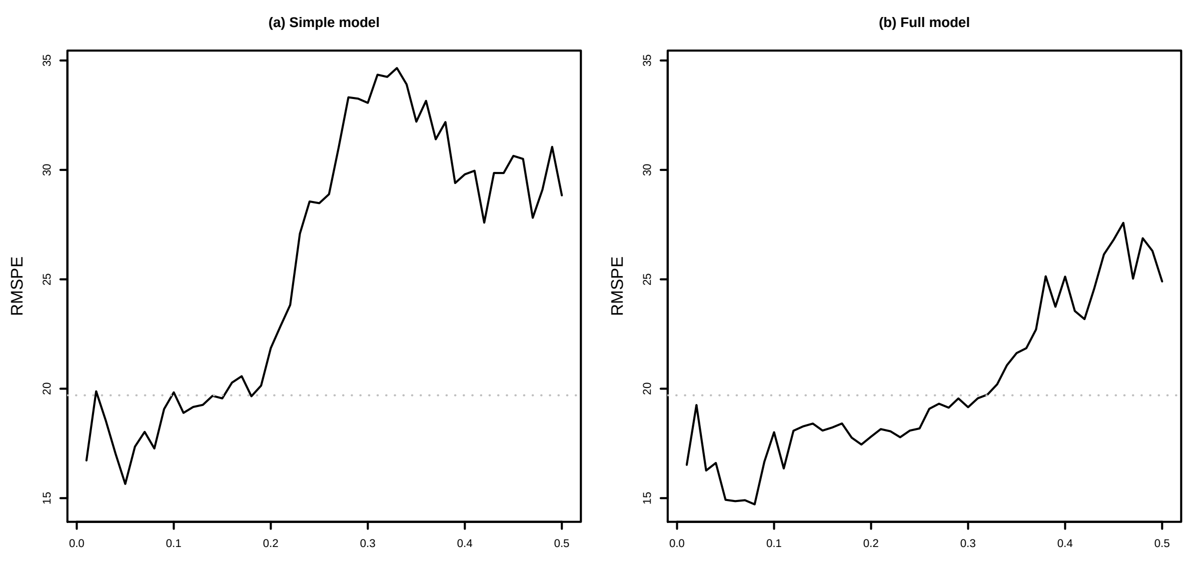

Figure 7: RMSPE for simple and full model. Horizontal line represents RMSPE of GDA5 model.

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.12	0.11	0.66	16.87	0.22	0.33	-0.32	1.09	0.66	15.01
	(4.14)	(0.71)			(0.81)	(0.99)	(-1.41)	(2.53)		
0.05	2.31	0.04	0.66	15.73	-0.22	2.15	0.23	0.05	0.66	15.08
	(4.87)	(0.21)			(-0.44)	(4.23)	(0.69)	(0.10)		
0.1	1.36	0.56	0.66	18.91	-0.84	2.44	0.73	-0.42	0.66	17.53
	(2.28)	(3.65)			(-1.41)	(2.92)	(1.42)	(-0.46)		
0.15	1.40	0.71	0.66	18.35	0.09	1.44	0.22	0.39	0.66	18.07
	(2.43)	(4.78)			(0.15)	(1.62)	(0.38)	(0.36)		
0.25	0.94	1.18	0.66	17.79	-0.10	1.42	0.43	0.41	0.66	17.31
	(1.44)	(7.75)			(-0.21)	(1.55)	(1.10)	(0.60)		

Table 4: Estimated coefficients. Reported are estimated prices of risk of simple 3-factor and full 5-factor model on 30 Fama-French value-weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds. Market price of risk is not estimated but imposed to be equal to the average market return. Volatility is computed as realized volatility from daily data.

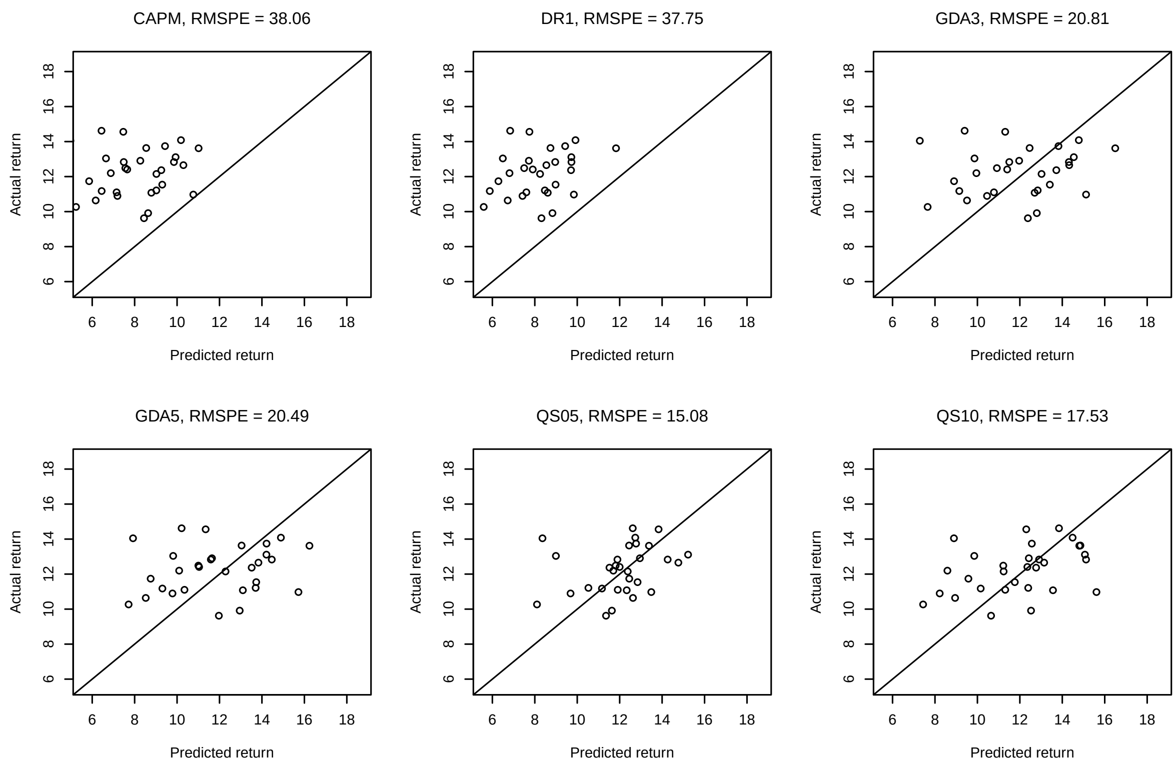

Figure 8: Predicted returns. Plots of predicted versus actual returns for competing models.

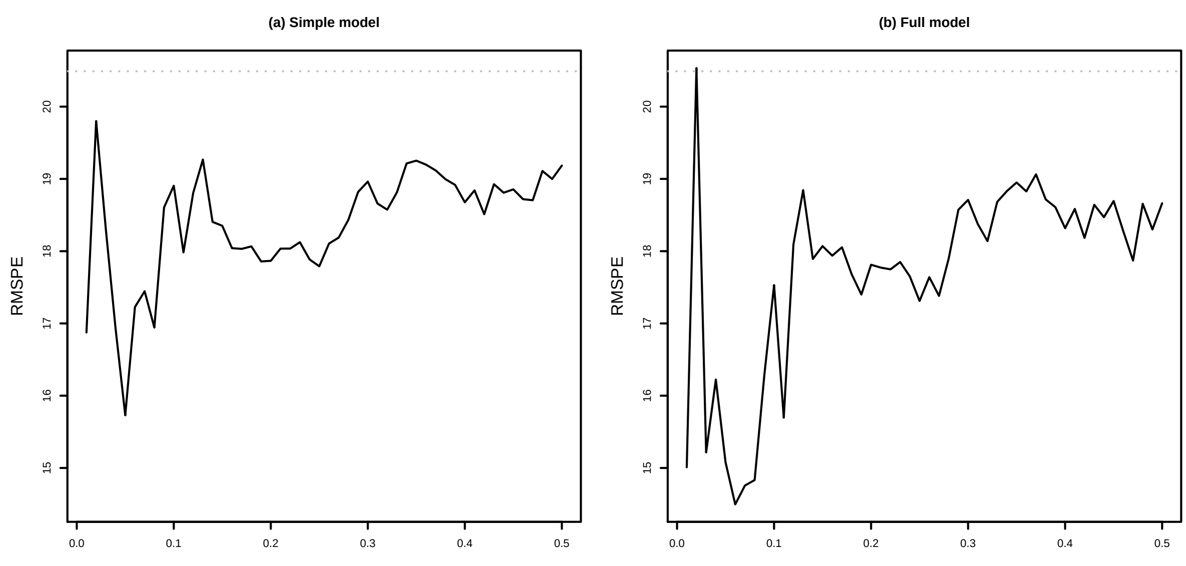

Figure 9: RMSPE for simple and full model. Horizontal line represents RMSPE of GDA5 model.

C Results for 25 F-F portfolios sorted on size and book-to-market

C.1 Equal weighted

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.81	-0.05	0.66	28.28	-0.25	0.86	0.38	0.20	0.66	22.80
	(5.14)	(-0.05)			(-0.72)	(1.99)	(1.42)	(0.27)		
0.05	3.66	0.10	0.66	21.15	-0.30	2.84	0.32	-1.93	0.66	20.60
	(5.40)	(0.13)			(-0.60)	(1.90)	(1.85)	(-1.30)		
0.1	-0.09	3.52	0.66	21.97	0.53	-0.23	0.21	6.12	0.66	21.25
	(-0.10)	(4.59)			(0.87)	(-0.15)	(1.35)	(1.90)		
0.15	0.32	4.80	0.66	22.64	-0.39	1.04	0.40	6.58	0.66	22.17
	(0.25)	(3.93)			(-0.41)	(0.42)	(1.80)	(2.39)		
0.25	-2.46	13.08	0.66	22.16	-1.49	-0.39	0.92	11.29	0.66	19.12
	(-1.31)	(4.58)			(-1.91)	(-0.17)	(6.22)	(3.57)		

Table 5: Estimated coefficients. Reported are estimated prices of risk of simple 3-factor and full 5-factor model on 30 Fama-French value-weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds. Market price of risk is not estimated but imposed to be equal to the average market return.

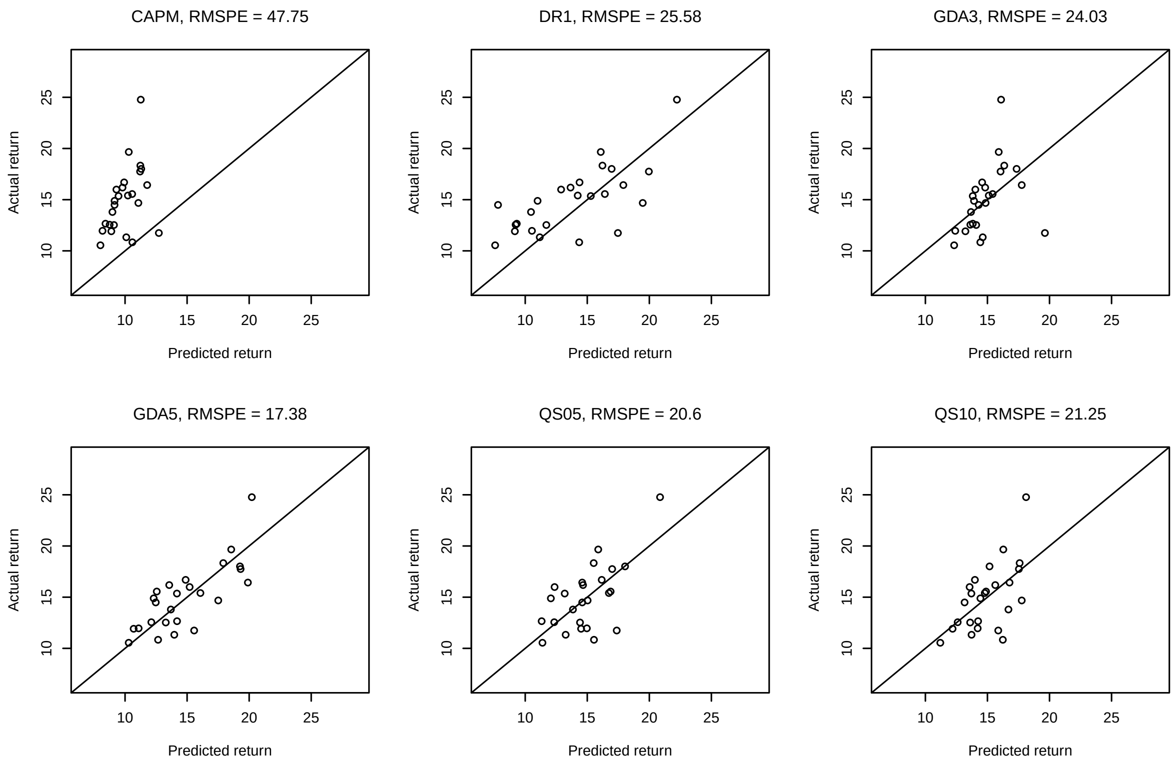

Figure 10: Predicted returns. Plots of predicted versus actual returns for competing models.

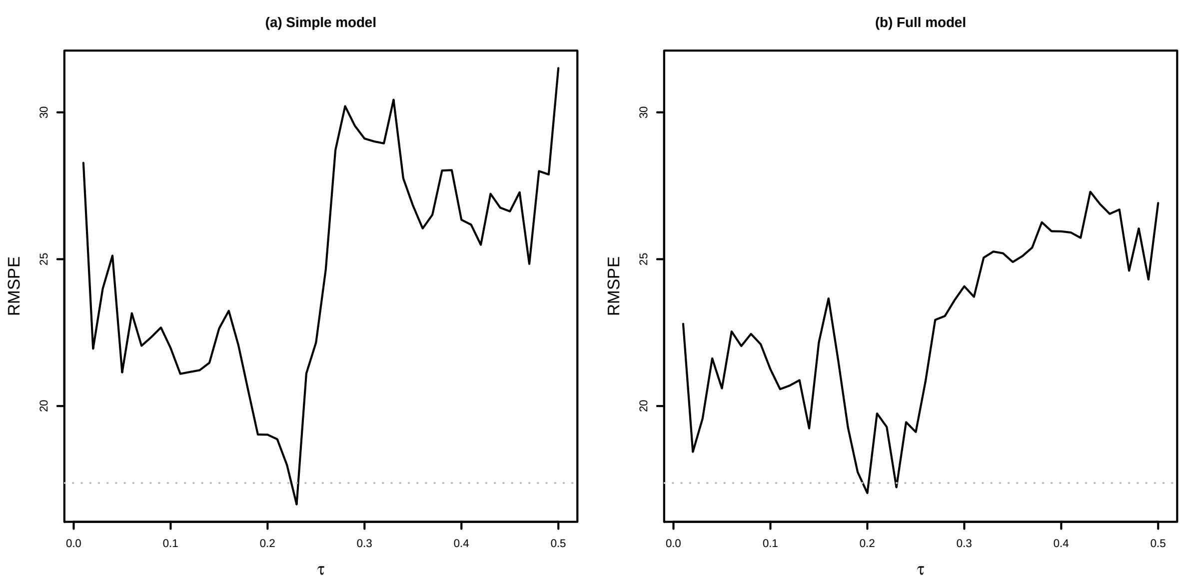

Figure 11: RMSPE for simple and full model. Horizontal line represents RMSPE of GDA5 model.

τ_{r_m}	Simple model λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	Full model λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	0.58	0.67	0.66	24.41	-0.25	1.18	0.40	-0.19	0.66	22.64
	(1.16)	(2.81)			(-0.71)	(1.63)	(1.23)	(-0.32)		
0.05	3.00	0.18	0.66	20.93	-0.16	3.42	0.00	0.73	0.66	20.77
	(2.69)	(0.69)			-0.26	(2.18)	(0.00)	(0.65)		
0.1	-0.32	1.01	0.66	23.55	1.46	-3.41	-0.96	2.43	0.66	22.03
	(-0.28)	(3.92)			(1.50)	(-1.41)	(-1.01)	(1.49)		
0.15	1.00	0.91	0.66	23.49	-1.39	4.03	1.46	-1.22	0.66	21.66
	(0.83)	(3.56)			(-1.47)	(1.63)	(2.07)	(-0.89)		
0.25	0.54	1.28	0.66	23.88	-1.42	2.79	1.79	-1.75	0.66	20.59
	(0.36)	(3.86)			(-1.73)	(1.30)	(2.68)	(-1.45)		

Table 6: Estimated coefficients. Reported are estimated prices of risk of simple 3-factor and full 5-factor model on 30 Fama-French value-weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds. Market price of risk is not estimated but imposed to be equal to the average market return. Volatility is computed as realized volatility from daily data.

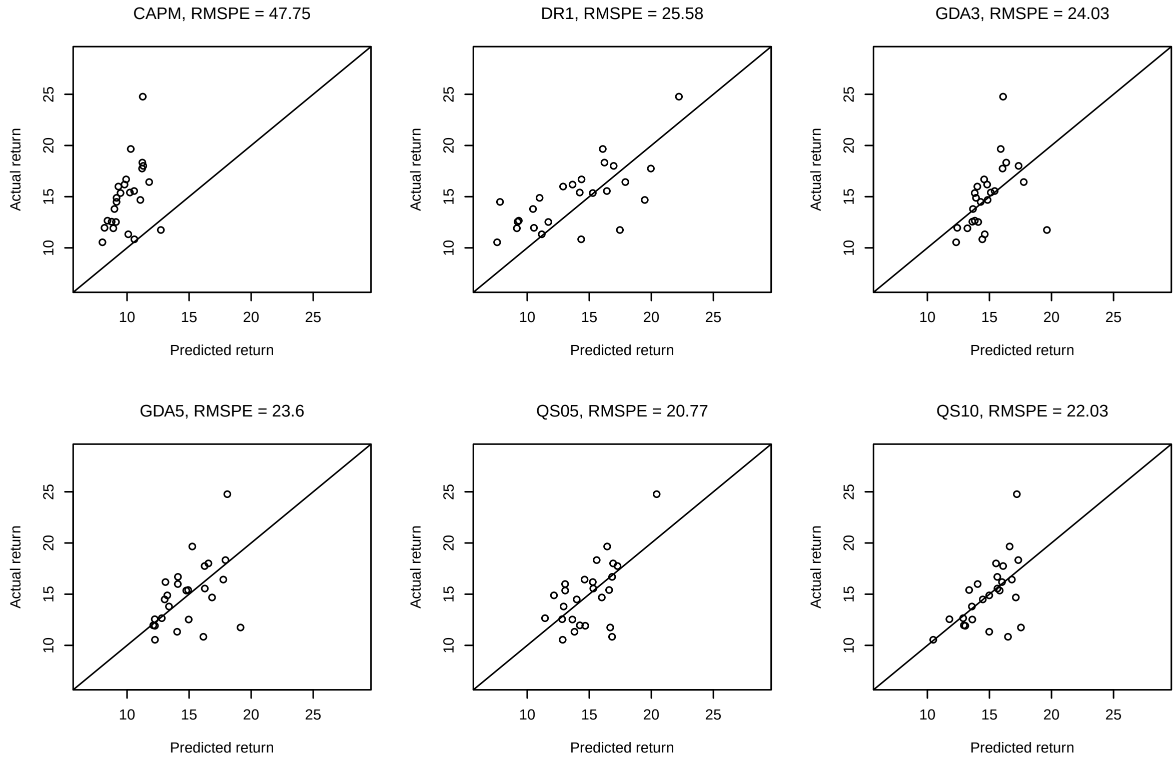

Figure 12: Predicted returns. Plots of predicted versus actual returns for competing models.

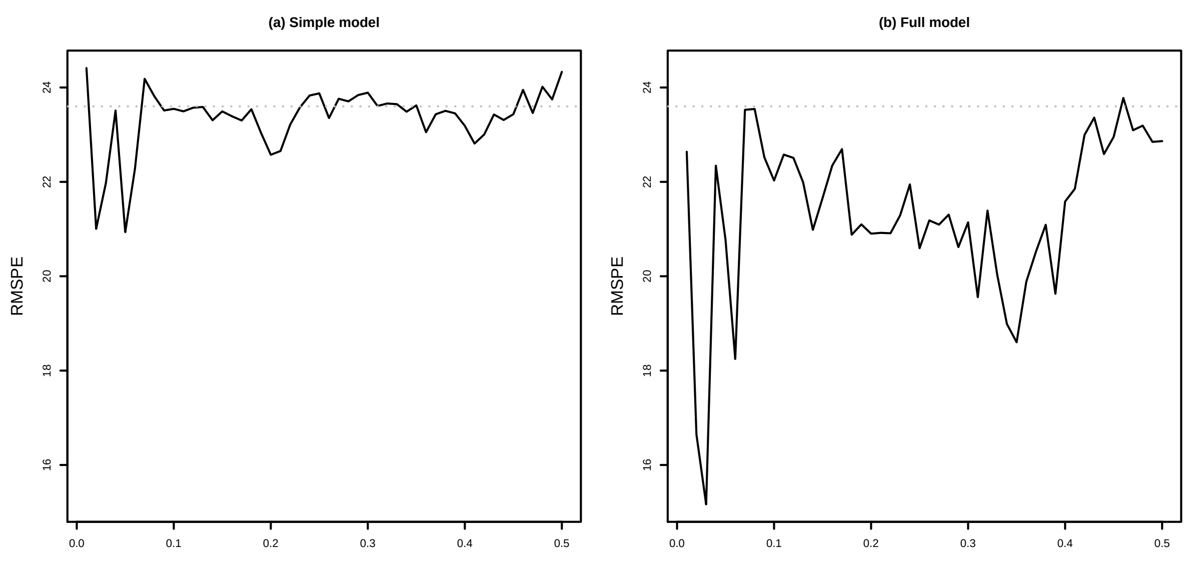

Figure 13: RMSPE for simple and full model. Horizontal line represents RMSPE of GDA5 model.

C.2 Value weighted

	Simple model				Full model					
τ_{r_m}	λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	1.59	0.74	0.66	24.39	-0.45	0.58	0.50	0.41	0.66	18.81
	(5.18)	(0.92)			(-1.51)	(1.43)	(2.44)	(0.68)		
0.05	2.90	0.84	0.66	21.56	-0.87	0.57	0.59	-2.69	0.66	19.25
	(2.38)	(0.70)			(-2.02)	(0.40)	(3.55)	(-1.74)		
0.1	-1.55	4.04	0.66	18.40	-0.03	-3.03	0.51	-2.21	0.66	16.89
	(-1.93)	(6.67)			(-0.05)	(-2.54)	(4.70)	(-0.87)		
0.15	-1.89	5.98	0.66	20.02	-0.81	-1.25	0.55	4.87	0.66	19.32
	(-1.65)	(5.57)			(-1.08)	(-0.82)	(3.46)	(1.56)		
0.25	-0.32	8.54	0.66	24.33	-1.26	-0.58	0.86	3.69	0.66	19.55
	(-0.26)	(4.38)			(-1.59)	(-0.38)	(5.59)	(1.47)		

Table 7: Estimated coefficients. Reported are estimated prices of risk of simple 3-factor and full 5-factor model on 30 Fama-French value-weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds. Market price of risk is not estimated but imposed to be equal to the average market return.

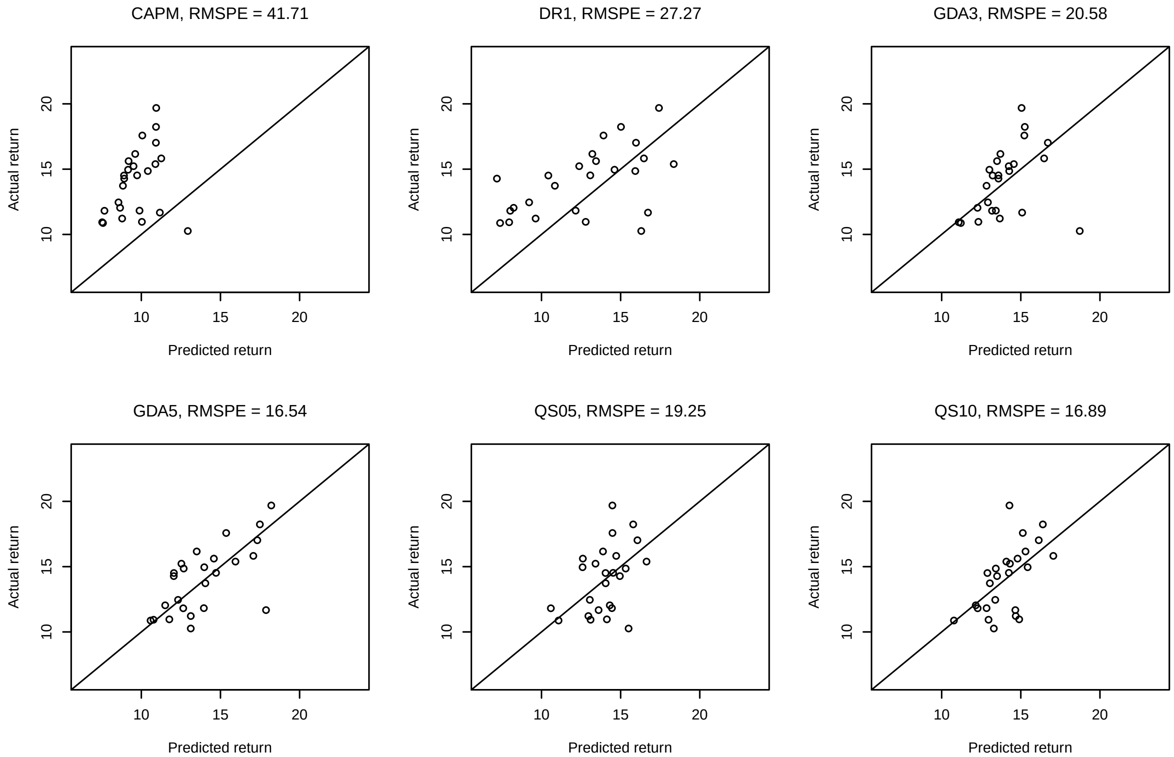

Figure 14: Predicted returns. Plots of predicted versus actual returns for competing models.

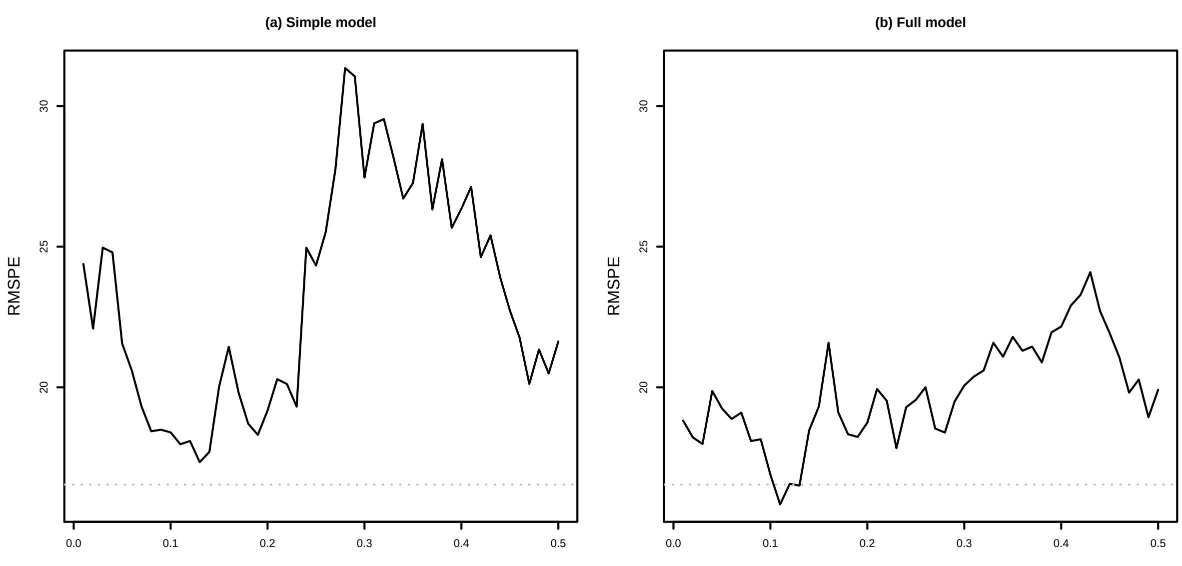

Figure 15: RMSPE for simple and full model. Horizontal line represents RMSPE of GDA5 model.

τ_{r_m}	Simple model λ^{TR}	λ^{EVR}	λ^{CAPM}	RMSPE	Full model λ^{TR}_{long}	λ^{TR}_{short}	λ^{EVR}_{long}	λ^{EVR}_{short}	λ^{CAPM}	RMSPE
0.01	0.34	0.68	0.66	21.38	-0.68	1.67	0.84	-0.94	0.66	15.49
	(0.63)	(2.83)			(-3.02)	(3.23)	(4.18)	(-2.27)		
0.05	1.31	0.49	0.66	20.74	-0.86	0.21	0.52	0.39	0.66	19.83
	(0.81)	(1.54)			(-1.47)	(0.14)	(1.08)	(0.47)		
0.1	-2.53	1.30	0.66	17.72	0.05	-2.96	0.18	0.80	0.66	17.06
	(-2.87)	(7.06)			(0.07)	(-2.31)	(0.29)	(0.76)		
0.15	-1.47	1.23	0.66	19.44	-1.43	0.52	1.60	-1.47	0.66	16.68
	(-1.42)	(5.85)			(-2.10)	(0.36)	(2.88)	(-1.49)		
0.25	-1.80	1.59	0.66	18.59	-1.42	-0.37	1.26	-0.57	0.66	15.86
	(-1.79)	(7.02)			(-2.22)	(-0.29)	(2.71)	(-0.67)		

Table 8: Estimated coefficients. Reported are estimated prices of risk of simple 3-factor and full 5-factor model on 30 Fama-French value-weighted industry portfolios sampled between July 1927 and November 2017. Model is estimated for various values of thresholds. Market price of risk is not estimated but imposed to be equal to the average market return. Volatility is computed as realized volatility from daily data.

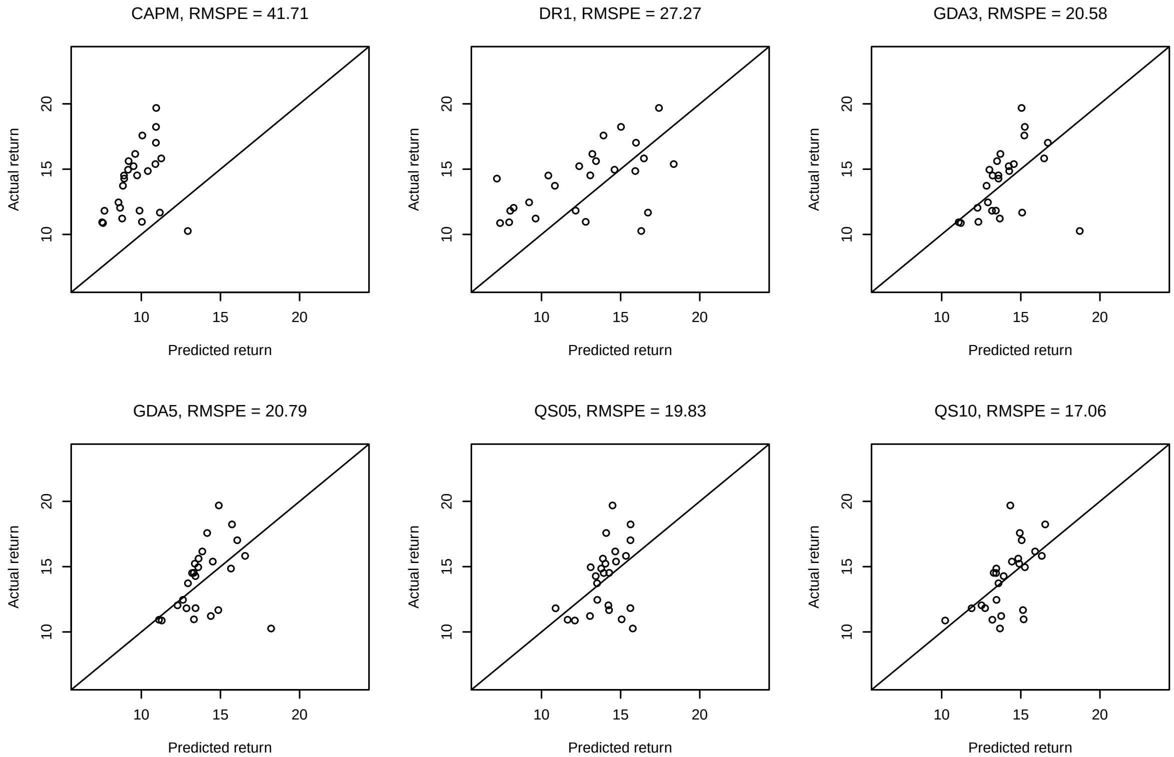

Figure 16: Predicted returns. Plots of predicted versus actual returns for competing models.

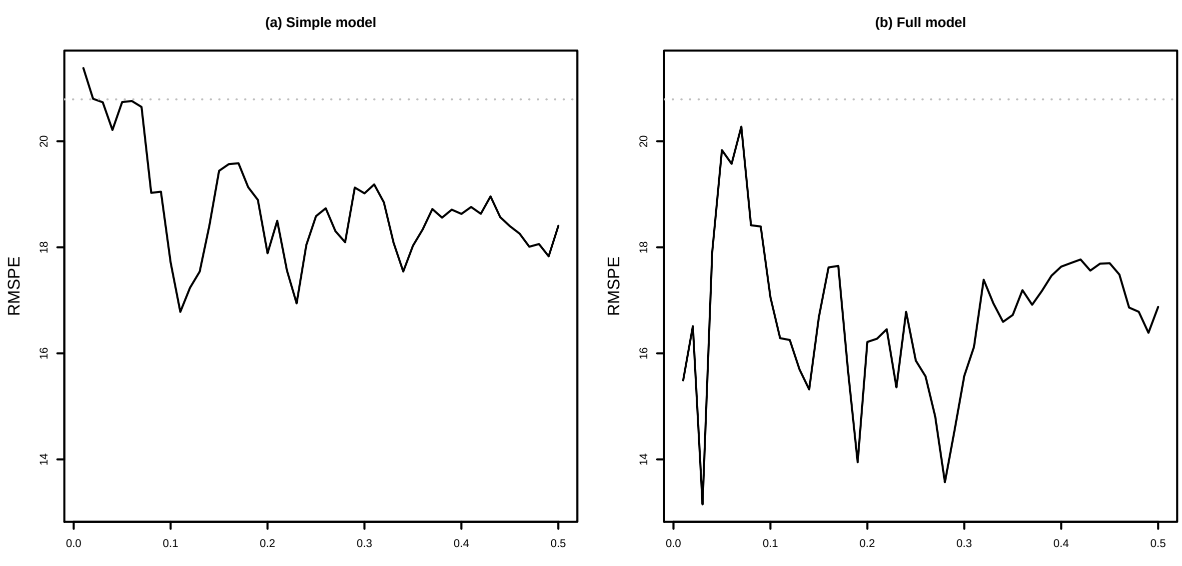

Figure 17: RMSPE for simple and full model. Horizontal line represents RMSPE of GDA5 model.