Modeling the Default Risk Charge (DRC) using the intensity model

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Abstract

The last regulation of the Fundamental Review of the Trading Book (FRTB) proposed to replace the Incremental Risk Charge (IRC) with the Default Risk Charge (DRC). Since, many studies were implemented to give the adequate model and the impact of this change. As we know in the modeling area, we always deem many assumptions during the conception and the implementation. These assumptions impact the results of the model output and they are sometimes verified or not in the market. The two commonly assumptions considered on the DRC modeling are: the default is implemented in the structural model (for example Merton model) and the Gaussian copula for correlations between issuers. However, the Merton model could not catch the default for the positions with very small probability of default. Hence, this approach arises a model risk, and we study in this paper the impact of this assumption using the intensity model to compute the value of the DRC to quantify this risk kind.

JEL classification: C15, G10, G18, G19.

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1. *Introduction*

The Fundamental Review of the Trading Book (FRTB) text defines the Default Risk Charge (DRC) as a new measurement of the default and deletes the Incremental Risk Charge (IRC). This measure deems only the default state and includes the equity scope. This risk is measured by a VaR that is based on one-year horizon and a 99.9% confidence level. The computing frequency is weekly, and the DRC capital requirement is equal to:

$$KDRC = \max\left(\frac{1}{N}\sum_{i=1}^{N} DRC_i\,, DRC_{N+1}\right); N = 12$$

To summarize the regulation, there are four components that we should calibrate and model to build the DRC measurement:

- The first one is the obligor's correlation. Initially, the regulator allows the use of the credit spread or the listed equity price historical data. This historical data must include at least 10 years and the stressed period, as defined in the ES model. The chosen liquidity horizon is one-year and 60 days minimum for the equities. These data must give a higher correlation for portfolios, including a short and long position. On the other hand, a low correlation is assigned to the portfolios that contain only long exposures. Next, the obligor's default has to be modeled, using two types of systematic factors to deduce the model correlation. Finally, the correlation measurement must be done on the one-year liquidity horizon.
- The second component is the Probability of Default (PD). The FRTB defines some conditions and priorities for the PD estimation. The first two conditions are: (1) the market PDs are not allowed; and (2) all default probabilities are floored to 0.03%. The Internal Ratings-Based (IRB) PDs come to the top of the choice when the model is validated. Otherwise, they have to develop a conform model to the IRB methodology. Therefore, the historical market PDs should not be used for calibration. The institutions must be based on a historical default and uploaded from a 5-year observation, as minimum of the calibration period. Banks could also use the external rating, provided by the rating agencies (e.g., S&P, Fitch, or Moody's) to estimate PDs. In this case, they must define the priority ranking choice.
- The third component is the Loss Given Default (LGD) model. The LGD model must catch the correlation between the recovery and the systematic factors. The model has to be calibrated by the IRB data if the institution already has a homologated model, and the historical data should be relevant to get accurate estimates. All LGDs must be floored to zero, and the external LGDs could be used too, respective to some defined ranking choice.
- The final component is the Jump to Default (JTD) model. The JTD model must catch the long and short positions for each obligor; and the set assets must contain the credit (i.e., sovereign and corporate credit) and the equity

exposures. This measure can be defined as a function of the LGD and the Exposure at the Default (EAD) for credit assets. However, it must measure the P&L for equities when the default occurs, since we know that the LGD is equal to 100% for the equity assets. The model includes the pricing of the equity derivatives within the zero value of the stock price. The JTD of non-linear products must integrate the multi-default obligors in the case of the derivative products with a multiple underlying. The linear approach could be used for these products, like the sensitivities approach, according to the solely obligor default, which is subject to the supervisor's approval.

There are many studies present and suggest frameworks to model the DRC. The first one was made by Laurent, J.-P., Sestier, M. and Thomas, S. (2016) where they use the Hoeffding decomposition to explain the loss function. The second one was implemented by Wilkens, S. and Predescu, M. (2017) and they propose a complete framework to build the DRC model. However, they all use the Merton model with multifactor and we know that is a structural approach. Hence, the model assumption could make vanish some obligor loss on the DRC, especially for those who have a good rating. In fact, if we take the obligor with a very good rating for example, then the PDs are equal to 0.03% since it is the regulation floor. Therefore, they have more chance to appear in the extreme of tail loss distribution on the Merton Model when we use the Monte Carlo simulation and the DRC could not catch it since this measure is a VaR at 99.9%. Contrariwise, the intensity models allow resolving this issue.

In this paper, we will use the Credit Risk+ as an intensity model to make comparison with the Merton one. The next section describes the two frameworks model and their differences.

2. *The DRC frameworks modeling*

2.1. The comparison between Merton and Credit Risk+ models

The literature defines two approaches for modeling the obligor default: the structural and the intensity approach. On the first hand, the most widely used structural model is Merton's, which defines the default at maturity when the value of the asset is less than the value of the liability. This condition allows writing the default variable for an obligor as $D = \mathbb{1}_{\{X<\Phi^{-1}(PD)\}}$, where PD represents the probability of default and also the health of the obligor since it depends on his rating, and X defines the asset return value and follows a Gaussian distribution. On the second hand, we have the Credit Risk+ as an intensity model and this model defines the default variable as $D = \mathbb{1}_{\{N\geq 1\}}$, with N is the default frequency and follows a Poisson distribution. Basing on the default definition, the first difference between the two models is the type of the variable to simulate the default. As consequence, the Merton model could not catch the default for high rating obligor, because the PD is very close to zero and the default occurs rarely when we use the Monte Carlo simulation. However, the default appears frequently in the Credit Risk+ model, since we don't use the rating as variable to generate the default. We also have the same behavior for obligors who have very small maturity. Indeed, we can use the distance to the default formula for this conclusion and we get the following results:

$$\begin{cases} \lim_{T\to 0} \frac{P(\tau \le T)}{T} = 0 \; Merton\; model \\ \lim_{T\to 0} \frac{P(\tau \le T)}{T} = \lambda > 0 \; Credit\; Risk + model \end{cases}$$

Where τ represents the default time and λ is the default intensity.

The second difference comes from the way how we define the systematic factors. The Merton model supposes that these factors are independent and follow the Gaussian distribution. The asset return is given by:

$$X = \beta Z' + \sigma \times \varepsilon$$

Where β represents the implied correlation vector between the obligor and the systematic factors $Z \sim N(0,1)$, $\sigma = \sqrt{1 - \beta\beta'}$, β' is the transposed vector, and $\varepsilon \sim N(0,1)$ is the specific risk

However, the Credit Risk+ model deems these factors follow the Gamma distribution and are independent. The default intensity is written as:

$$\lambda^Y = \lambda \times (\, \omega^0 + \omega Y')$$

Where λ gives the non-conditional obligor intensity, $\sum_k \omega^k = 1$, $Y \sim Gamma\left(\alpha, \theta, \alpha = {}^1\!/_\theta\right)$, and Y' is the transposed vector.

Therefore, the conditional default probability to the systematic factors has the following formula for each model:

$$\begin{cases} PD(Z) = \Phi\left(\frac{\Phi^{-1}(PD) - \beta Z'}{\sigma}\right) Merton\; model \\ PD(Y) = 1 - \exp(-\,\lambda^Y) \; Credit\; Risk + model \end{cases}$$

2.2. Model definition

The FRTB requires two types of systematic factors to simulate the obligor default. We suggest using the same configuration used in Slime, B. (2017) “Concentration Risk Under the Default Risk Charge (DRC)”. Hence, we deem two types of factors: (1) Global factors and (2) Sectorial factors. To summarize, the first set of factors is built by one global factor and two global asset types: (1) sovereign and (2) corporate. The second asset type contains regional and industry factors. We note these sets, respectively, by $\mathrm{GA} = \{\mathrm{GS}, \mathrm{GC}\}$, $\mathrm{R} = \{1 \ldots \mathrm{r}\}$ and $\mathrm{I} = \{1 \ldots \mathrm{s}\}$. In the last paper, we have proposed the Merton (1974) model with multifactors as a framework and we will keep it for our study. However, we also build an intensity model to establish the comparison with the structural one and we will use the Credit Risk+ model. The Merton model defines the return variable for an obligor i as:

$$X_i = \beta_i \times Z_G + \beta_i^g \times Z_g + \beta_i^j \times Z_j^R + \beta_i^l \times Z_l^I + \sigma_i \varepsilon_i$$

Where Z_G, Z_g, Z_j^R, Z_l^I are independent by set and follow $N(0,1)$, with $g \in GA, j \in R, l \in I$. β gives the correlation between obligors and systematic factors, whereas $\varepsilon_i \sim N(0,1)$ represents the specific risk, and they are independent and identically distributed for $i \in \{1 \dots N\}$ and are independent from all systematic factors. Also, the following formula is used to keep $X_i \sim N(0,1)$:

$$\sigma_i = \sqrt{1 - \left({\beta_i}^2 + {\beta_i^g}^2 + {\beta_i^j}^2 + {\beta_i^l}^2\right)}$$

Therefore, the implied correlation between obligors can be deduced by:

$$\rho^I = \beta \times \rho^F \times \beta' + \sigma^2 \times I$$

where ρ^I represents the obligor implied correlation matrix, $N \times N$; ρ^F is the systematic factor intra-correlation matrix, $K \times K$ and $K = (3 + r + s)$; β represents the correlation factors between the obligors matrix, $N \times (3 + r + s)$, and the systematic factors; β' represents the transposed matrix; σ^2 is the vector of σ_i^2; and I is the identity matrix.

The Credit Risk+ uses the intensity default to simulate the default since the frequency of default follows the Poisson distribution. Hence, we denote N_i the number of default for an obligor i. We keep the same structure of systematic factors and we denote them Y. We deem that the variables of the default numbers between obligors are idiosyncratically independent. The systematic part of these variables conditional on Y follows a Poisson distribution with the following intensity:

$$\lambda_i^Y = \lambda_i \times \left(\omega_i^0 + \omega_i \times Y_G + \omega_i^g \times Y_g + \omega_i^j \times Y_j^R + \omega_i^l \times Y_l^I\right)$$

Where Y_G, Y_g, Y_j^R, Y_l^I are independent by set and are Gamma distributed given parameters $\left(\alpha, \theta, \alpha = {}^1\!/_\theta\right)$, with $g \in GA, j \in R, l \in I$. ω_i verify the following condition $\omega_i^0 + \omega_i + \omega_i^g + \omega_i^j + \omega_i^l = 1$. Finally, λ_i represents the non-conditional intensity for the obligor i.

The first result is coming from the normalization and we have $\mathbb{E}[\lambda_i^Y] = 1$. The second one allows computing the expectation of N_i conditionally to Y, $\mathbb{E}[N_i \,|\mathrm{Y}] = \lambda_i^Y$. The covariance between two obligors is given as follow:

$$COV(N_i, N_j) = \mathbb{E}\left[COV(N_i, N_j|Y)\right] + COV\left(\mathbb{E}[N_i \,|\mathrm{Y}], \mathbb{E}\left[N_j \,|\mathrm{Y}\right]\right)$$

$$= \mathbb{E}\left[\mathbb{1}_{\{i=j\}} \times \lambda_i^Y\right] + COV\left(\lambda_i^Y, \lambda_j^Y\right)$$

$$= \mathbb{1}_{\{i=j\}} \times \lambda_i + COV\left(\lambda_i^Y, \lambda_j^Y\right)$$

We then can write the implied covariance matrix of obligors as:

$$C^I = \omega \times C^F \times \omega' + \lambda \times I$$

Where C^I represents the obligor implied covariance matrix; C^F is the systematic factor intra-covariance matrix; ω represents the matrix of ω_i; ω' represents the transposed matrix; λ is the vector of λ_i; and I is the identity matrix.

Using these results, we can deduce the implied correlation between two obligors:

$$\rho_{i,j}^I = \begin{cases} \dfrac{C_{i,j}^I}{\sqrt{\lambda_i \times \lambda_j}} & i \neq j \\ 1 & i = j \end{cases}$$

Hence, the conditional default probability on systematic factors, in both of models, can be written as follows:

$$\begin{cases} PD_i(Z) = \Phi\left(\dfrac{\Phi^{-1}(PD_i) - \beta_i Z'}{\sigma_i}\right) \\ PD_i(Y) = 1 - \exp(-\lambda_i \times (\omega_i^0 + \omega_i Y')) \end{cases}$$

Given that β_i and ω_i are respectively the obligor lines of the β and ω matrices, Z' and Y' are the systematic vectors transpose and is defined as: $Z = (Z_G, Z_{GS}, Z_{GC}, Z_1^R, \dots, Z_r^R, Z_1^I, \dots, Z_s^I)$; $Y = (Y_G, Y_{GS}, Y_{GC}, Y_1^R, \dots, Y_r^R, Y_1^I, \dots, Y_s^I)$.

We keep the same model in our first study (Concentration Risk Under the DRC (2018)) for the LGD and the JTD, and we deem the following relationship between the LGD and the probability of default conditional on systematic factors:

$$\begin{cases} LGD(Z) = 1 - b \times e^{-a \times PD(Z)} \\ LGD(Y) = 1 - b \times e^{-a \times PD(Y)} \end{cases}; a, b \geq 0$$

Where

$$\begin{cases} a = -ln\left(\dfrac{1 - LGD_{max}}{1 - LGD_{min}}\right) \\ b = 1 - LGD_{min} \end{cases}$$

The JTD for the obligor i is given by:

$$\begin{cases} JTD_i(Z) = LGD_i(Z) \times EAD_i^{Credit} + EAD_i^{Equity} \\ JTD_i(Y) = LGD_i(Y) \times EAD_i^{Credit} + EAD_i^{Equity} \end{cases}$$

With $\mathrm{EAD}_\mathrm{i}^\mathrm{Credit}$ and $\mathrm{EAD}_\mathrm{i}^\mathrm{Equity}$ represent respectively the credit and the equity exposure for the given obligor i.

The loss function is given by the following equations for each model:

$$\begin{cases} L = \sum_{i=1}^{N} JTD_i(Z) \times \mathbb{1}_{\{X_i < \Phi^{-1}(PD_i)\}} \\ L = \sum_{i=1}^{N} JTD_i(Y) \times \mathbb{1}_{\{N_i \geq 1\}} \end{cases}$$

The loss inducing from the systematic factors is defined as follows for the two models:

$$\begin{cases} L_Z = \mathbb{E}[L|Z] = \sum_{i=1}^{N} JTD_i(Z) \times PD_i(Z) \\ L_Y = \mathbb{E}[L|Y] = \sum_{i=1}^{N} JTD_i(Y) \times PD_i(Y) \end{cases}$$

In the next section, we will present the results of the models calibration and compare the DRC values for the two models to make conclusions of the model choice.

3. *Numerical results*

We deem a set of 1,342 issuers within a 10-year historical spread for Merton model and intensity for Credit Risk+ in 6 regions and 11 industries. Our population contains 115 obligors with very small PDs that equal two 0.03% and this also means that for those the default appears rarely in the Merton. However, they have a default intensity that could bring these obligors frequently to the default on the Poisson distribution. In additional, the total exposure summing the long and short positions for these obligors is equal to half million euros and we can have an idea on the difference magnitude between the two models. Figure 1 gives the exposure density of the portfolio used in this paper:

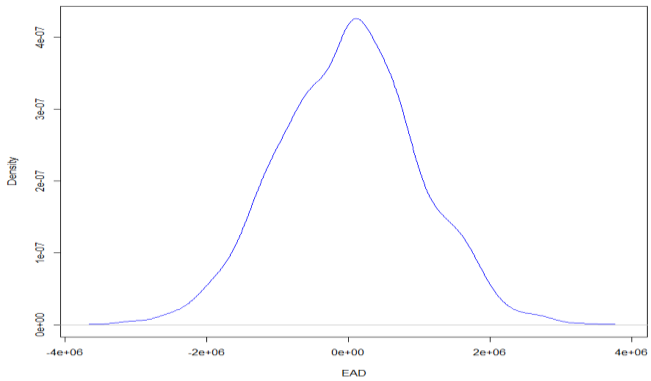

Figure 1: EAD density.

Therefore, we compute the implied correlation for the two models and we compare it with the historical correlation using the following plots:

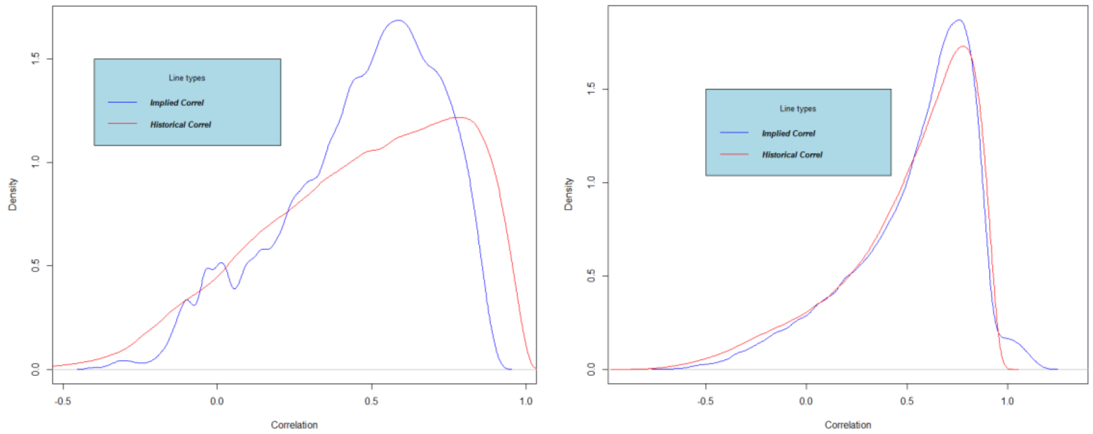

Figure 2: Correlation densities.

The plot in the right represents Merton model correlation density and the left one gives the correlation density for Credit Risk+. Hence, the correlation on the Merton model fit better than the Credit Risk+, we then get another model risk on this part of modeling.

As we see the systematic factors Y follow the Gamma distribution, we should calibrate the factor α for each one using the Maximum Likelihood Estimation (MLE). We deem n observation of $Y = (y_1 \dots y_n)$ and the likelihood function is defined by:

$$l(x,\alpha) = \prod_{i=1}^{n} \frac{\alpha^{\alpha}}{\Gamma(\alpha)} e^{-\alpha y_i} y_i^{\alpha-1} = \left(\frac{\alpha^{\alpha}}{\Gamma(\alpha)}\right)^{n} \times e^{-\alpha \sum_{i=1}^{n} y_i} \times \prod_{i=1}^{n} y_i^{\alpha-1}$$

We then should compute the maximum of the logarithmic function:

$$L(x,\alpha) = n \times \big(\alpha \times ln(\alpha) - ln(\Gamma(\alpha))\big) + \alpha \times \left(\sum_{i=1}^{n} (ln(y_i) - y_i)\right) - \sum_{i=1}^{n} ln(y_i)$$

It remains to develop the first order derivative to find the maximum. The calculation leads to the following results:

$$\frac{\partial L(x,\alpha)}{\partial \alpha} = 0 \Rightarrow ln(\hat{\alpha}) + 1 - \frac{\Gamma'(\hat{\alpha})}{\Gamma(\hat{\alpha})} = \frac{1}{n} \times \left(\sum_{i=1}^{n} (y_i - ln(y_i))\right)$$

We use the Stirling approximation to resolve this equation:

$$ln(\Gamma(\alpha)) \approx \left(\alpha - \frac{1}{2}\right) \times ln(\alpha) - \alpha - ln(\sqrt{2\pi}) \Rightarrow \hat{\alpha} = \frac{n}{2} \times \left(\sum_{i=1}^{n} (y_i - ln(y_i)) - 1\right)^{-1}$$

Once the calibration is complete, we launch computations using Monte Carlo approach with one million simulations to draw the loss densities for the both of models. The results are plotted on the following figures:

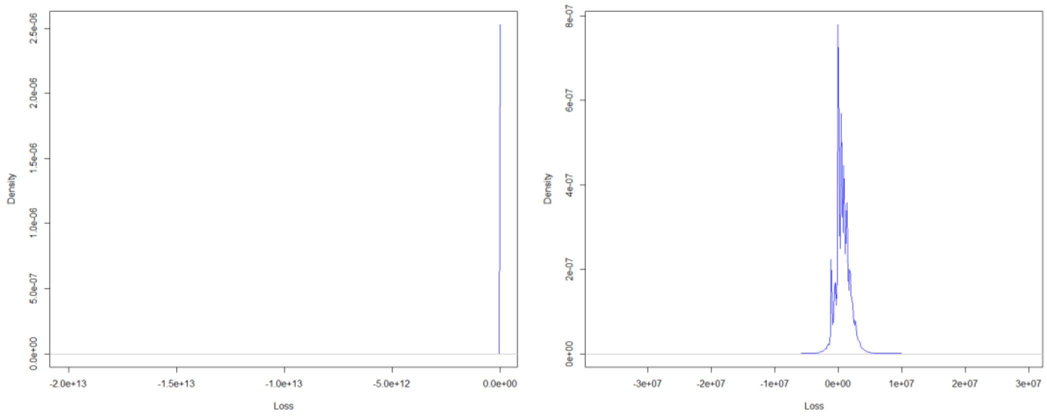

Figure 3: Loss densities.

The graph in the right represents the Merton model loss density and the left one gives the loss density for Credit Risk+. The DRC values are equal respectively 7 396 194 for Merton model and 7 527 208 for Credit Risk+. The value of the relative difference is 1.77% and it seems small because of the small amount of obligor's with small PDs since they are the origin of this difference. However, we can obtain more important difference between the two models and it can arise a risk model that comes from the choice model assumption.

4. Conclusion

This study shows the risk model that could arise from the type choice model. Indeed, we handle two types of model conception. We introduce in the first section the DRC IMA FRTB guidelines. The second section was dedicated to draw the comparison of the Merton and the Credit Risk+ model. We then define our framework model to make implementation and explain results. The Merton is deemed as a structural approach, and theoretically it could not catch the default when the default probabilities are very small. The second one is the Credit Risk+ which is part of intensity model and brings obligors to the default despite the fact that their PDs are very small. The results lead on the same conclusion of the theoretical one since we find that that DRC of the Credit Risk+ model is more important than Merton's. The model risk remains always an issue for all Internal Model Approach and we have to challenge these models since there are always assumptions that could be not verified.

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