Credit Risk, Liquidity, and Bubbles

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Abstract

This paper presents an arbitrage-free valuation model for a credit risky security where credit risk coexists and interacts with an asset price bubble and liquidity risk (or liquidity costs). As an illustration, this model is applied to determine the fair rate for microfinance loans.

1 Introduction

In the derivatives literature, models exist for pricing securities with credit risk (see Jarrow [10] for a review), liquidity risk (see Cetin, Jarrow, Protter [4]), and asset price bubbles (see Protter [17] for a review). However, these risks are considered separately. As is well known in the economics literature, these three risks coexist and interact. Indeed, these interacting risks are one of the underlying causes of financial crisis. A case in point was the 2007 credit crisis, proceeded by a housing price bubble with expanded home loan credit risk. When the housing bubble collapsed, massive mortgage loan defaults occurred, and a liquidity crisis resulted (see Brunnermeier, Eisenbach, Sannikov [3] for a review of the related economics literature).

The purpose of this paper is to provide an arbitrage-free valuation model for a credit risky security where there is also liquidity risk and an asset price bubble is present. This valuation methodology applies to any credit risky

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security, although for clarity and illustration, we will apply this model to microfinance loans in this paper.

An outline of this paper is as follows. Section 2 provides the mathematical setup of the model, and section 3 provides the details of the market and the traded securities. Section 3 studies a model with credit risk and bubbles, and section 5 adds liquidity risk. Section 6 applies the model to microfinance loans, and section 7 concludes.

2 The Setup

We begin by considering a stochastic differential equation evolving under a risk neutral measure (to be subsequently defined), so that there is no drift. The classical Black Scholes paradigm gives rise to an equation of the form

$$dX_t = \sigma(X_t)dB_t \qquad X_0 = 1. \tag{1}$$

A more interesting situation is when stochastic volatility also plays a role. In this case the equation for X would be of the form

$$\begin{aligned} dX_t &= \sigma(v_t, X_t)dB_t \qquad X_0 = 1 \\ dv_t &= s(v_t)dW_t + b(v_t)dt \end{aligned} \tag{2}$$

where W is another Brownian motion, either independent or correlated to B.

In this model, the process X represents the evolution of the value of the assets underlying a credit risky loan (or security). X can alternatively be viewed as the value of the business, or the value of the collateral supporting the loan. In the case of microfinance, the loans are often small and without collateral. In these cases what is important is the reputation of the group who is borrowing, especially if there is a past history of successful microfinance loans. In cases such as this, the process X is best interpreted as the value of the business.

We now discuss the role of the Markov process X. Note that in equations (1) and (2) the assets could evolve according to a familiar diffusion, or with "badly behaved" functions σ, s, and b they could even behave like a financial bubble. Put simply, due to the already developed theory of the modeling of financial bubbles (cf, eg, [18]) means that the price process X modeled in (1) or alternatively in (2) must be a strict local martingale under the risk neutral measure (or under the chosen risk neutral measure in the case of (2)). A strict local martingale is a local martingale which is not a martingale.

Delbaen and Sharikawa [8] showed for σ continuous showed that equation (1) gives rise to a strict local martingale if and only if X is nonnegative and one has the condition

$$\int_a^\infty \frac{x}{\sigma(x)^2}dx < \infty \quad \text{for some} \quad a > 0. \tag{3}$$

This was later extended by Mijatovic and Urusov [15] without requiring σ to be continuous but rather satisfying the conditions of Engelbert and Schmidt for weak existence and uniqueness.

The stochastic volatility case does not lend itself to such a simple and elegant formulation, but partial results exist, and are general enough to be useful. In the one dimensional case we have the results of Andersen-Piterbarg [1], Bernard-Cui-McLeish [2], and Lions-Musiela [14]. For the multidimensional case the theory becomes difficult. So far the best results are due to Xue-Mei Li [13] and Dandapani and Protter [6]. A tangentially related result is that of Ruf [19].

As is explained in our companion paper [11] one typically uses a reduced form Cox model to interpret credit risk. We introduce the function

$$g : [0, \infty) \to [0, \infty)$$

which is concave and strictly decreasing in x for all x. We combine g with X when we consider the usual Cox framework, where the stopping time τ represents the time of a default event:

$$\tau = \inf\{t > 0 : \int_0^t g(X_s)ds \geq Z\} \tag{4}$$

where Z is an independent exponential random variable with parameter 1. What's nice about this construction is that the stopping time τ is a "totally inaccessible" stopping time[1] and it has a compensator. Indeed the process

$$1_{\{t \geq \tau\}} - \int_0^t g(X_s)ds \tag{5}$$

[1]See for example [16] for all definitions of terms, such as totally inaccessible stopping times.

is a martingale for its natural filtration. Giving (5) an intuitive interpretation using the idea of hazard rates, we have that

$$P(\tau \in [t, t + dt]) \approx g(X_t),$$

given there has been no default prior to the time t, which of course makes no strict sense since the left side is non-random and the right side is random. However one thinks of the right side as non-random due to the realization of a particular $\omega \in \Omega$.

We can now give our economic interpretation of the function g. Given that $g(X_t)$ represents the default intensity and X_t the value of the assets underlying the loan, g decreasing implies that as the value of the assets increase, the default intensity declines. This is consistent with the structural models of credit risk (see Lando [12]). The concavity of the function g means that as the value of the assets X increases, the default probability decreases but at a decreasing rate. Alternatively stated, as the value of the assets X decreases, the default probability increases at an increasing rate.

To include liquidity costs, following Cetin, Jarrow, Protter [4] we define the *liquidity cost function* $l : [0, \infty) \to [0, \infty)$ which is concave and strictly decreasing in x for all x with $l(0) = 0$ and $0 \le l(x) \le x$ for all x. This represents the value of the assets, after being sold or liquidated. In the event of default, the lender would receive the value after liquidity costs $l(X_t)$, and not the market price X_t for a marginal trade. As the size of the asset sold increases the liquidity cost of a forced sale also increases, and thus the difference $x - l(x)$ will increase with the size of x. This is why l is taken to be concave.

Then, the default intensity with liquidity costs is given by the composition of these two functions, i.e.

$$g(l(X_t)).$$

Note that $g(l(x)) < g(x)$ for all x. With liquidity costs, the default time is therefore

$$\tau = \inf\{t > 0 : \int_0^t g(l(X_s))ds \ge Z\} \tag{6}$$

with

$$1_{\{t \ge \tau\}} - \int_0^t g(l(X_s)ds$$

being a martingale.

3 The Market

Let us assume we have a time varying spot interest rate denoted as the process $(r_t)_{t\geq 0}$, which for now could be taken either to be random or deterministic. For short time durations it is often a reasonable simplifying approximation to assume that $r_t \equiv r$, that is a constant, and deterministic. We initialize the money market account with a dollar at time 0 and denote its time t value by[2]

$$B_t = e^{\int_0^t r_s ds}. \tag{7}$$

We let the time t value of a default-free zero-coupon bond paying a dollar at time T be strictly positive and denoted by $p(t,T) > 0$. Let V_t be the value of the microfinace loan.

We next make the assumption of an absence of arbitrage opportunities by using the now classic theory of Delbaen and Schachermayer [7].

ASSUMPTION. *There exists an equivalent probability measure Q such that*

$$\frac{p(t,\mathcal{T})}{B_t} \text{ for all } \mathcal{T} \in [0,T] \quad \text{and} \quad \frac{V_t}{B_t}$$

are Q local martingales.

As stated earlier, the process X represents the value of the business supporting the payments to the loan.

We need to introduce a new process $(\kappa_t)_{t\geq 0}$, which represents systemic issues rather than just idiosyncratic issues. The process $\kappa_t(\omega)$ represents a default jump risk premium. This leads us to change the Cox construction of (6) to become

$$\tau = \inf\{t > 0 : \int_0^t g(l(X_s))\kappa_s ds \geq Z\}$$

and by analogy equation (5) now becomes

$$1_{\{t\geq\tau\}} - \int_0^t g(l(X_s))\kappa_s ds. \tag{8}$$

We now refine our model for the spot interest rate which was taken *a priori* to be an arbitrary process $r_t = r(Y_t)$ for a given stochastic process Y.

[2] Of course, we assume the necessary measurability and integrability such that the following expression is well-defined.

For illustrative purposes, letting Y equal X, the formula for the price of a default-free zero-coupon bond is

$$p(0,t) = E^Q\{e^{-\int_0^t r(X_s)ds}\}$$

and the price from time t to maturity time T is given by

$$p(0,t) = E^Q\{\, e^{-\int_0^t r(X_s)ds} \Big|\, \mathcal{G}_t\}$$

where $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the underlying filtration relative to which (8) is a martingale.

4 No Liquidity Costs

This section develops a pricing model for credit risk that includes price bubbles and where there is no liquidity risk. Liquidity risk will be included in the next section. For simplicity of notation, we also assume in this section that there is no jump risk premium and $\kappa_s \equiv 1$. In this case the default time is

$$\tau = \inf\{t > 0 : \int_0^t g(X_s)ds \geq Z\} \tag{9}$$

where

$$1_{\{t\geq\tau\}} - \int_0^t g(X_s)ds \tag{10}$$

is a martingale.

If there is a bubble in the risky assets then the diffusion X that satisfies an equation of the form (1) is a local strict martingale. We know that a nonnegative diffusion following an equation such as (1) or (2) is a strict local martingale if and only there exists a point t_0 where $E(X_{t_0}) < 1$, given that $X_0 \equiv 1$. One can prove that for equations of the form (1) or (2) by using the time homogenous strong Markov property that the function $t \mapsto E(X_t)$ is strictly decreasing. That is, for arbitrary t_0 and t_1 with $t_0 < t_1$ we have $E(X_{t_1}) < E(X_{t_0})$. Of course when X is a true martingale then $t \mapsto E(X_t) \equiv 1$. So in some sense the expectation of the asset value process X determines whether or not it is undergoing bubble pricing. Constant means no bubble, and strictly decreasing (the only other possibility) means bubble pricing is in force. To aid the tractability of our analysis we add the following assumption.

Assumption. We replace the process X_t in our model with its expectation $E(X_t)$.

This assumption is a gross simplification, and yet for short time spans it may not be that unreasonable in practice. It certainly does make the analysis more tractable. For the martingale case, we have $\lambda_t = E(X_t)$ is constant: $\lambda_t \equiv \lambda$ for all $t, 0 \leq t \leq 1$. For the strict local martingale case we have $\lambda_t = E(X_t)$ is strictly decreasing in t.

Equations (9) and (10) become

$$\tau = \inf\{t > 0 : \int_0^t g(\lambda)ds \geq Z\} \tag{11}$$

where Z follows an exponential distribution with parameter 1, and

$$1_{\{t \geq \tau\}} - \int_0^t g(\lambda)ds \tag{12}$$

is a martingale. In this case the probability that default does not occur is equivalent to

$$Q(Z \geq \sup_{t \leq 1} g(\lambda)t) = Q(Z \geq g(\lambda)) \tag{13}$$

and therefore the probability of a default occurring is $1 - Q(Z \geq g(\lambda)) = e^{-g(\lambda)}$. [3]

In the bubble case we have the non-random default intensity $t \mapsto \lambda_t$ which as previously discussed is equal to $\lambda_t^Q = E^Q(X_t)$ and is strictly decreasing as a function of t. Note that the value of λ_t depends *a priori* on the choice of the risk neutral measure Q so we denote this dependence on Q with a superscript.

$$\tau = \inf\{t > 0 : \int_0^t g(\lambda_s^Q)ds \geq Z\} \tag{14}$$

where

$$1_{\{t \geq \tau\}} - \int_0^t g(\lambda_s^Q)ds \tag{15}$$

is a martingale.

[3] So in this case, in section 6 below, $\theta = e^{-g(\lambda)}$.

Since in the bubble case $\lambda_s < \lambda_0 = \lambda$ we see by inspection, given that g is a decreasing function, that it takes longer for τ to occur in general when X exhibits a bubble.

In analogy with (13) we have

$$Q(Z \geq \sup_{t \leq 1} \int_0^t g(\lambda_s^Q) ds) = 1 - e^{-\int_0^1 g(\lambda_s^Q) ds)} \tag{16}$$

and this leads to the probability of there being no default to be $e^{-\int_0^1 g(\lambda_s^Q) ds)}$. [4]

5 Liquidity Costs

This section adds liquidity costs to the model in the previous section. Here we need to include the liquidity cost function l. We recall that

$$\tau_l = \inf\{t > 0 : \int_0^t g(l(X_s))\kappa_s ds \geq Z\}. \tag{17}$$

For simplicity, we again assume that jump risk has no risk premium, i.e. $\kappa_s \equiv 1$. In addition, we take things one step further. Recalling that l is concave, Jensen's inequality gives us

$$E(l(X_t)) \leq l(E(X_t)).$$

Also, g is assumed to be concave and decreasing, hence by Jensen's inequality again:

$$E(g(l(X_t)) \leq g(E(l(X_t))) \leq g(l(\lambda_t^Q)) \text{ where } \lambda_t^Q = E^Q(X_t).$$

Next, using our simplifying assumption, we replace $g(l(X_t))$ by $E^Q(g(l(X_t))$. In analogy to the previous calculations we have that

$$\begin{aligned} Q(Z \geq \sup_{t \leq 1} \int_0^t (g(l(E^Q(X_s)))ds) &= Q(Z \geq \int_0^1 g(l((E^Q(X_s)))ds) \\ &= Q(Z \geq \int_0^1 g(l((\lambda_s^Q)))ds) \\ &\leq Q(Z \geq \int_0^1 E^Q(g(l((X_s)))ds). \end{aligned} \tag{18}$$

[4] In this case, in section 6 below, $\theta = e^{-\int_0^1 g(\lambda_s^Q) ds)}$.

And, the probability of a default is

$$1 - (\text{probability of no default}) = 1 - Q(Z \geq \int_0^1 E^Q(g(l((X_s)))ds)$$

$$\leq 1 - Q(Z \geq \int_0^1 g(l((\lambda_s^Q))ds). \tag{19}$$

The probability of default is the complement, hence

$$Q(\text{default}) = e^{-\int_0^1 E^Q(g(l((X_s)))ds)} \geq e^{-\int_0^t g(l(\lambda_s^Q))ds)}. \tag{20}$$

where the second inequality above follows from using (16) with its successive approximations. We conclude that $e^{-\int_0^t g(l(\lambda_s^Q))ds}$ is a crude underestimate of the probability of default, but in spite of this it might be useful as a first pass. [5]

6 A Simple Example

This section applies the previous model to determine a microfinance loan's fair borrowing rate. The borrower is a collection of small business people connected by the microfinance loan, see our companion article [11]. We can mimic the "back of the envelope calculation" developed in that paper. Adding to our standing assumptions we discretize time as finely as one might like. Our interval is divided into increments labeled $1, 2, \ldots, k, \ldots T$.

We calculated the fair lending rate in [11] and found it under a risk neutral measure Q to be equal to

$$i_Q = \frac{1 + \frac{C}{L} - \theta^T p(0, T)}{\sum_{k=1}^T \theta^k p(0, k)} \tag{21}$$

where θ is the probability of default over $[0, 1]$ and $p(0, k) = E^Q(e^{-\int_0^k r_u du})$ for $k = 1, \ldots, T$, and where $p(t, T)$ in general represents the price of a zero coupon bond at time t and expiration date T. The constant L is the principal of the microfinance loan, and the constant C is the cost to the lender in making the loan.

To compute this fair lending rate, needed are: (i) the term structure of zero-coupon bond prices $(p(0, k))$, easily computed in most government

[5] In this case, in section 6 below, $\theta = e^{-\int_0^1 g(l(\lambda_s^Q))ds)}$.

bond markets (see Jarrow [9] for more details), (ii) an estimate of the cost of issuing the loan as a percent of the loan's principal ($\frac{C}{L}$), and (iii) an estimate or conjecture of the probability of default over a year (θ). The one year default probability in a model with both bubbles and liquidity risk is given in expression (20) in the previous section.

For group lending programs these quantities are sometimes readily available or can be obtained. For example, Conlin [5] documents that for the Calmeadow Metrofund, a group lending program in Toronto Canada, administration fees charged were .065 (p. 257, Conlin [5]),[6] and the actual default frequencies for loans between 1990 and 1996 ranged from .0004 - .011 (Table 1, Conlin [5]).

7 Conclusion

This paper presents an arbitrage-free valuation model for a credit risky security where credit risk coexists and interacts with an asset price bubble and liquidity risk (or liquidity costs). As an illustration, this model is applied to determine the fair rate for microfinance loans. The application of this model to practice and its empirical implementation awaits subsequence research.

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